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• Supplementary File •

Necessary and sufficient conditions of controllability of multi-layer networked systems with non-uniform layer scales

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Appendix A Related work

The concept of controllability was first proposed by Kalman in 1960s, see [1] and [2]. In the past decades, controllability has become the most fundamental aspect in system control, and some criteria for verifying controllability were presented, [3–6] and the references therein. With the development of network science, it has been discovered that a large number of practical systems can be modeled as networks to analyze, such as traffic networks in [7], grid networks in [8], social networks in [9], and communication networks in [10] and [11]. Therefore, the study of controllability of networks will help control the networks to serve human society. Initially, some results of controllability of networks were derived with the assumption that the nodes are one dimensional, see [12] and [13].

However, the assumption that nodes are one-dimensional cannot describe the dynamics of nodes, where node systems usually have high-dimensional states in reality. Then, some conditions for verifying the controllability of networked systems with high-dimensional multi-input-multi-output (MIMO) nodes were derived in [14], in which the influence of node dynamics on the controllability was illustrated. In 2017, the controllability of networked systems with single-input-single-output (SISO) nodes was analyzed in [15], where the methods are economical in practice. New methods based on Jordan chain theory for verifying the controllability of networked systems with linear-time-invariant (LTI) node systems were presented in [16] and [17], where the relationship between the eigenvalues and eigenvectors of node systems and networked systems was clarified. Furthermore, the controllability of networked sampled-data systems was studied in [18] with single and multirate sampling.

The above studies assume that nodes in the networks have the same dynamics. Considering a more general situation where nodes in networks may have different dynamics, some conditions of controllability are derived. In [19], the controllability of networked MIMO systems were considered, in which the nodes are heterogeneous high-dimensional dynamical systems. Besides, the special case that networked systems with controllable nodes was also considered. Furthermore, the controllability of heterogeneous networked systems with nonidentical inner-coupling matrices was investigated in [20], in which the influence of inner-coupling matrices on the controllability of networked systems was clarified. Besides, the methods in [16] and [17] were extended to networked sampled-data systems with some assumptions in [21]. Then, the controllability of networked systems with nonidentical dimensional nodes was studied in [22], and some necessary and sufficient conditions were provided.

Nevertheless, the above studies do not take into account the interaction between different networks. However, the networks in reality are usually not independent of each other. For example, the network composed of buses and the network composed of subways together form a multi-layer transportation network due to the common stations, see [23]. The controllability of networked systems with multiple coupling relationships was cnonsidered in [24]. Then, the controllability of multilayer snapback networks was studied in [25], and some conditions for verifying the controllability by small scale factor networks were proposed.

Appendix B Definitions used in the body of the letter

Definition 1. ([26]) For two matrices $A_{n\times m}$ and $B_{n\times m}$, Hadamard product is just the multiplication of each element of one matrix by the corresponding element of the other matrix. Hadamard product is denoted by $\circ$, i.e.,

$$A\circ B=\begin{bmatrix} a_{11}b_{11} & \cdots & a_{1m}b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}b_{n1} & \cdots & a_{nm}b_{nm}\end{bmatrix}. \tag{B1}$$

Furthermore, an extended Hadamard product is defined as follows: if $A=[A_{ij}]\in\mathbb{C}^{ns\times mt}$, where $A_{ij}\in\mathbb{C}^{s\times t}$, $B=[b_{ij}]\in\mathbb{C}^{n\times m}$, then

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$$A \circ B = \begin{bmatrix} A_{11}b_{11} & \cdots & A_{1m}b_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1}b_{n1} & \cdots & A_{nm}b_{nm} \end{bmatrix}.$$

Definition 2. ([27]) A row vector v_m is called the mth-order generalized left eigenvector of matrix $A \in \mathbb{C}^{n\times n}$ corresponding to $\mu \in \sigma(A)$ if

$$v_m(A-\mu I_n)^m = 0 \text{ and } v_m(A-\mu I_n)^{m-1} \neq 0,$$

then $v_1, v_2, \cdots, v_\alpha$ form a left Jordan chain of A about μ on top of v_1, where the maximum value of α is the length of this Jordan chain.

Definition 3. ([17]) Let $A \in \mathbb{C}^{n\times n}$, $H \in \mathbb{C}^{n\times n}$, $\theta \in \sigma(A)$. If row vectors $\xi_1, \xi_2, \cdots, \xi_r$ satisfy

$$\xi_1(\theta I_n - A) = 0 \text{ and } \xi_j(\theta I_n - A) = \xi_{j-1}H,\ j = 2, \cdots, r,$$

then $\xi_1, \xi_2, \cdots, \xi_r$ form a generalized left Jordan chain of A about H corresponding to θ, where ξ_1 is the top vector and the maximum value of r is the length of this generalized Jordan chain.

Appendix C Supplementary of general multi-layer networked systems

Appendix C.1 Proof of Lemma 1

Proof. According to PBH criteria, the networked system (2) is controllable if and only if $\alpha(sI_{MNn} - \Phi, \Psi) = 0$, that is,

$$\begin{cases} \alpha_i^K(sI - A^K) - \sum\limits_{j=1, j\neq i}^{N_K} w_{ji}^K \alpha_j^K H^K C^K = \sum\limits_{J=1, J\neq K}^{M} \sum\limits_{j=1}^{N_J} d_{ji}^{JK} \alpha_j^J H^{JK} C^K, \\ \delta_i^K \alpha_i^K B^K = 0, K = 1, \cdots, M, i = 1, \cdots, N_K \end{cases} \tag{C1}$$

have the unique solution $\alpha = [\alpha^1, \cdots, \alpha^M] = 0$, where $\alpha^K \in \mathbb{C}^{1\times N_K n}$. It is easily to verify that (C1) is equivalent to (3).

Appendix C.2 Proof of Theorem 1

Proof. According to PBH criteria, for $s \in \mathbb{C}$, there exists a vector $\alpha = (\alpha^1, \cdots, \alpha^M) \in \mathbb{C}^{1\times MNn}$, where $\alpha^K = (\alpha_1^K, \cdots, \alpha_{N_K}^K)$, $\alpha_i^K \in \mathbb{C}^{1\times n}$, $K = 1, \cdots, M$, $i = 1, \cdots, N_K$, satisfying that $\alpha(sI - \Phi, \Psi) = 0$, i.e.,

$$\alpha_i^K(sI - A^K) - \sum_{j=1, j\neq i}^{N_K} w_{ji}^K \alpha_j^K H^K C^K - \sum_{J=1, J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \alpha_j^J H^{JK} C^K = 0, \tag{C2}$$

$$\delta_i^K \alpha_i^K B^K = 0, \text{ for } K = 1, \cdots, M,\ i = 1, \cdots, N_K. \tag{C3}$$

If $s \notin \sigma(A^K)$, $K = 1, \cdots, M$, according to (C2), one can get that

$$\alpha_i^K = (\sum_{j=1, j\neq i}^{N_K} w_{ji}^K \alpha_j^K H^K + \sum_{J=1, J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \alpha_j^J H^{JK}) C^K (sI - A^K)^{-1}.$$

For $K = 1, \cdots, M$ and $i = 1, \cdots, N_K$, let

$$\xi_i^K =: \sum_{j=1, j\neq i}^{N_K} w_{ji}^K \alpha_j^K H^K + \sum_{J=1, J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \alpha_j^J H^{JK}, \tag{C4}$$

then

$$\alpha_i^K = \xi_i^K C^K (sI - A^K)^{-1}. \tag{C5}$$

Substituting (C5) to (C4) leads to

$$\xi_i^K = \sum_{j=1, j\neq i}^{N_K} w_{ji}^K \xi_j^K C^K (sI - A^K)^{-1} H^K + \sum_{J=1, J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \xi_j^J C^J (sI - A^J)^{-1} H^{JK}.$$

