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Necessary and sufficient conditions for controllability of multilayer networked systems with nonuniform layer scales

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Many complex systems can be modeled as complex networks for analysis [1]. Recently, the controllability of multilayer complex network systems has become a topic of considerable interest. The controllability of deep-coupling network systems was investigated in [2], and the influence of interlayer couplings on the controllability was clarified. Moreover, the controllability of some special structural networks was also considered. Furthermore, the target controllability of multilayer network systems was considered in [3]. The relationships between controllability, target controllability, and output controllability were clarified. The controllability of multilayer network sampled-data systems was also studied in [4], and some sufficient conditions were proposed.

Although the aforementioned studies consider the interactions between different networks, these approaches require that different layers have the same number of nodes. In practice, the number of nodes in different layers is usually different. For example, in the bus-subway network, the number of buses and subways is usually different, constituting a multilayer network system with nonuniform layer scales. It may not be possible to describe network systems if the layer scales are still assumed to be the same.

The main contributions of this study are fourfold. First, a model of multilayer network systems with nonuniform layer scales is established, in which nodes in different layers can be heterogeneous, and the interconnections can be arbitrary. This model is more general. Second, some necessary and/or sufficient conditions for verifying the controllability of multilayer network systems with nonuniform layer scales are proposed. Compared to the conditions of controllability proposed in [2–4], the conditions of controllability derived herein are applicable to more general multilayer network systems with nonuniform layer scales. Moreover, the conditions presented in this study only involve low-dimensional matrix operations, which have lower computational complexity than classical criteria. Third, some easily verified conditions for network systems with drive-response mode are also given, and the effects of the controllability of the drive layers on the controllability of the entire network are clarified. Finally, the influence of node dynamics on the controllability of multilayer network systems is analyzed. The study may lead to a better understanding of complex network theory, and the realistic model may facilitate the control of practical networks.

Notations. $\sigma(A)$ denotes the set of the eigenvalues of matrix A. Let $[A_{ij}]$ denote a matrix A, where A_{ij} is the element in the ith row and jth column of matrix A. $M(\theta|A)$ denotes the eigenspace of A corresponding to eigenvalue θ. $A \otimes B$ denotes the Kronecker product of matrices A and B. $M_1 \oplus M_2$ denotes the direct sum of space M_1 and M_2. $\emptyset$ denotes the null set.

Model formulation. Consider an M-layer network system with nonuniform layer scales, where layer K consists of N_K node systems. The network system can be described as follows:

$$\begin{cases} \dot{x}_i^K = A^K x_i^K + \sum\limits_{j=1, j\neq i}^{N_K} w_{ij}^K H^K C^K x_j^K \\ \qquad + \sum\limits_{J=1, J\neq K}^{M} \sum\limits_{j=1}^{N_J} d_{ij}^{KJ} H^{KJ} C^J x_j^J + \delta_i^K B^K u_i^K, \\ y_i^K = C^K x_i^K, i = 1, \cdots, N_K, K = 1, \cdots, M, \end{cases} \tag{1}$$

where $x_i^K \in \mathbb{R}^n$, $u_i^K \in \mathbb{R}^p$, and $y_i^K \in \mathbb{R}^m$ denote the state vector, input vector, and output vector of node i in layer K, respectively. $A^K \in \mathbb{R}^{n\times n}$, $B^K \in \mathbb{R}^{n\times p}$, and $C^K \in \mathbb{R}^{m\times n}$ denote the state matrix, input matrix, and output matrix of node i in layer K, respectively. $H^K \in \mathbb{R}^{n\times m}$ is the inner-coupling matrix, which describes the inner-couplings of nodes in layer K. $H^{KJ} \in \mathbb{R}^{n\times m}$ is the interlayer matrix, which describes the intercouplings of nodes in layer K and layer J. $w_{ij}^K \neq 0$ if there exists an edge from node j to node i in layer K, otherwise $w_{ij}^K = 0$. $d_{ij}^{KJ} \neq 0$ if there exists an edge from node j in layer J to node i in layer K, otherwise $d_{ij}^{KJ} = 0$. $\delta_i^K = 1$ if node i in layer K is under control, otherwise $\delta_i^K = 0$.

