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• **Supplementary File** •

Predefined-time adaptive neural network optimal output feedback control for nonlinear systems

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Appendix A Proof of Theorem 1

Proof: Define $\tilde{x}(t) = x(t) - \hat{x}(t)$, $\tilde{\Lambda} = \Lambda^* - \hat{\Lambda}$ and $\tilde{\epsilon} = \epsilon - \hat{\epsilon}$. The observe error dynamic equation is

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t) + B(\tilde{\Lambda}^\top \psi(\hat{x}(t)) + \tilde{\epsilon}) + \sigma(x, \hat{x}), \tag{A1}$$

where $\sigma(x, \hat{x}) = B(M(x) - M(\hat{x})) + (G(x) - G(\hat{x}))v(u(t))$. From Assumption 1, we obtain $\|\sigma(x, \hat{x})\| \leqslant \sigma_m \|\tilde{x}(t)\|$, where $\sigma_m = l_M\|B\| + 2G_m\bar{u} > 0$, l_M and G_m are positive constants.
Consider the Lyapunov function

$$V_0(t) = \frac{1}{2}\tilde{x}^\top P\tilde{x} + \frac{1}{2\eta_1}\mathrm{tr}(\tilde{\Lambda}^\top\tilde{\Lambda}) + \frac{1}{2\eta_2}\tilde{\epsilon}^\top\tilde{\epsilon}, \tag{A2}$$

where $\eta_1 > 0$ and $\eta_2 > 0$ are the learning rates of $\hat{\Lambda}$ and $\hat{\epsilon}$.
Taking the derivative of $V_0(t)$ with respect to time, we have

$$\begin{aligned} V_0(t) &= \frac{1}{2}\dot{\tilde{x}}^\top P\tilde{x} + \frac{1}{2}\tilde{x}^\top P\dot{\tilde{x}} + \frac{1}{\eta_1}\mathrm{tr}(\tilde{\Lambda}^\top\dot{\tilde{\Lambda}}) + \frac{1}{\eta_2}\tilde{\epsilon}^\top\dot{\tilde{\epsilon}} \\ &= \frac{1}{2}\tilde{x}^\top((A - LC)^\top P + P(A - LC))\tilde{x} + \tilde{x}^\top P(B(\tilde{\Lambda}^\top\psi(\hat{x}(t)) + \tilde{\epsilon}) + \sigma) + \frac{1}{\eta_1}\mathrm{tr}(\tilde{\Lambda}^\top\dot{\tilde{\Lambda}}) + \frac{1}{\eta_2}\tilde{\epsilon}^\top\dot{\tilde{\epsilon}}. \end{aligned} \tag{A3}$$

From (7), one has

$$V_0(t) = -\frac{c_0}{2}\|\tilde{x}\|^2 + \tilde{y}^\top(\tilde{\Lambda}^\top\psi(\hat{x}(t)) + \tilde{\epsilon}) + \tilde{x}^\top P\sigma(x, \hat{x}) + \frac{1}{\eta_1}\mathrm{tr}(\tilde{\Lambda}^\top\dot{\tilde{\Lambda}}) + \frac{1}{\eta_2}\tilde{\epsilon}^\top\dot{\tilde{\epsilon}}. \tag{A4}$$

By $\|\sigma(x, \hat{x})\| \leqslant \sigma_m\|\tilde{x}(t)\|$ and the Young' s inequality, we have

$$\tilde{x}^\top P\sigma(x, \hat{x}) \leqslant \|\tilde{x}\|^2\|P\|\sigma_m \leqslant \|\tilde{x}\|^2(\frac{\|P\|^2}{2v_0} + \frac{v_0\sigma_m^2}{2}), \tag{A5}$$

where v_0 is a positive constant. From the properties of projection operator [1], we can know that

$$\begin{cases} \mathrm{Proj}(\eta_1\psi(\hat{x})\tilde{y}^\top, \hat{\Lambda}) \geqslant \tilde{\Lambda}^\top\eta_1\psi(\hat{x})\tilde{y}^\top, \\ \mathrm{Proj}(\eta_2\tilde{y}, \hat{\epsilon}) \geqslant \tilde{\epsilon}^\top\eta_2\tilde{y}^\top. \end{cases} \tag{A6}$$