Define $\beta^K =: C^K(sI - A^K)^{-1}H^K$, $\beta^{KJ} =: C^K(sI - A^K)^{-1}H^{KJ}$, $K, J = 1, \cdots, M$ and $J \neq K$. Besides, let $\xi^K = [\xi_1^K, \cdots, \xi_{N_K}^K]$, then

$$\xi^K(I - W^K\beta^K) - \sum_{J=1, J\neq K}^{M} \xi^J D^{JK} \beta^{JK} = 0,\ K, J = 1, \cdots, M \text{ and } J \neq K.$$

One can get

$$[\xi^1,\cdots,\xi^M](I_{MN}-W\circ\beta)=0. \tag{C6}$$

Let $\xi=[\xi^1,\cdots,\xi^M]$. (C6) can be written as

$$\xi(I_{MN}-W\circ\beta)=0, \tag{C7}$$

where β is defined in condition 3).

Substituting (C5) to (C3), it is able to get

$$\delta_i^K\xi_i^K C^K(sI-A^K)^{-1}B^K=0,\ \text{for}\ K=1,\cdots,M,\ i=1,\cdots,N_K. \tag{C8}$$

Define $\gamma^K=: C^K(sI-A^K)^{-1}B^K$, then (C8) can be written as

$$\xi\Delta\gamma=0, \tag{C9}$$

where γ is defined as condition 3).

Combining (C7) and (C9), one can get

$$\xi(I_{MN}-W\circ\beta,\Delta\gamma)=0.$$

Since the condition 3) that $\text{rank}(I-W\circ\beta,\Delta\gamma)=MN$, it is easy to obtain $\xi=0$ and hence $\xi_i^K=0$, where $K=1,\cdots,M$, $i=1,\cdots,N_K$. From (C5), this is equivalent to $\alpha=0$.

If $s\in\sigma(A^K)$, $K=1,\cdots,M$, then $\text{rank}(sI-A^K)<n$. Based on (C2), it can be got

$$\sum_{j=1,j\neq i}^{N_K}w_{ji}^K\alpha_j^K H^K+\sum_{J=1,J\neq K}^{M}\sum_{j=1}^{N_J}d_{ji}^{JK}\alpha_j^J H^{JK}=0. \tag{C10}$$

If (C10) is not satisfied, then

$$\text{rank}\begin{bmatrix} sI-A^K\\ C^K\end{bmatrix}=\text{rank}(sI-A^K)<n,$$

which contradicts the observability of (A^K,C^K). Combing (C2) and (C10), it can be obtain that for $K=1,\cdots,M$, $i=1,\cdots,N_K$,

$$\alpha_i^K(sI-A^K)=0, \tag{C11}$$

which means that $\alpha^K=[\alpha_1^K,\cdots,\alpha_{N_K}^K]\in M^K(s)$. Let $\alpha_i^K H^K=\eta_i^K\in\mathbb{C}$, In light of (C10) and $H^{KJ}=H^K$, one can get

$$\sum_{j=1,j\neq i}^{N_K}w_{ji}^K\eta_j^K+\sum_{J=1,J\neq K}^{M}\sum_{j=1}^{N_J}d_{ji}^{JK}\eta_j^J=0.$$

Let $\eta^K=[\eta_1^K,\cdots,\eta_{N_K}^K]$, $\eta=[\eta^1,\cdots,\eta^M]$, then $\eta W=0$. Based on condition 4), it is concluded that $\eta_i^K=0$, that is,

$$\alpha_i^K H^K=0,\ \text{for}\ K=1,\cdots,M,\ i=1,\cdots,N_K. \tag{C12}$$

Combining (C11), (C12) and the controllability of (A^K,H^K), one can draw that $\alpha_i^K=0$, holds for $K=1,\cdots,M$, $i=1,\cdots,N_K$, that is, $\alpha=0$ holds.

If $s\in\sigma(A^L)$, $L\in\{1,\cdots,M\}$, then

$$\alpha_i^L(sI-A^L)-\sum_{j=1,j\neq i}^{N_L}w_{ji}^L\alpha_j^L H^L C^L-\sum_{J=1,J\neq L}^{M}\sum_{j=1}^{N_J}d_{ji}^{JL}\alpha_j^J H^{JL}C^L=0, \tag{C13}$$

$$\alpha_i^L B^L=0. \tag{C14}$$

Since $\text{rank}(sI-A^L)<n$ and the observability of (A^L,C^L), one has

$$\sum_{j=1,j\neq i}^{N_L}w_{ji}^L\alpha_j^L H^L+\sum_{J=1,J\neq L}^{M}\sum_{j=1}^{N_J}d_{ji}^{JL}\alpha_j^J H^{JL}=0.$$

Furthermore,

$$\alpha_i^L(sI-A^L)=0. \tag{C15}$$

Combining (C14), (C15) and the controllability of (A^L,B^L), one can draw that

$$\alpha_i^L=0,\ \text{for}\ i=1,\cdots,N_L. \tag{C16}$$

Obviously, one has

$$\alpha_i^K(sI-A^K)-\sum_{j=1,j\neq i}^{N_K}w_{ji}^K\alpha_j^K H^K C^K-\sum_{J=1,J\neq K}^{M}\sum_{j=1}^{N_J}d_{ji}^{JK}\alpha_j^J H^{JK}C^K=0, \tag{C17}$$

$$\delta_i^K\alpha_i^K B^K=0, \tag{C18}$$

hold for $K=1,\cdots,M$ and $K\neq L$. Then,

$$\alpha_i^K = (\sum_{j=1,j\neq i}^{N_K} w_{ji}^K \alpha_j^K H^K + \sum_{J=1,J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \alpha_j^J H^{JK}) C^K (sI - A^K)^{-1}.$$

Let

$$\xi_i^K =: \sum_{j=1,j\neq i}^{N_K} w_{ji}^K \alpha_j^K H^K + \sum_{J=1,J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \alpha_j^J H^{JK}. \tag{C19}$$

Substituting (C19) to (C17) leads to

$$\alpha_i^K = \xi_i^K C^K (sI - A^K)^{-1}. \tag{C20}$$

Then, substituting (C20) to (C19), it implies that

$$\xi_i^K = \sum_{j=1,j\neq i}^{N_K} w_{ji}^K \xi_j^K C^K (sI - A^K)^{-1} H^K + \sum_{J=1,J\neq K}^{M} \sum_{j=1}^{N_J} d_{ji}^{JK} \xi_j^J C^J (sI - A^J)^{-1} H^{JK}.$$

Define $\beta^K =: C^K(sI - A^K)^{-1}H^K$, $\beta^{KJ} =: C^K(sI - A^K)^{-1}H^{KJ}$, for $K, J = 1, \cdots, M$, $K \neq L$ and $J \neq K$. Besides, let $\xi^K = [\xi_1^K, \cdots, \xi_{N_K}^K]$, one has

$$\xi^K (I - W^K \beta^K) - \sum_{J=1,J\neq K,L}^{M} \xi^J D^{JK} \beta^{JK} = 0, \text{for } K = 1, \cdots, M \text{ and } K \neq L. \tag{C21}$$