Let $W^K = [w_{ij}^K] \in \mathbb{R}^{N_K\times N_K}$ and $D^{KJ} = [d_{ij}^{KJ}] \in \mathbb{R}^{N_K\times N_J}$ denote the intralayer network topology matrix of layer K and the interlayer topology matrix of layer K and layer J, respectively. Also, let $\Delta^K = \mathrm{diag}\left\{\delta_1^K, \cdots, \delta_{N_K}^K\right\}$. Set $x^K = [(x_1^K)^{\mathrm{T}}, \cdots, (x_{N_K}^K)^{\mathrm{T}}]^{\mathrm{T}}$ and $u^K = [(u_1^K)^{\mathrm{T}}, \cdots, (u_{N_K}^K)^{\mathrm{T}}]^{\mathrm{T}}$. Let $x = [(x^1)^{\mathrm{T}}, \cdots, (x^M)^{\mathrm{T}}]^{\mathrm{T}}$ and $u = [(u^1)^{\mathrm{T}}, \cdots, (u^M)^{\mathrm{T}}]^{\mathrm{T}}$. Then the multilayer network system can be represented as follows:

$$\dot{x} = \Phi x + \Psi u, \tag{2}$$

where $\Phi = [\Phi^{KJ}]$, $\Psi = [\Psi^{KJ}]$, $\Phi^{KK} = I_{N_K} \otimes A^K + W^K \otimes H^K C^K$, $\Phi^{KJ} = D^{KJ} \otimes H^{KJ} C^J$, $\Psi^{KK} = \Delta^K \otimes B^K, \Psi^{KJ} = 0$, for $K, J = 1, \cdots, M, K \neq J$.

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Let $N_1 + \cdots + N_M = MN$, then $\Phi \in \mathbb{R}^{MNn \times MNn}$ and $\Psi \in \mathbb{R}^{MNn \times MNp}$. Also, let

$$W = \begin{bmatrix} W^1 \cdots & \cdots & D^{1M} \\ \vdots & \ddots & \vdots \\ D^{M1} & \cdots & W^M \end{bmatrix}, \ \Delta = \begin{bmatrix} \Delta^1 & & \\ & \ddots & \\ & & \Delta^M \end{bmatrix}.$$

General multilayer network systems. Here, we derive some controllability conditions suitable for general multilayer network systems.

Lemma 1. The network system (2) is controllable, if and only if

$$\begin{cases} F^K(sI_n - A^K) - (W^K)^{\mathrm{T}} F^K H^K C^K \\ \quad = \sum\limits_{J=1, J\neq K}^{M} (D^{JK})^{\mathrm{T}} F^J H^{JK} C^K, \\ \Delta^K F^K B^K = 0, \ K = 1, \cdots, M \end{cases} \tag{3}$$

have a unique solution $F^K = 0$, where $F^K = [(\alpha_1^K)^{\mathrm{T}}, \cdots, (\alpha_{N_K}^K)^{\mathrm{T}}]^{\mathrm{T}}$, $\alpha_i^K \in \mathbb{C}^{1\times n}$, $i = 1, \cdots, N_K$.

See Appendix C.1 for the proof of Lemma 1.

Remark 1. The necessary and sufficient condition is given in Lemma 1, which is applicable to multilayer networks with an interlayer structure. Compared with MNn dimensional matrix operations while directly using PBH criteria, Lemma 1 only involves N_K $(K = 1, \cdots, M)$ and n dimensional matrix operations.

The index set of the controlled nodes is denoted by

$$U^K = \left\{ i | \delta_i^K \neq 0, i = 1, \cdots, N_K \right\}, K = 1, \cdots, M.$$

Define a vector set

$$M^K(s) = \left\{ [\alpha_1^K, \cdots, \alpha_{N_K}^K] \middle| \begin{matrix} \alpha_i^K \in M_i^{K1}(s), i \notin U^K \\ \alpha_i^K \in M_i^{K2}(s), i \in U^K \end{matrix} \right\},$$

where $K = 1, \cdots, M$, with

$$M_i^{K1}(s) = \left\{ \xi_i^K \in \mathbb{C}^{1\times n} | \xi_i^K (sI - A^K) = 0 \right\}, \text{ and}$$

$$M_i^{K2}(s) = \left\{ \xi_i^K \in \mathbb{C}^{1\times n} | \xi_i^K B^K = 0, \xi_i^K \in M_i^{K1}(s) \right\}.$$