Substituting (7), (A5) and (A6) into (A4), we obtain

$$\dot{V}_0(t) \leqslant -\lambda_0\|\tilde{x}\|_2^2, \tag{A7}$$

where $\lambda_0 = \frac{1}{2}c_0 - \frac{\|P\|^2}{2v_0} - \frac{v_0\sigma_m^2}{2} > 0$.

From (A7), we have $\dot{V}_0(t) < 0$. According to Barbalat's lemma [2-3], it can be confirmed that the observer error $\tilde{x}(t)$ asymptotically converges to zero, that is, $\lim_{t\to\infty}\|\tilde{x}(t)\| = 0$.

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Appendix B Proof of Theorem 2

To prove the Theorem 2, we introduce the following lemma.

Lemma 1 ([4])**.** For the nonlinear system (1), if there exists a continuously differentiable function $V(x):\mathbb{R}^m\to\mathbb{R}$ and $V(x)$ is positive definite such that

$$\dot{V}(x)\leqslant-\frac{\gamma}{T_c}(\varpi_1V^{\delta_1}(x)+\varpi_2V^{\delta_2}(x))^{\delta},\tag{B1}$$

where $T_c>0$, ϖ_1, ϖ_2, δ_1, δ_2 and δ are positive constants such that $\delta_1\delta<1$, $\delta_2\delta>1$ and

$$\gamma=\frac{\Gamma(\frac{1-\delta\delta_1}{\delta_2-\delta_1})\Gamma(\frac{\delta\delta_2-1}{\delta_2-\delta_1})}{\varpi_1^{\delta}\Gamma(\delta)(\delta_2-\delta_1)}\left(\frac{\varpi_1}{\varpi_2}\right)^{\frac{1-\delta\delta_1}{\delta_2-\delta_1}}.$$

Then, the nonlinear systems (1) is predefined-time stable with predefined time T_c.

Proof: By Theorem 1, we know that $\hat{x}(t)$ can estimate the system state $x(t)$, thus the NN observer can be further written as

$$\dot{\hat{x}}(t)=Z(\hat{x},y)+G(\hat{x}(t))v(u(t)),\tag{B2}$$

where $Z(\hat{x},y)=A\hat{x}+B(\hat{\Lambda}^\top\psi(\hat{x})+\hat{\epsilon})+L(y-\hat{y})$.

The Lyapunov function is defined as:

$$V(t)=V_1^*(\hat{x})+V_c(\tilde{\Lambda}_c),\tag{B3}$$

where $V_1^*(\hat{x})$ is defined by (10) and $V_c(\tilde{\Lambda}_c)=\frac{1}{2\varrho}\tilde{\Lambda}_c^\top\tilde{\Lambda}_c$, $\varrho>0$ is the learning rate.

The time derivative of $V_1^*(\hat{x})$ is formulated as:

$$\begin{aligned}\dot{V}_1^*(\hat{x})&=\nabla V_1^{*\top}\Big[Z(\hat{x},y)-G(\hat{x})D^*+G(\hat{x})(\hat{u}+\xi+D^*)\Big]\\&\leqslant-\frac{\gamma}{T_c}(\alpha{V_1^*}^{\frac{p+1}{2}}(\hat{x})+\beta{V_1^*}^{\frac{q+1}{2}}(\hat{x}))+\|\nabla V_1^{*\top}G(\hat{x})(\hat{u}+D^*)\|_2+\nabla V_1^{*\top}G(\hat{x})\xi.\end{aligned}\tag{B4}$$