Define $\xi^L =: [0, \cdots, 0] \in \mathbb{C}^{1\times N_L}$, $\beta^L =: 0$ and $\beta^{JL} =: 0$, for $J = 1, \cdots, M$ and $J \neq L$. According to (C21), it can be obtained that

$$(\xi^1, \cdots, 0, \cdots, \xi^M)(I - W \circ \overline{\beta}) = 0. \tag{C22}$$

Let $\overline{\xi} = [\xi^1, \cdots, 0, \cdots, \xi^M]$, then (C22) is equivalent to

$$\overline{\xi}(I - W \circ \overline{\beta}) = 0, \tag{C23}$$

where $\overline{\beta}$ is defined in condition 5). Substituting (C20) to (C18) leads to

$$\delta_i^K \xi_i^K C^K (sI - A^K)^{-1} B^K = 0. \tag{C24}$$

Define

$$\gamma^K =: C^K (sI - A^K)^{-1} B^K, \text{ for } K \neq L. \tag{C25}$$

Substituting (C25) to (C24), one can draw $\delta_i^K \xi_i^K \gamma^K = 0$, holds for $K \neq L$, that is,

$$\xi^K \Delta^K \gamma^K = 0, \text{ for } K \neq L. \tag{C26}$$

Let $\gamma^L = 0$, and define $\overline{\gamma} = \mathrm{diag}\{\gamma^1, \cdots, 0, \cdots, \gamma^M\}$. Based on (C26), one can get

$$\overline{\xi}\Delta\overline{\gamma} = 0. \tag{C27}$$

In view of (C23) and (C27), one has $\overline{\xi}(I - W \circ \overline{\beta}, \Delta\overline{\gamma}) = 0$. Since $\mathrm{rank}(I - W \circ \overline{\beta}, \Delta\overline{\gamma}) = MN$, one can obtain $\overline{\xi} = 0$, i.e., $\xi_i^K = 0$, holds for $K = 1, \cdots, M$ and $K \neq L$, $i = 1, \cdots, N_K$. Then,

$$\alpha_i^K = 0, \text{ for } K = 1, \cdots, M, K \neq L, i = 1, \cdots, N_K. \tag{C28}$$

Combining (C28) and (C16), it is concluded that $\alpha_i^K = 0$, holds for $K = 1, \cdots, M$, $i = 1, \cdots, N_K$, i.e., $\alpha = 0$.

Appendix C.3 Proposition C1

Proposition C1. If there exists node i in layer K without incoming edges from other nodes in layer K or other layers, the node system can be described as

$$\dot{x}_i^K = A^K x_i^K + \delta_i^K B^K u_i^K. \tag{C29}$$

The node system is controllable, if and only if it is equivalent to the following form:

$$\dot{\hat{x}}_i^K = \hat{A}_c^K \hat{x}_i^K + \hat{B}_c^K u_i^K, \tag{C30}$$

where $\hat{x}_i^K = \begin{bmatrix} \hat{x}_{i1}^K \\ \vdots \\ \hat{x}_{ip}^K \end{bmatrix}$, $\hat{x}_{ik}^K = \begin{bmatrix} \hat{x}_{ik}^K(1) \\ \vdots \\ \hat{x}_{ik}^K(\mu_k) \end{bmatrix} \in \mathbb{R}^{\mu_k}$, $u_i^K = \begin{bmatrix} u_{i1}^K \\ \vdots \\ u_{ip}^K \end{bmatrix}$, $\hat{A}_c^K = \begin{bmatrix} \hat{A}_{11}^K & \cdots & \hat{A}_{1p}^K \\ \vdots & \ddots & \vdots \\ \hat{A}_{p1}^K & \cdots & \hat{A}_{pp}^K \end{bmatrix}_{(n\times n)}$, $\hat{B}_c^K = \begin{bmatrix} B_1^K \\ \vdots \\ B_p^K \end{bmatrix}_{(n\times p)}$, in which

$$\hat{A}_{kk}^K = \left[\begin{array}{c|cccc} 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & & 1 \\ \hline -\alpha_{kk}^K(0) & -\alpha_{kk}^K(1) & \cdots & & -\alpha_{kk}^K(\mu_k-1) \end{array}\right]_{(\mu_k \times \mu_k)} \tag{C31}$$

$$\hat{A}_{kj}^K = \begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ -\alpha_{kj}^K(0) & \cdots\cdots & -\alpha_{kj}^K(\mu_j-1) \end{bmatrix}_{(\mu_k \times \mu_j)} \tag{C32}$$

$$B_k^K = \begin{bmatrix} 0 & \cdots\cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots\cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 1 \quad b_{k,k+1}^K & \cdots & b_{k,p}^K \end{bmatrix}_{(\mu_k \times p)} \tag{C33}$$

Proof. If node i in layer K without incoming edges from other nodes in layer K or other layers, then the node system is controllable, if and only if the node is under control and the system (A^K, B^K) is controllable. Not difficult to find (C30) is the controllable canonical form of (A^K, B^K).

Appendix C.4 Theorem C1

Before giving Theorem C1, a lemma is introduced.

Lemma C1. ([28]) The linear system (C29) is controllable, if and only if it can be equivalently converted into a linear high-order fully actuated (HOFA) system in the form of

$$A_\mu^K z_i^{K(\mu)} + A_{\mu-1}^K z_i^{K(\mu-1)} + \cdots + A_1^K \dot{z}_i^K + A_0^K z_i^K = \tilde{B}^K u_i^K,$$

where z_i^K is a p dimensional vector, $A_j^K \in \mathbb{R}^{p\times p}$, $j = 0, \cdots, \mu$, are the cofficient matrices, and

$$\tilde{B}^K = \begin{bmatrix} 1 & b_{12}^K & b_{13}^K & \cdots & b_{1p}^K \\ & 1 & b_{23}^K & \cdots & b_{2p}^K \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}.$$

Since some matrices A_j^K may be singular, the above system can generally written as

$$\begin{bmatrix} x_{i1}^K(\mu_1) \\ x_{i2}^K(\mu_2) \\ \vdots \\ x_{ip}^K(\mu_p) \end{bmatrix} = \begin{bmatrix} L_1(x_{ik}^K(0\sim\mu_k-1)|_{k=1\sim p}) \\ L_2(x_{ik}^K(0\sim\mu_k-1)|_{k=1\sim p}) \\ \vdots \\ L_p(x_{ik}^K(0\sim\mu_k-1)|_{k=1\sim p}) \end{bmatrix} + \tilde{B}^K u_i^K,$$

where

$$x_{ik}^K(0\sim\mu_k-1)|_{k=1\sim p} = \begin{bmatrix} x_{i1}^K(0) \\ \vdots \\ x_{i1}^K(\mu_1-1) \\ \vdots \\ x_{ip}^K(0) \\ \vdots \\ x_{ip}^K(\mu_p-1) \end{bmatrix},$$

$L_k(\cdot)$, for $k = 1, \cdots, p$, are some linear functions. μ_k, for $k = 1, \cdots, p$, are a set of integers, and $\mu_1 + \cdots + \mu_p = n$.