Theorem 1. Assume $H^K \in \mathbb{C}^{n\times 1}$, $B^K \in \mathbb{C}^{n\times 1}$, $C^K \in \mathbb{C}^{1\times n}$, $H^{KJ} \in \mathbb{C}^{n\times 1}$, and $H^{KJ} = H^K$, where $K, J = 1, \cdots, M, J \neq K$. The network system (2) is controllable if the following conditions hold:

(1) (A^K, C^K) is observable, $K = 1, \cdots, M$;

(2) (A^K, H^K) is controllable, $K = 1, \cdots, M$;

(3) For any $s \notin \sigma(A^K)$, rank $(I - W \circ \beta, \Delta\gamma) = MN$, where $\beta^K = C^K(sI - A^K)^{-1}H^K, \beta^{KJ} = C^K(sI - A^K)^{-1}H^{KJ}, \gamma^K = C^K(sI - A^K)^{-1}B^K, K, J = 1, \cdots, M$, $J \neq K$, and

$$\beta = \begin{bmatrix} \beta^1 & \cdots & \beta^{1M} \\ \vdots & \ddots & \vdots \\ \beta^{M1} & \cdots & \beta^M \end{bmatrix}, \gamma = \begin{bmatrix} \gamma^1 & & \\ & \ddots & \\ & & \gamma^M \end{bmatrix};$$

(4) For any $s \in \sigma(A^K)$ and $v^K = [v_1^K, \cdots, v_{N_K}^K] \in M^K(s)$, if $\eta W = 0$, then $\eta = 0$, where $\eta = [\eta^1, \cdots, \eta^M]$, $\eta^K = [\eta_1^K, \cdots, \eta_{N_K}^K]$, $\eta_i^K = v_i^K H^K$, $K = 1, \cdots, M$, $i = 1, \cdots, N_K$;

(5) For any $s \in \sigma(A^L)$, $L \in \{1, \cdots, M\}$, then $\Delta^L = I_{N_L}$; (A^L, B^L) is controllable; rank$(I - W \circ \overline{\beta}, \Delta\overline{\gamma}) = MN$, where

$$\overline{\beta} = \begin{array}{c} \begin{array}{ccccc} & & Lth & & \end{array} \\ \begin{bmatrix} \beta^1 & \cdots & 0 & \cdots & \beta^{1M} \\ \vdots & & \vdots & & \vdots \\ \beta^{M1} & \cdots & 0 & \cdots & \beta^M \end{bmatrix} \end{array},$$

$$\overline{\gamma} = \mathrm{diag}\left\{ \gamma^1, \cdots, \gamma^{L-1}, 0, \gamma^{L+1}, \cdots, \gamma^M \right\}.$$

Remark 2. In multilayer network systems, the nodes are divided into different layers based on the node dynamics, leading to a network system with nonuniform layer scales. Because the node dynamics are heterogeneous between different layers, the eigenvalues of $A^K (K = 1, \cdots, M)$ may be different. Theorem 1 takes this into consideration and provides the controllability conditions verified by low-dimensional matrix operations. The computational complexity is no more than $O(Mn^4 + M^3N^3 + MNn^3)$, verifying that controllability is achievable with Theorem 1, whereas the computational complexity of applying the PBH criteria is $O(M^4N^4n^4)$. Note that Theorem 1 is easier to apply in large-scale networks than the classical criteria. Moreover, the relationship between the controllability of the network systems and the controllability and observability of subsystems is also analyzed in Theorem 1.

See Appendix C.2 for the proof of Theorem 1.

See Appendixes C.3 and C.4 for the other controllability conditions suitable for general multilayer network systems.

The conditions suitable for multilayer network systems with drive-response mode are derived, including directed chain structures and directed star structures, and the conditions for verifying the controllability of multilayer network systems with special node dynamics are presented. Furthermore, some simulated examples are given (Appendixes D–F).

Conclusion. The controllability of multilayer network systems with nonuniform layer scales is investigated herein. In the future, the controllability of networked systems with replay attacks will be considered, as described in [5].

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Supporting information Appendixes A–F. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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