According to [5-9], we assume that $\|\nabla\phi(\hat{x})\|_2\leqslant\phi_{dm}$ and $\|\nabla\epsilon_c\|_2\leqslant\epsilon_c^{dm}$, where ϕ_{dm} and ϵ_c^{dm} are positive constants. Thus $\|\nabla V_1^*(\hat{x})\|_2\leqslant\phi_{dm}\|\Lambda_c^*\|_2+\epsilon_c^{dm}$. Then, we have

$$\dot{V}_1^*(\hat{x})\leqslant-\frac{\gamma}{T_c}(\alpha{V_1^*}^{\frac{p+1}{2}}(\hat{x})+\beta{V_1^*}^{\frac{q+1}{2}}(\hat{x}))+\mu_1\|\tilde{\Lambda}_c\|_2+\mu_2+d_1G_m\xi,\tag{B5}$$

where $d_1=\phi_{dm}\|\Lambda_c^*\|_2+\epsilon_c^{dm}$, $\mu_1=\frac{1}{2}\lambda_{max}(\mathcal{R}^{-1})G_m^2\phi_{dm}d_1$, $\mu_2=\mu_1\epsilon_c^{dm}/\phi_{dm}+d_1G_m(\bar{u}+\|D^*\|)$.

Case 1: $\|\delta_{HJB}(t)\|_2\equiv0$. By calculating the time derivative of $V_c(\tilde{\Lambda}_c)=\frac{1}{2\varrho}\tilde{\Lambda}_c^\top\tilde{\Lambda}_c$ and using the adaptive law (18), we obtain

$$\begin{aligned}\dot{V}_c(\tilde{\Lambda}_c)=&-\frac{\tilde{\Lambda}_c^\top\Xi(t)}{\bar{\Psi}(t)}\left\lceil\frac{\Xi(t)^\top\tilde{\Lambda}_c}{\bar{\Psi}(t)}\right\rfloor^{\gamma_1}-\sum_{k=1}^{d}\frac{\tilde{\Lambda}_c^\top\Xi(t_k)}{\bar{\Psi}(t_k)}\left\lceil\frac{\Xi(t_k)^\top\tilde{\Lambda}_c}{\bar{\Psi}(t_k)}\right\rfloor^{\gamma_1}\\&-\frac{\tilde{\Lambda}_c^\top\Xi(t)}{\bar{\Psi}(t)}\left\lceil\frac{\Xi(t)^\top\tilde{\Lambda}_c}{\bar{\Psi}(t)}\right\rfloor^{\gamma_2}-\sum_{k=1}^{d}\frac{\tilde{\Lambda}_c^\top\Xi(t_k)}{\bar{\Psi}(t_k)}\left\lceil\frac{\Xi(t_k)^\top\tilde{\Lambda}_c}{\bar{\Psi}(t_k)}\right\rfloor^{\gamma_2}.\end{aligned}\tag{B6}$$

Because of $\frac{\tilde{\Lambda}_c^\top\Xi}{\bar{\Psi}}\left\lceil\frac{\Xi^\top\tilde{\Lambda}_c}{\bar{\Psi}}\right\rfloor^{\gamma_i}=\left|\frac{\tilde{\Lambda}_c^\top\Xi}{\bar{\Psi}}\right|^{\gamma_i+1}$, $i=1,2$, then (B6) can be formulated as

$$\dot{V}_c(\tilde{\Lambda}_c(t))\leqslant-\sum_{k=1}^{d}\left|\frac{\tilde{\Lambda}_c^\top\Xi(t_k)}{\bar{\Psi}(t_k)}\right|^{\gamma_1+1}-\sum_{k=1}^{d}\left|\frac{\tilde{\Lambda}_c^\top\Xi(t_k)}{\bar{\Psi}(t_k)}\right|^{\gamma_2+1}.\tag{B7}$$

Noting that

$$\left(\sum_{k=1}^{d}\left|\frac{\tilde{\Lambda}_c^\top\Xi(t_k)}{\bar{\Psi}(t_k)}\right|^{\gamma_i+1}\right)^{\frac{1}{\gamma_i+1}}=\|H^\top\tilde{\Lambda}_c\|_{\gamma_i+1},\quad i=1,2.$$