Theorem C1. Consider the networked system (2). If there exists node i in layer L without incoming edges from other nodes in layer L or other layers, where $L \in \{1, \cdots, M\}$, $i \in \{1, \cdots, N_L\}$, then the multi-layer networked system is controllable, only if the following conditions hold:

1) $\delta_i^L = 1$;

2) (A^L, B^L) can be converted into an HOFA system:

$$A_\mu^L z_i^{L(\mu)} + A_{\mu-1}^L z_i^{L(\mu-1)} + \cdots + A_1^L \dot{z}_i^L + A_0^L z_i^L = \tilde{B}^L u_i^L,$$

where $z_i^L = [z_{i1}^L, \cdots, z_{ip}^L]^T$, $\hat{x}_{it}^L(j) = z_{it}^{L(j-1)}$, $t = 1, \cdots, p$, $j = 1, \cdots, \mu_t$, $\mu = \max\{\mu_i, i = 1, \cdots, p\}$, $\mu_1 + \cdots + \mu_p = n$, $A_k^L = [\alpha_{sl}^L(k)]_{p \times p}$, $k = 0, \cdots, \mu$, and

$$\tilde{B}^L = \begin{bmatrix} 1 & b_{12}^L & \cdots & \cdots & b_{1p}^L \\ & \ddots & b_{23}^L & \cdots & b_{2p}^L \\ & & \ddots & & \vdots \\ & & & & \\ & & & & 1 \end{bmatrix}_{p \times p},$$

where $\alpha_{sl}^L(k)$ and b_{sl}^L are defined as (C31)-(C33), $s, l = 1, \cdots, p$, $k = 1, \cdots, \mu_i$ and

$$\begin{cases} \alpha_{sl}^L(k) = 0, \text{if } k > \mu_s \text{ or } k = \mu_s \text{ and } s \neq l \\ \alpha_{ss}^L(\mu_s) = 1, \text{if } s = 1, \cdots, p. \end{cases}$$

Proof. Since node i in layer L without incoming edges from the other nodes in layer L or other layers, the ith row block of Φ^{LL} is $[0 \cdots A^L \cdots 0]$. If condition 1) is not satisfied, then $\delta_i^L = 0$. the ith row block of Ψ^{LL} is $[0 \cdots 0]$. For any $s_0 \in \sigma(A^L)$, the rank of $(s_0 I - \Phi, \Psi)$ will reduce at least 1, which means the networked system is uncontrollable. If condition 2) is not satisfied, one can get (A^L, B^L) is uncontrollable based on Lemma C1. For any $s_0 \in \sigma(A^L)$, $\mathrm{rank}(s_0 I - A^L, B^L) < n$, then, $\mathrm{rank}(s_0 I - \Phi, \Psi) < MNn$. One can get the networked system is uncontrollable.

Remark 1. In multi-layer networked systems with non-uniform layer scales, the controllability of networked systems is usually not related to the controllability of node systems because of the influence of couplings of other nodes in the same layer or the other layers. Nevertheless, if a node without incoming edges, the controllability of this node system have a great influence on the controllability of the entire networked system according to Theorem C1, which is convenient for verifying the uncontrollable networked systems. Besides, some physical systems with high-order original models may detach from the physical background if it is converted into first-order systems. In addition, it may increase the difficulty of research, see [28]. This situation is considered in Theorem C1, which makes it applicable in practice. The controllability of multi-layer networked systems with malicious attacks will be further considered, as described in [29].

Appendix D Multi-layer networked systems with drive-response mode

Many practical networks only have edges pointing from the upper layers to the lower layers, which is called drive-response mode. Among the multi-layer networked systems with drive-response mode, the most typical types are directed chain and directed star structures, see [30] and [31]. Besides, for the one-layer networked systems, directed chain and directed star are the typical structures, see [18] and [20]. The typical multi-layer networks will be analyzed.

Based on Lemma D1 and Lemma D2, the eigenvalues and left eigenvectors of state matrix of each layer can be obtained.

Lemma D1. ([17]) Denote $\sigma(W^K) = \left\{\mu_1^K, \cdots, \mu_{r_K}^K\right\}$, $\sigma(A^K + \mu_i^K H^K) = \left\{\lambda_{i1}^K, \cdots, \lambda_{i,p_i^K}^K\right\}$, for $i = 1, \cdots, r_K$, $K = 1, \cdots, M$. Then $\sigma(\Phi^{KK}) = \left\{\lambda_{11}^K, \cdots, \lambda_{1,p_1^K}^K, \cdots, \lambda_{r_K,1}^K, \cdots, \lambda_{r_K,p_{r_K}^K}^K\right\}$.

Lemma D2. ([17]) Suppose the left Jordan chain of W^K about μ_i^K are $\alpha_i^K(1), \cdots, \alpha_i^K(\gamma_i^K)$, and the generalized left Jordan chain of $A^K + \mu_i^K H^K$ about H^K corresponding to λ_{ij}^K are $\xi_{ij}^K(1), \cdots, \xi_{ij}^K(\theta_{ij}^K)$, where $i = \{1, \cdots, r_K\}$, $j = \{1, \cdots, p_i^K\}$. Then the left eigenvectors of Φ^{KK} corresponding to λ_{ij}^K are $\eta_{ij}^K(1) = \alpha_i^K(1) \otimes \xi_{ij}^K(1)$, $\eta_{ij}^K(2) = \alpha_i^K(1) \otimes \xi_{ij}^K(2) + \alpha_i^K(2) \otimes \xi_{ij}^K(1), \cdots, \eta_{ij}^K(\epsilon_{ij}^K) = \alpha_i^K(1) \otimes \xi_{ij}^K(\epsilon_{ij}^K) + \cdots + \alpha_i^K(\epsilon_{ij}^K) \otimes \xi_{ij}^K(1)$, where $\epsilon_{ij}^K = \min\left\{\gamma_i^K, \theta_{ij}^K\right\}$, $i = 1, \cdots, \gamma_K$, $j = 1, \cdots, p_i^K$.

The left eigenspace of Φ^{KK} corresponding to λ_{ij}^K is $M(\lambda_{ij}^K | \Phi^{KK}) = V(\lambda_{ij}^K) = \mathrm{span}\left\{\eta_{ij}^K(1), \cdots, \eta_{ij}^K(\epsilon_{ij}^K)\right\}$. If $\lambda_{i_1 j_1}^K = \cdots = \lambda_{i_q j_q}^K$, where $q > 1$, the left eigenspace about λ_{ij}^K is the direct sum of all the left eigenspace of $\lambda_{i_k j_k}^K$, $k = 1, \cdots, q$, i.e., $M(\lambda_{ij}^K | \Phi^{KK}) = \oplus_{k=1}^q V(\lambda_{i_k j_k}^K)$.

Appendix D.1 Directed chain structure

For multi-layer networked systems, if there only exists edges from upper layers to the lower layers, i.e., all the inter-topology matrices are zero matrices, except $D^{K,K-1}(K = 2, \cdots, M)$, then this type of interlayer coupling mode is called directed chain structure. Firstly, the controllability of directed chain multi-layer networks is analyzed, where each layer of the networks has directed chain structure. This type of networked system is as shown in Fig. D1, and the state matrix and input matrix in (2) are as follows:

$$\Phi = \begin{bmatrix} \Phi^{11} & & & \\ \Phi^{21} & \Phi^{22} & & \\ & \ddots & \ddots & \\ & & \Phi^{M,M-1} & \Phi^{MM} \end{bmatrix}, \Psi = \begin{bmatrix} \Psi^{11} & & \\ & \ddots & \\ & & \Psi^{MM} \end{bmatrix} \tag{D1}$$

where

$$\Phi^{KK} = \begin{bmatrix} A^{K} & & & \\ w_{21}^{K} & A^{K} & & \\ & \ddots & \ddots & \\ & & w_{N_K,N_K-1}^{K} & A^{K} \end{bmatrix}, K = 1, \cdots, M.$$