Because of $1<\gamma_1+1<2$ and the monotonicity of the $\|\cdot\|_p$, we have $\|H^\top\tilde{\Lambda}_c\|_2\leqslant\|H^\top\tilde{\Lambda}_c\|_{\gamma_1+1}$. By using the norm equivalence in $\mathbb{R}$, there exists a number $c>0$ such that $\|H^\top\tilde{\Lambda}_c\|_2\leqslant c\|H^\top\tilde{\Lambda}_c\|_{\gamma_2+1}$. Thus, $\dot{V}_c(\tilde{\Lambda}_c)$ can be represented as

$$\dot{V}_c(\tilde{\Lambda}_c)\leqslant-\|H^\top\tilde{\Lambda}_c\|_2^{\gamma_1+1}-c^{-(\gamma_2+1)}\|H^\top\tilde{\Lambda}_c\|_2^{\gamma_2+1}.\tag{B8}$$

From Assumption 2, it follows that $\mathrm{rank}(H)=\mathrm{rank}(HH^\top)=N'$, then $HH^\top$ is positive definite matrix, it is implies that the following inequality holds

$$\lambda_{\min}^2(H)\|\tilde{\Lambda}_c\|_2^2\leqslant\tilde{\Lambda}_c^\top HH^\top\tilde{\Lambda}_c\leqslant\lambda_{\max}^2(H)\|\tilde{\Lambda}_c\|_2^2.\tag{B9}$$

Based on (B8), (B9) and $V_c(\tilde{\Lambda}_c)=\frac{1}{2\varrho}\|\tilde{\Lambda}_c\|_2^2$, we obtain

$$\dot{V}_c(\tilde{\Lambda}_c)\leqslant-\lambda_{\min}^{\gamma_1+1}(H)(2\varrho)^{\frac{\gamma_1+1}{2}}V_c^{\frac{\gamma_1+1}{2}}(\tilde{\Lambda}_c)-c^{-(\gamma_2+1)}\lambda_{\min}^{\gamma_2+1}(H)(2\varrho)^{\frac{\gamma_2+1}{2}}V_c^{\frac{\gamma_2+1}{2}}(\tilde{\Lambda}_c).\tag{B10}$$

By combining (B5) with (B10), we have

$$\begin{aligned}\dot{V} \leqslant & -\frac{\gamma}{T_c}(\alpha {V_1^*}^{\frac{p+1}{2}}(\hat{x}) + \beta {V_1^*}^{\frac{q+1}{2}}(\hat{x})) - \lambda_{\min}^{\gamma_1+1}(H)(2\varrho)^{\frac{\gamma_1+1}{2}} V_c^{\frac{\gamma_1+1}{2}}(\tilde{\Lambda}_c) \\ & - c^{-(\gamma_2+1)}\lambda_{\min}^{\gamma_2+1}(H)(2\varrho)^{\frac{\gamma_2+1}{2}} V_c^{\frac{\gamma_2+1}{2}}(\tilde{\Lambda}_c) + \mu_1\|\tilde{\Lambda}_c\|_2 + \mu_2 + d_1 G_m \xi.\end{aligned} \tag{B11}$$

Then, choose κ such that $0 < \kappa < \lambda_{\min}^{\gamma_1+1}(H)$ and by using the Young's inequality, we have

$$\mu_1\|\tilde{\Lambda}_c\|_2 \leqslant \frac{\kappa}{\gamma_1+1}\|\tilde{\Lambda}_c\|_2^{\gamma_1+1} + k_0 \mu_1^{1+\frac{1}{\gamma_1}}, \tag{B12}$$

where $k_0 = \frac{\gamma_1}{\gamma_1+1}\kappa^{-\frac{\gamma_1+1}{\gamma_1+\gamma_1^2}}$.
From (B11) and (B12), we obtain

$$\begin{aligned}\dot{V} \leqslant & -\frac{\gamma}{T_c}(\alpha {V_1^*}^{\frac{p+1}{2}}(\hat{x}) + \beta {V_1^*}^{\frac{q+1}{2}}(\hat{x})) - (\lambda_{\min}^{\gamma_1+1}(H) - \frac{\kappa}{\gamma_1+1})(2\varrho)^{\frac{\gamma_1+1}{2}} V_c^{\frac{\gamma_1+1}{2}}(\tilde{\Lambda}_c) \\ & - c^{-(\gamma_2+1)}\lambda_{\min}^{\gamma_2+1}(H)(2\varrho)^{\frac{\gamma_2+1}{2}} V_c^{\frac{\gamma_2+1}{2}}(\tilde{\Lambda}_c) + k_0\mu_1^{1+\frac{1}{\gamma_1}} + \mu_2 + d_1 G_m \xi.\end{aligned} \tag{B13}$$

Similar to [2-3] and [10], choose the design parameter a in robustifying term ξ such that

$$a\,\|\hat{x}\|^2 \geqslant \frac{b+\|\hat{x}\|^2}{d_1 G_m}\left[k_0\mu_1^{1+\frac{1}{\gamma_1}} + \mu_2\right]. \tag{B14}$$

Then, (B13) can be represented as

$$\dot{V} \leqslant -\frac{\gamma}{T_c}(\alpha {V_1^*}^{\frac{p+1}{2}}(\hat{x}) + \beta {V_1^*}^{\frac{q+1}{2}}(\hat{x})) - \left(\alpha_c V_c^{\frac{\gamma_1+1}{2}}(\tilde{\Lambda}_c) + \beta_c V_c^{\frac{\gamma_2+1}{2}}(\tilde{\Lambda}_c)\right), \tag{B15}$$

where $\alpha_c = (\lambda_{\min}^{\gamma_1+1}(H) - \frac{\kappa}{(\gamma_1+1)})(2\varrho)^{\frac{\gamma_1+1}{2}}$ and $\beta_c = c^{-(\gamma_2+1)}\lambda_{\min}^{\gamma_2+1}(H)(2\varrho)^{\frac{\gamma_2+1}{2}}$.
Let $\gamma_1 = p$, $\gamma_2 = q$, and select ϱ such that $\alpha_c \geqslant \frac{\gamma}{T_c}\alpha$, $\beta_c \geqslant \frac{\gamma}{T_c}\beta$, we obtain

$$\begin{aligned}\dot{V} \leqslant & -\frac{\gamma}{T_c}\alpha({V_1^*}^{\frac{p+1}{2}}(\hat{x}) + V_c^{\frac{p+1}{2}}(\tilde{\Lambda}_c)) - \frac{\gamma}{T_c}\beta({V_1^*}^{\frac{q+1}{2}}(\hat{x}) + V_c^{\frac{q+1}{2}}(\tilde{\Lambda}_c)) \\ \leqslant & -\frac{\gamma}{T_c}(\alpha V^{\frac{p+1}{2}} + 2^{\frac{1-q}{2}}\beta V^{\frac{q+1}{2}}).\end{aligned} \tag{B16}$$

Case 2: $\|\delta_{HJB}(t)\|_2 \not\equiv 0$. We consider the same Lyapunov function candidate as *Case 1*, its derivative changes to