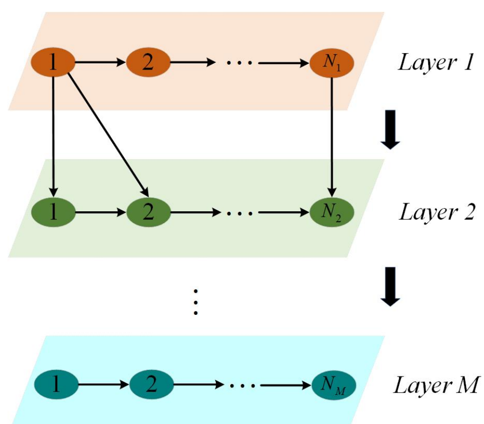

Figure D1 A directed chain multi-layer networked system with directed chain in each layer

Theorem D2. Suppose $\sigma(A^K)\bigcap\sigma(A^J)=\emptyset$, for $K, J = 1, \cdots, M$, and $K \neq J$. The networked system (D1) is controllable, if the following conditions hold simultaneously:

1) for $\eta_1 \in M(\theta_{ij}^{1}|\Phi^{11})$ and $\eta_1 \neq 0$, then $\eta_1(\Delta^1 \otimes B^1) \neq 0$, $i = 1, \cdots, r_1$, $j = 1, \cdots, p_i^1$;

2) for $\eta_K \in M(\theta_{ij}^{K}|\Phi^{KK})$, $\eta_K \neq 0$ and $\eta_J \in \Xi_{ij}^{J}$, then $[\eta_P(\Delta^P \otimes B^P), \eta_K(\Delta^K \otimes B^K)] \neq 0$, where $\Xi_{ij}^{J} = \left\{\eta_J \in \mathbb{C}^{1\times N_J n} | \eta_J(\theta_{ij}^{K} I - \Phi^{JJ}) = \eta_{J+1}\Phi^{J+1,J}\right\}$, $2 \leqslant K \leqslant M$, $1 \leqslant J \leqslant K-1$, $P \in \{1, \cdots, K-1\}$, $i = 1, \cdots, r_K$, $j = 1, \cdots, p_i^K$.

Proof. Firstly, $\sigma(\Phi) = \sigma(\Phi^{11})\bigcup\sigma(\Phi^{22})\bigcup\cdots\bigcup\sigma(\Phi^{MM})$. According to PBH criteria, the networked system (Φ, Ψ) is uncontrollable, then there exists $\theta \in \sigma(\Phi)$, such that $\text{rank}([\theta I - \Phi, \Psi]) < MNn$, i.e., there exists a nonzero vector $\eta = [\eta_1, \cdots, \eta_M]$ satisfying

$$\eta(\theta I - \Phi) = 0, \tag{D2}$$

$$\eta\Psi = 0. \tag{D3}$$

where $\eta_K \in \mathbb{C}^{1\times N_K n}$, $K = 1, \cdots, M$.

If $\theta \in \sigma(\Phi^{MM})$, then $\eta_M \in M(\theta|\Phi^{MM})$. If $\eta_M \neq 0$, then $\eta_M(\theta I - \Phi^{MM}) = 0$, $\eta_{M-1}(\theta I - \Phi^{M-1,M-1}) = \eta_M\Phi^{M,M-1}, \cdots, \eta_1(\theta I - \Phi^{11}) = \eta_2\Phi^{21}$, that is, $\eta_J(\theta_{ij}^{M} I - \Phi^{JJ}) = \eta_{J+1}\Phi^{J+1,J}$, $1 \leqslant J \leqslant M-1$. Since $\eta_M(\Delta^M \otimes B^M) = 0$ and $\eta_J(\Delta^J \otimes B^J) = 0$, condition 2) is not satisfied. If $\eta_M = 0$, then $\eta_{M-1}(\theta I - \Phi^{M-1,M-1}) = 0$. If $\eta_{M-1} \neq 0$, one can draw $\theta \in \sigma(\Phi^{M-1,M-1})$, which contradicts the assumption. If $\eta_{M-1} = 0$, there exists $L \in \{1, \cdots, M-2\}$, such that $\eta_L(\theta I - \Phi^{LL}) = 0$, where $\eta_L \neq 0$. Since $\theta \in \sigma(\Phi^{LL})$, the assumption is not satisfied. If $\theta \in \sigma(\Phi^{KK})$, where $2 \leqslant K \leqslant M-1$, then $\theta \notin \sigma(\Phi^{SS})$, i.e., $\eta_S = 0$, $K < S \leqslant M$. $\eta_k(\theta I - \Phi^{KK}) = 0, \cdots, \eta_1(\theta I - \Phi^{11}) = \eta_2\Phi^{21}$, where $\eta_K \in M(\theta|\Phi^{KK})$ and $\eta_J(\theta_{ij}^{K} I - \Phi^{JJ}) = \eta_{J+1}\Phi^{J+1,J}$, $1 \leqslant J \leqslant K-1$. If $\eta_K = 0$, there exists $L \in \{1, \cdots, K-1\}$ satisfying $\eta_L(\theta I - \Phi^{LL}) = 0$ with $\eta_L \neq 0$, that is, $\theta \in \sigma(\Phi^{LL})$. It is concluded that $\eta_K \neq 0$, $\eta_K(\Delta^K \otimes B^K) = 0$, and $\eta_J(\Delta^J \otimes B^J) = 0$, $2 \leqslant K \leqslant M-1$, $1 \leqslant J \leqslant K-1$. If $\theta \in \sigma(\Phi^{11})$, it is obviously $\eta_M = 0, \cdots, \eta_2 = 0, \eta_1 \neq 0$, and $\eta_1(\theta I - \Phi^{11}) = 0$. One can obtain $\eta_1 \in M(\theta_{ij}^{1}|\Phi^{11})$ and $\eta_1(\Delta^1 \otimes B^1) = 0$, which contradicts condition 1).

Remark 2. Based on the relationship of the eigenvectors and eigenvalues of $\Phi^{KK}(K=1,\cdots,M)$ and the eigenvectors and eigenvalues of the subsystems established in Lemma D1 and Lemma D2, Theorem D2 gives conditions to verify the controllability of the networked systems by verifying the controllability of single layers. The controllability of drive layers can be verified independently, while the controllability of response layers can not be verified independently due to the inter-layer couplings.

Corollary 1. The networked system (D1) is controllable, only if the following conditions hold:

1) $\delta_1^1 = 1$;

2) (A^1, B^1) can be converted into an HOFA system:

$$A_\mu^1 z_1^{1(\mu)} + A_{\mu-1}^1 z_1^{1(\mu-1)} + \cdots + A_1^1 \dot{z}_1^1 + A_0^1 z_1^1 = \tilde{B}^1 u_i^1,$$

where $z_1^1 = [z_{11}^1, \cdots, z_{1p}^1]^T$, $\hat{x}_{1t}^1(j) = z_{1t}^{1\,(j-1)}$, $t=1,\cdots,p$, $j=1,\cdots,\mu_t$, $\mu = \max\{\mu_i, i=1,\cdots,p\}$, $\mu_1+\cdots+\mu_p = n$, $A_k^1 = [\alpha_{sl}^1(k)]_{p\times p}$, $k=0,\cdots,\mu$,

$$\tilde{B}^1 = \begin{bmatrix} 1 & b_{12}^1 & \cdots & \cdots & b_{1p}^1 \\ & \ddots & b_{23}^1 & \cdots & b_{2p}^1 \\ & & \ddots & & \vdots \\ & & & & 1 \end{bmatrix}_{p\times p},$$

in which $\alpha_{sl}^1(k)$ and b_{sl}^1 are defined as (C31)-(C33), $s,\ l=1,\cdots,p,\ k=1,\cdots,\mu_i$, and

$$\begin{cases} \alpha_{sl}^1(k) = 0, \ \text{if } k > \mu_s \text{ or } k = \mu_s \text{ and } s \neq l, \\ \alpha_{ss}^1(\mu_s) = 1, \ \text{if } s = 1,\cdots,p. \end{cases}$$

Proof. Obviously, node 1 in layer 1 without incoming edges from other nodes in layer 1 or other layers. Based on Theorem C1, the proof is omitted.