$$\begin{aligned}\dot{V}_c(\tilde{\Lambda}_c) = & -\tilde{\Lambda}_c^\top\left\lceil\frac{\Xi(t)^\top\tilde{\Lambda}_c + \delta_{HJB}(t)}{\bar{\Psi}(t)}\right\rfloor^{\gamma_1}\frac{\Xi(t)}{\bar{\Psi}(t)} - \sum_{k=1}^{d}\tilde{\Lambda}_c^\top\left\lceil\frac{\Xi(t_k)^\top\tilde{\Lambda}_c + \delta_{HJB}(t_k)}{\bar{\Psi}(t_k)}\right\rfloor^{\gamma_1}\frac{\Xi(t_k)}{\bar{\Psi}(t_k)} \\ & -\tilde{\Lambda}_c^\top\left\lceil\frac{\Xi(t)^\top\tilde{\Lambda}_c + \delta_{HJB}(t)}{\bar{\Psi}(t)}\right\rfloor^{\gamma_2}\frac{\Xi(t)}{\bar{\Psi}(t)} - \sum_{k=1}^{d}\tilde{\Lambda}_c^\top\left\lceil\frac{\Xi(t_k)^\top\tilde{\Lambda}_c + \delta_{HJB}(t_k)}{\bar{\Psi}(t_k)}\right\rfloor^{\gamma_2}\frac{\Xi(t_k)}{\bar{\Psi}(t_k)},\end{aligned} \tag{B17}$$

where $\delta_{HJB}(t) = \mathcal{H}(\hat{x}(t), \hat{u}(t), \Lambda_c^{*\top}(t)\nabla\phi(\hat{x}(t)))$. Assume that $\|\delta_{HJB}(t)\|_2 \leqslant \delta_{HJB}^m$ and $\|\frac{\Xi(t)}{\bar{\Psi}(t)}\|_2 \leqslant \bar{\Psi}_m$, where δ_{HJB}^m, $\bar{\Psi}_m$ are positive constants. There exists a $N_\eta > 0$ such that

$$\left|\Xi^\top(t)\tilde{\Lambda}_c\right| \geqslant |\delta_{HJB}(t)|, \forall N' \geqslant N_\eta. \tag{B18}$$

Then, one has

$$\operatorname{sgn}\left(\frac{\Xi^\top(t)\tilde{\Lambda}_c + \delta_{HJB}(t)}{\bar{\Psi}(t)}\right) = \operatorname{sgn}\left(\frac{\Xi^\top(t)\tilde{\Lambda}_c}{\bar{\Psi}(t)}\right). \tag{B19}$$

Note that the following inequalities hold.

$$\begin{cases}\left|\frac{\Xi^\top(t)\tilde{\Lambda}_c + \delta_{HJB}(t)}{\bar{\Psi}(t)}\right|^{\gamma_1} \geqslant \left|\frac{\Xi^\top(t)}{\bar{\Psi}(t)}\tilde{\Lambda}_c\right|^{\gamma_1} - \left|\frac{\delta_{HJB}(t)}{\bar{\Psi}(t)}\right|^{\gamma_1}, \\ \left|\frac{\Xi^\top(t)\tilde{\Lambda}_c + \delta_{HJB}(t)}{\bar{\Psi}(t)}\right|^{\gamma_2} \geqslant 2^{1-\gamma_2}\left|\frac{\Xi^\top(t)}{\bar{\Psi}(t)}\tilde{\Lambda}_c\right|^{\gamma_2} - \left|\frac{\delta_{HJB}(t)}{\bar{\Psi}(t)}\right|^{\gamma_2}.\end{cases} \tag{B20}$$

From (B10), (B19) and (B20), we have

$$\begin{aligned}\dot{V}_c(\tilde{\Lambda}_c) \leqslant & -\lambda_{\min}^{\gamma_1+1}(H)\|\tilde{\Lambda}_c\|_2^{\gamma_1+1} - 2^{1-\gamma_2}c^{-(\gamma_2+1)}\lambda_{\min}^{\gamma_2+1}(H)\|\tilde{\Lambda}_c\|_2^{\gamma_2+1} \\ & + \left(\sum_{k=1}^{d}\left|\frac{\tilde{\Lambda}_c^\top\Xi(t_k)}{\bar{\Psi}(t_k)}\right| + \left|\frac{\tilde{\Lambda}_c^\top\Xi(t)}{\bar{\Psi}(t)}\right|\right)\left(\left|\frac{\delta_{HJB}}{\bar{\Psi}(t)}\right|^{\gamma_1} + \left|\frac{\delta_{HJB}}{\bar{\Psi}(t)}\right|^{\gamma_2}\right).\end{aligned} \tag{B21}$$

Moreover, $\|\frac{\Xi(t)}{\Psi(t)}\|_2 \leqslant \bar{\Psi}_m$ and $\|\delta_{HJB}(t)\|_2 \leqslant \delta_{HJB}^m$, where $\bar{\Psi}_m$ and δ_{HJB}^m are positive constants. (B21) can be formulated as

$$\begin{aligned}\dot{V}_c(\tilde{\Lambda}_c) \leqslant & -\lambda_{\min}^{\gamma_1+1}(H)(2\varrho)^{\frac{\gamma_1+1}{2}} V_c^{\frac{\gamma_1+1}{2}}(\tilde{\Lambda}_c) - 2^{1-\gamma_2} c^{-(\gamma_2+1)} \lambda_{\min}^{\gamma_2+1}(H)(2\varrho)^{\frac{\gamma_2+1}{2}} V_c^{\frac{\gamma_2+1}{2}}(\tilde{\Lambda}_c) \\ & + \bar{\Psi}_m(d+1)(|\delta_{HJB}^m|^{\gamma_1} + |\delta_{HJB}^m|^{\gamma_2})\|\tilde{\Lambda}_c\|_2. \end{aligned} \tag{B22}$$

By combining (B5) with (B22), we have

$$\begin{aligned}\dot{V} \leqslant & -\frac{\gamma}{T_c}(\alpha V_1^{*\frac{p+1}{2}}(\hat{x}) + \beta V_1^{*\frac{q+1}{2}}(\hat{x})) - \lambda_{\min}^{\gamma_1+1}(H)(2\varrho)^{\frac{\gamma_1+1}{2}} V_c^{\frac{\gamma_1+1}{2}}(\tilde{\Lambda}_c) \\ & - 2^{1-\gamma_2} c^{-(\gamma_2+1)} \lambda_{\min}^{\gamma_2+1}(H)(2\varrho)^{\frac{\gamma_2+1}{2}} V_c^{\frac{\gamma_2+1}{2}}(\tilde{\Lambda}_c) + c_1\|\tilde{\Lambda}_c\|_2 + \mu_2 + d_1 G_m \xi, \end{aligned} \tag{B23}$$

where $c_1 = [\bar{\Psi}_m(d+1)(\left|\delta_{HJB}^m\right|^{\gamma_1} + \left|\delta_{HJB}^m\right|^{\gamma_2}) + \mu_1]$.
Then, by using similar arguments as in *Case 1*, we obtain

$$\dot{V} \leqslant -\frac{\gamma}{T_c}(\alpha V^{\frac{p+1}{2}} + 2^{\frac{1-q}{2}} \beta V^{\frac{q+1}{2}}). \tag{B24}$$

Finally, combining *Case 1* and *Case 2*, then by utilizing (B16) and (B24) and Lemma 1, we conclude that the system (1) is predefined-time stable within a predefined time T_c.

Appendix C Simulation

Example: Consider a spacecraft system given by [10]

$$\begin{cases} \dot{x}_1(t) = L_1^{-1}(L_2 - L_3)k_3 x_2(t) + \mathrm{sat}(u_1(t)), \\ \dot{x}_2(t) = L_2^{-1}(L_3 - L_1)k_3 x_1(t) + \mathrm{sat}(u_2(t)), \\ y(t) = Cx, \end{cases} \tag{C1}$$