Appendix D.2 Directed star structure

For multi-layer networked systems, if there only exists edges from upper layers to the lower layers, i.e., all the inter-topology matrices are zero matrices, except $D^{K1} \neq 0$ $(K=2,\cdots,M)$, then this type of interlayer coulping mode is called directed star structure. The controllability of directed star multi-layer networks is considered, in which each layer of the networks has directed star structure. This type of networked system is as shown in Fig. D2, and the state matrix and input matrix in (2) are as follows:

$$\Phi = \begin{bmatrix} \Phi^{11} & & & \\ \Phi^{21} & \Phi^{22} & & \\ \vdots & & \ddots & \\ \Phi^{M1} & & & \Phi^{MM} \end{bmatrix}, \Psi = \begin{bmatrix} \Psi^{11} & & \\ & \ddots & \\ & & \Psi^{MM} \end{bmatrix}, \tag{D4}$$

where

$$\Phi^{KK} = \begin{bmatrix} A^K & & & \\ w_{21}^K & \ddots & & \\ \vdots & & \ddots & \\ w_{N_K 1}^K & & & A^K \end{bmatrix}.$$

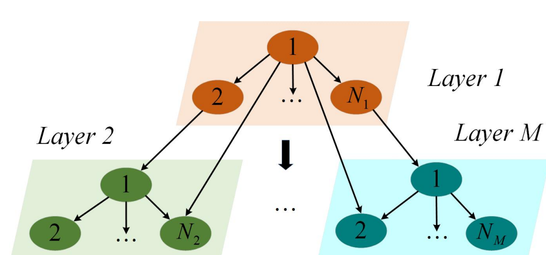

Figure D2 A directed star multi-layer networked system with directed star in each layer

Theorem D3. Suppose $\sigma(A^K)\bigcap\sigma(A^J)=\emptyset$, for $K,J=1,\cdots,M$, and $K\neq J$. The networked system (D4) is controllable, if the following conditions hold:

1) for $\eta_1\in M(\theta^1_{ij}|\Phi^{11})$ and $\eta_1\neq 0$, $\eta_1(\Delta^1\otimes B^1)\neq 0$, $i=1,\cdots,r_1$, $j=1,\cdots,p^1_i$;

2) for $\eta_K\in M(\theta^K_{ij}|\Phi^{KK})$, $\eta_K\neq 0$ and $\eta_1\in\Xi^K_{ij}$, then $[\eta_1(\Delta^1\otimes B^1),\ \eta_K(\Delta^K\otimes B^K)]\neq 0$, in which $\Xi^K_{ij}=\left\{\eta_1\in\mathbb{C}^{1\times N_1 n}|\eta_1(\theta^K_{ij}I-\Phi^{11})=\eta_K\Phi^{K1}\right\}$, $2\leqslant K\leqslant M$, $i=1,\cdots,r_K$, $j=1,\cdots,p^K_i$.

Proof. First of all, $\sigma(\Phi)=\sigma(\Phi^{11})\bigcup\cdots\bigcup\sigma(\Phi^{MM})$. If the networked system (Φ,Ψ) is uncontrollable, there exist $\theta\in\sigma(\Phi)$ and a nonzero vector $\eta=[\eta_1,\cdots,\eta_M]$ satisfying

$$\eta(\theta I-\Phi)=0, \tag{D5}$$

$$\eta\Psi=0, \tag{D6}$$

where $\eta_K\in\mathbb{C}^{1\times N_K}$, $K=1,\cdots,M$. That is,

$$\eta_1(\theta I-\Phi^{11})-\eta_2\Phi^{21}-\cdots-\eta_M\Phi^{M1}=0, \tag{D7}$$

$$\eta_2(\theta I-\Phi^{22})=0,\cdots,\eta_M(\theta I-\Phi^{MM})=0. \tag{D8}$$

If $\theta\in\sigma(\Phi^{KK})$, then $\eta_K\in M(\theta|\Phi^{KK})$ and $\eta_J=0$, in which $2\leqslant K\leqslant M$, $2\leqslant J\leqslant M$, $J\neq K$. (D7) is equivalent to $\eta_1(\theta I-\Phi^{11})=\eta_K\Phi^{K1}$. If $\eta_K=0$, one can deduce $\eta_1\neq 0$ and $\theta\in\sigma(\Phi^{11})$, which contradicts the assumption. Then, $\eta_K\neq 0$. Since $\eta_K(\Delta^K\otimes B^K)=0$ and $\eta_1(\Delta^1\otimes B^1)=0$, condition 2) is not satisfied. If $\theta\in\sigma(\Phi^{11})$, based on (D8), it can be get $\eta_K=0$, $K=2,\cdots,M$. Otherwise, there exists $L\in\{2,\cdots,M\}$, such that $\eta_L\neq 0$ and $\eta_L(\theta I-\Phi^{LL})=0$. Then one can draw that $\theta\in\sigma(\Phi^{LL})$, which contradicts the assumption. It is obviously that $\eta_1\in M(\theta|\Phi^{11})$ and $\eta_1\neq 0$. Since $\eta_1(\Delta^1\otimes B^1)=0$, condition 1) is not satisfied.

Remark 3. Based on the relationship of the eigenvectors and eigenvalues of $\Phi^{KK}(K=1,\cdots,M)$ and the eigenvectors and eigenvalues of the subsystems established in Lemma D1 and Lemma D2, Theorem D3 gives conditions to verify the controllability of the networked systems by verifying the controllability of single layer networks. The directed star network structure makes the verification of controllability easy, which motivates these controllability conditions.

Corollary 2. The networked system (D4) is controllable, only if the following conditions hold:

1) $\delta^1_1=1$;

2) (A^1,B^1) can be converted into an HOFA system:

$$A^1_\mu z^{1\,(\mu)}_1+A^1_{\mu-1}z^{1\,(\mu-1)}_1+\cdots+A^1_1\dot{z}^1_1+A^1_0z^1_1=\tilde{B}^1u^1_i,$$

where $z^1_1=[z^1_{11},\cdots,z^1_{1p}]^T$, $\hat{x}^1_{1t}(j)=z^{1\,(j-1)}_{1t}$, $t=1,\cdots,p$, $j=1,\cdots,\mu_t$, $\mu=\max\{\mu_i,i=1,\cdots,p\}$, $\mu_1+\cdots+\mu_p=n$, $A^1_k=[\alpha^1_{sl}(k)]_{p\times p}, k=0,\cdots,\mu$,

$$\tilde{B}^1=\begin{bmatrix}1 & b^1_{12} & \cdots & \cdots & b^1_{1p}\\ & \ddots & b^1_{23} & \cdots & b^1_{2p}\\ & & \ddots & & \vdots\\ & & & & 1\end{bmatrix}_{p\times p},$$

in which $\alpha^1_{sl}(k)$ and b^1_{sl} are defined as (C31)-(C33), $s,\ l=1,\cdots,p,\ k=1,\cdots,\mu_i$, and

$$\begin{cases}\alpha^1_{sl}(k)=0,\ \text{if}\ k>\mu_s\ \text{or}\ k=\mu_s\ \text{and}\ s\neq l,\\ \alpha^1_{ss}(\mu_s)=1,\ \text{if}\ s=1,\cdots,p.\end{cases}$$

Proof. Obviously, node 1 in layer 1 without incoming edges from other nodes in layer 1 or other layers. Based on Theorem C1, the proof is omitted.

Appendix E Multi-layer networked systems with special node dynamics

Multi-layer networked systems with special node dynamics are considered here, where each node has chain dynamics, i.e., for $K=1,\cdots,M$,

$$A^K=\begin{bmatrix}0 & & & \\ a^K_1 & \ddots & & \\ & \ddots & \ddots & \\ & & a^K_{n-1} & 0\end{bmatrix}\in\mathbb{R}^{n\times n}.$$

Define $Q^K=\left\{[0,\cdots,0,q^K]|q^K\in\mathbb{R},q^K\neq 0\right\}$ and $G^K=\left\{[g^K,0,\cdots,0]^T|g^K\in\mathbb{R},g^K\neq 0\right\}$.

Corollary 3. Suppose $H^K \in \mathbb{C}^{n\times 1}$, $B^K \in \mathbb{C}^{n\times 1}$, $C^K \in \mathbb{C}^{1\times n}$, $H^{KJ} = H^K$, for $K, J = 1, \cdots, M$, $J \neq K$. The networked system (2) with chain node dynamics is controllable, if the following conditions hold simultaneously:

1) $C^K \in Q^K$ and $H^K \in G^K$, $K = 1, \cdots, M$;

2) for $s = 0$ and any $v^K = [v_1^K, \cdots, v_{N_K}^K] \in M^K(s)$, if $\eta W = 0$, then $\eta = 0$, where $\eta = [\eta^1, \cdots, \eta^M]$, $\eta^K = [\eta_1^K, \cdots, \eta_{N_K}^K]$, $\eta_i^K = v_i^K H^K$, $K = 1, \cdots, M$, $i = 1, \cdots, N_K$;

3) if $s \neq 0$, rank $(I - W \circ \beta, \Delta\gamma) = MN$, where $\beta^K = C^K(sI - A^K)^{-1}H^K$, $\beta^{KJ} = C^K(sI - A^K)^{-1}H^{KJ}$, $\gamma^K = C^K(sI - A^K)^{-1}B^K$, $K, J = 1, \cdots, M$, $J \neq K$, and

$$\beta = \begin{bmatrix} \beta^1 & \cdots & \beta^{1M} \\ \vdots & \ddots & \vdots \\ \beta^{M1} & \cdots & \beta^M \end{bmatrix}, \gamma = \begin{bmatrix} \gamma^1 & & \\ & \ddots & \\ & & \gamma^M \end{bmatrix}.$$

Proof. For $s \in \mathbb{C}$ and $C^K \in Q^K$, there exists a vector $\xi^K = [\xi_1^K, \cdots, \xi_n^K]^T \in \mathbb{C}^n$ satisfying that $(sI - A^K)\xi^K = 0$, $C^K\xi^K = 0$, $K = 1, \cdots, M$. Specifically,

$$\begin{cases} s\xi_1^K = 0 \\ s\xi_2^K - a_1^K\xi_1^K = 0 \\ \vdots \\ s\xi_n^K - a_{n-1}^K\xi_{n-1}^K = 0 \\ q^K\xi_n^K = 0 \end{cases}$$

If $s \neq 0$, then $\xi_1^K = \xi_2^K = \cdots = \xi_n^K = 0$. If $s = 0$, then $\xi_1^K = \cdots = \xi_{n-1}^K = 0$. Since $q^K \neq 0$, $\xi_n^K = 0$. In summary, $\xi^K = 0$, i.e., (A^K, C^K) is observable, $K = 1, \cdots, M$. Similarly, for $s \in \mathbb{C}$ and $H^K \in G^K$, there exists a vector $\eta^K = [\eta_1^K, \cdots, \eta_n^K]$ satisfying $\eta^K(sI - A^K) = 0$ and $\eta^K H^K = 0, K = 1, \cdots, M$. i.e.,

$$\begin{cases} s\eta_1^K - a_1^K\eta_2^K = 0 \\ \vdots \\ s\eta_{n-1}^K - a_{n-1}^K\eta_n^K = 0 \\ s\eta_n^K = 0 \\ g^K\eta_1^K = 0 \end{cases}$$

If $s \neq 0$, then $\eta_n^K = \cdots = \eta_1^K = 0$. If $s = 0$, then $\eta_2^K = \cdots = \eta_n^K = 0$. Since $g^K \neq 0$, it is easy to get $\eta_1^K = 0$. One can get $\eta^K = 0$, i.e., (A^K, H^K) is controllable, $K = 1, \cdots, M$. Based on Theorem 1, the rest of the proof is omitted.

Remark 4. For the node systems with chain dynamics, the controllability of the subsystem (A^K, H^K) and the observability of the subsystem (A^K, C^K) are not need to be verified specifically if $H^K \in G^K$ and $C^K \in Q^K$, where $K = 1, \cdots, M$. In addition, although the node dynamics of each layer are heterogeneous, the eigenvalues of the state matrix of nodes are the same, i.e., zero. Therefore, the situation of condition 5) in Theorem 1 does not exist.

Appendix F Simulated examples

Some examples are given to illustrated the applicability of the conditions proposed in this letter.

Appendix F.1 Numerical examples

Example 1. Consider a two-layer networked system with non-uniform layer scales, i.e., layer 1 with 3 nodes, and layer 2 with 4 nodes. $\delta_1^2 = 1$,

$$W^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, W^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D^{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H^{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C^1 = [1 \quad 0],$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, B^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C^2 = [0 \quad 1].$$

It is obviously that node 1 in layer 1 without incoming edges from other nodes of layer 1 or layer 2. The controllability of this network can be easily verified by Theorem C1. Since $\delta_1^1 = 0$, one can get the networked system is uncontrollable.

Example 2. Consider a two-layer directed chain structured network, where layer 1 with 2 nodes and layer 2 with 3 nodes. Each layer is a chain network. $\delta_1^1 = \delta_2^1 = \delta_1^2 = \delta_2^2 = 1$,

$$W^1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, W^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D^{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, H^{21} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, B^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C^1 = [1 \quad 0],$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}, B^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C^2 = [0 \quad 1].$$

Since $\sigma(A^1) = \{1, 2\}$, $\sigma(A^2) = \{0, 3\}$, and $\sigma(A^1) \bigcap \sigma(A^2) = \emptyset$, the controllability of this networked system can be verified by Theorem D2. $\sigma(W^1) = \{0, 0\}$, $\sigma(W^2) = \{0, 0, 0\}$. $\sigma(A^1 + \mu_1^1 H^1) = \{1, 2\}$, $\sigma(A^2 + \mu_1^2 H^2) = \{0, 3\}$. Then,

$$\lambda_{11}^1 = 1, \lambda_{12}^1 = 2; \lambda_{11}^2 = 0, \lambda_{12}^2 = 3.$$

$$\alpha_1^1(1) = [1 \quad 0], \alpha_1^1(2) = [0 \quad 1], \alpha_1^2(1) = [1 \quad 0 \quad 0], \alpha_1^2(2) = [0 \quad 1 \quad 0], \alpha_1^2(3) = [0 \quad 0 \quad 1],$$

$$\xi_{11}^1(1) = [1 \quad -1], \xi_{12}^1(1) = [0 \quad 1], \xi_{11}^2(1) = [1 \quad 0], \xi_{12}^2(1) = [1 \quad 3],$$

$$\eta_{11}^1(1) = \alpha_1^1(1) \otimes \xi_{11}^1(1) = [1 \quad -1 \quad 0 \quad 0], \eta_{12}^1(1) = \alpha_1^1(1) \otimes \xi_{12}^1(1) = [0 \quad 1 \quad 0 \quad 0],$$

$$\eta_{11}^2(1) = \alpha_1^2(1) \otimes \xi_{11}^2(1) = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \eta_{12}^2(1) = \alpha_1^2(1) \otimes \xi_{12}^2(1) = [1 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0].$$

Case 1: $\lambda \in \sigma(\Phi^{22})$. If $\lambda_1 = 0$, then $\eta_{11} = [-1 \quad \frac{1}{2} \quad 0 \quad 0]$. If $\lambda_2 = 3$, then $\eta_{12} = [2 \quad 2 \quad 0 \quad 0]$. For $\eta_{11}^2 \in M(\lambda_{11}^2 | \Phi^{22})$ and $\eta_{11} \in \Xi_{11}^1$, $[\eta_{11}(\Delta^1 \otimes B^1), \eta_{11}^2(\Delta^2 \otimes B^2)] = [\frac{1}{2}\alpha \quad 0 \quad \alpha \quad 0 \quad 0]$. If $\eta_{11}^2 \neq 0$, then $[\eta_{11}(\Delta^1 \otimes B^1), \eta_{11}^2(\Delta^2 \otimes B^2)] \neq 0$. For $\eta_{12}^2 \in M(\lambda_{12}^2 | \Phi^{22})$ and $\eta_{12} \in \Xi_{12}^1$, $[\eta_{12}(\Delta^1 \otimes B^1), \eta_{12}^2(\Delta^2 \otimes B^2)] = [2\alpha \quad 0 \quad \alpha \quad 0 \quad 0]$. If $\eta_{12}^2 \neq 0$, then $[\eta_{12}(\Delta^1 \otimes B^1), \eta_{12}^2(\Delta^2 \otimes B^2)] \neq 0$.

Case 2: $\lambda \in \sigma(\Phi^{11})$. For $\eta_{11} \in M(\lambda_{11}^1 | \Phi^{11})$, $\eta_{11}^1(\Delta^1 \otimes B^1) = [-\alpha \quad 0]$. That is, $\eta_{11}^1 \neq 0$, $\eta_{11}^1(\Delta^1 \otimes B^1) \neq 0$. For $\eta_{12}^1 \in M(\lambda_{12}^1 | \Phi^{11})$, $\eta_{12}^1(\Delta^1 \otimes B^1) = [\alpha \quad 0]$. One can get $\eta_{12}^1 \neq 0$, $\eta_{12}^1(\Delta^1 \otimes B^1) \neq 0$. In summary, the networked system is controllable.

Appendix F.2 A practical example

Consider a two-layer autonomous driving intelligent transportation networked system. The two vehicles in layer 1 receive control signals and keep driving side by side based on the positions of each other. In layer 2, vehicles 1 and 3 directly receive signals transmitted by vehicles in layer 1, while vehicle 2 maintains normal driving based on the signals transmitted by the other two vehicles, as shown in Fig. F1.

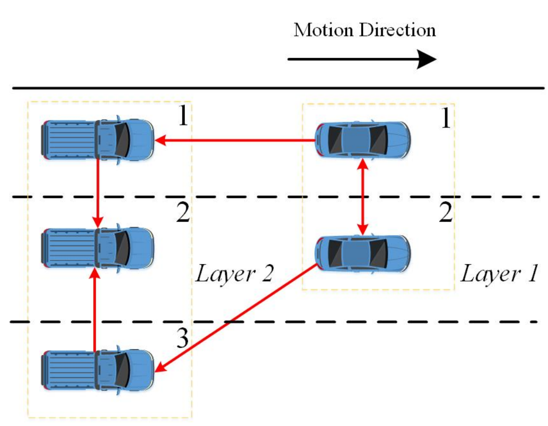

Figure F1 A two-layer autonomous driving intelligent transportation networked system

The dynamics of the vehicles in layer 1 and layer 2 can be described as:

$$A^1 = \begin{bmatrix} -\frac{1}{\tau_0} & \frac{1}{\tau_m} & \frac{1}{\tau_i} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} -\frac{1}{\tau_0} & \frac{\tau_p}{\tau_u} & \frac{1}{\tau_u} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

respectively, where $\tau_0, \tau_m, \tau_i, \tau_u$, and τ_p are time constants.

The state of the network is $x = [(x_1^1)^T\ (x_2^1)^T\ (x_1^2)^T\ (x_2^2)^T\ (x_3^2)^T]^T$, where $x_i^K = [a_i^K\ v_i^K\ d_i^K]^T$. a_i^K, v_i^K, and d_i^K represent acceleration, velocity, and displacement, respectively. For $K = 1, i = 1, 2$; for $K = 2, i = 1, 2, 3$. The topology matrices of layer 1 and layer 2 are

$$W^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, W^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. Let $\tau_0 = 0.5, \tau_m = 0.1, \tau_i = 0.25, \tau_u = 0.25, \tau_p = 2.5$, then

$$A^1 = \begin{bmatrix} -2 & 10 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} -2 & 10 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D^{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, H^{12} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, H^{21} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$B^1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, H^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, C^1 = [0\ \ 0\ \ 1], B^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, H^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, C^2 = [0\ \ 1\ \ 1].$$

The controllability of this network can be verified by Theorem 1. $\sigma(A^1) = \sigma(A^2) = \left\{-\frac{8329}{1998}, \frac{1507}{592}, -\frac{729}{1934}\right\}$. (A^K, C^K) is observable, (A^K, H^K) is controllable, $K = 1, 2$.

If $s \notin \sigma(A^K), K = 1, 2$, then

$$\beta = \begin{bmatrix} \frac{s^2+3s-7}{s^3+2s^2-10s-4} & \frac{s^2+3s-7}{s^3+2s^2-10s-4} \\ \frac{s^2+3s+2}{s^3+2s^2-10s-4} & \frac{s^2+3s+2}{s^3+2s^2-10s-4} \end{bmatrix}, \gamma = \begin{bmatrix} \frac{s^2+2s-9}{s^3+2s^2-10s-4} & 0 \\ 0 & \frac{s^2+4s+3}{s^3+2s^2-10s-4} \end{bmatrix}.$$

One can get that rank $(I - W \circ \beta, \Delta\gamma) = 5$.

If $s \in \sigma(A^K), K = 1, 2$, $v^1 = [0\ 0\ 0\ 0\ 0\ 0] \in M^1(s)$, $v^2 = [\frac{467}{481}v_1^2\ \ -\frac{739}{3173}v_1^2\ \ \frac{484}{8663}v_1^2\ \ \frac{1639}{1779}v_2^2\ \ \frac{958}{2647}v_2^2\ \ \frac{327}{2300}v_2^2\ \ \frac{114}{865}v_3^2\ \ -\frac{1108}{3169}v_3^2\ \ \frac{397}{428}v_3^2] \in M^2(s)$. One can get $\eta^1 = [0\ \ 0]$, $\eta^2 = [-\frac{739}{3173}v_1^2\ \ \frac{958}{2647}v_2^2\ \ -\frac{1108}{3169}v_3^2]$. Then, $\eta W = [-\frac{739}{3173}v_1^2\ \ -\frac{1108}{3169}v_3^2\ \ \frac{958}{2647}v_2^2\ \ 0\ \ \frac{958}{2647}v_2^2]$. It is obviously that if $\eta \neq 0$, then $\eta W \neq 0$. The networked system is controllable.

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