where $L_1 > 0$, $L_2 > 0$, and $L_3 > 0$ are the principal moments of inertia, $x = [x_1(t), x_2(t)]^\top$ is the state vector, $k_3 \in \mathbb{R}$ is angular velocity component, and $u = [u_1(t), u_2(t)]^\top$ are the spacecraft control moments. Let $f(x(t)) = [L_1^{-1}(L_2 - L_3)k_3 x_2(t), L_2^{-1}(L_3 - L_1)k_3 x_1(t)]^\top$, $G = I_2$ and $C = [1, 0]$, where $L_1 = 5$, $L_2 = 5$, $L_3 = 19$ and $k_3 = 1$.
Construct the NN $\Lambda^\top \psi(\hat{x})$ and critic NN $\Lambda_c^\top \phi(\hat{x})$, where the activation functions are selected $\psi(\hat{x}) = \phi(\hat{x}) = [\hat{x}_1^2, \hat{x}_1\hat{x}_2, \hat{x}_2^2]^\top$, the hidden layer nodes $N = N' = 3$. The $\hat{\Lambda}^\top \psi(\hat{x})$ and $\hat{\Lambda}_c^\top \phi(\hat{x})$ are applied to estimate $M(\hat{x})$ and $\hat{V}_1(\hat{x})$ in (5) and (15), respectively. We choose $A = \begin{pmatrix} 3 & 1 \\ -0.5 & -6 \end{pmatrix}$, $B = [5.8053, -7.0796]^\top$, $L = [9, -12]^\top$ and $c_0 = 1$ in the NN state observer (7). By solving the equation (7), one obtains the positive definite matrix $P = \begin{pmatrix} 0.3278 & 0.1276 \\ 0.1276 & 0.1046 \end{pmatrix}$.
In the cost function (8), select the saturation bound $\bar{u} = 0.5$ and $\mathcal{R}(\hat{x}) = \frac{1}{2} I_2$,

$$\mathcal{K}_2(\hat{x}) = \left[\left(\frac{\gamma}{T_c}(2^{-\frac{p+1}{2}} \lceil \hat{x} \rfloor^p + 2^{-\frac{q+1}{2}} 2^{\frac{q-1}{2}} \lceil \hat{x} \rfloor^q) + Z(\hat{x}, y)\right) - \hat{x} \right]^\top, \tag{C2}$$

$$\mathcal{K}_1(\hat{x}) = (\mathcal{K}_2(\hat{x}) + \hat{x}^\top)\bar{u} \tanh\left((\mathcal{K}_2(\hat{x}) + \hat{x}^\top)^\top/\bar{u}\right) - U\left(-\bar{u}\tanh\left((\mathcal{K}_2(\hat{x}) + \hat{x}^\top)^\top/\bar{u}\right)\right) - \hat{x}^\top Z(\hat{x}, y). \tag{C3}$$

We choose $b = 1$ and $a = 0.27$ in the robustifying term ξ, then condition (B14) is satisfied. To guarantee the control system predefined-time stable within $T_c = \frac{2\Gamma(\frac{1-p}{q-p})\Gamma(\frac{q-1}{q-p})}{\Gamma(1)(q-p)} \left(\frac{1}{2^{\frac{1-q}{2}}}\right)^{\frac{1-p}{q-p}}$. Let $T_c = 5.8097$ so that $p = 0.4$, $q = 1.6$, $\gamma_1 = 0.4$, $\gamma_2 = 1.6$, $\varrho = 60$. Choose $\eta_1 = 10$, $\eta_2 = 15$ in weight adaptive law (8).

The initial conditions of the system states, the observer system states, and the weight of NN state observer and critic NN are given as: $x(0) = [0.5, -0.5]^\top$, $\hat{x}(0) = [0.6, -0.75]^\top$, $\hat{\epsilon}(0) = 0.5$, $\hat{\Lambda}(0) = [0.3, 0.2, 0.1]^\top$, $\hat{\Lambda}_c(0) = [0.7324, 0.1975, 0.3785]^\top$.
Figs. C1-C5 show the simulation results. Fig. C1 shows the curves of the NN state observer errors $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$. Fig. C2 shows the curves of the system state $x_1(t)$ and $x_2(t)$. Fig. C3 shows the curves of controllers $\hat{u}_1(t)$ and $\hat{u}_2(t)$. Fig. C4 and Fig. C5 are the curves of critic weight $\hat{\Lambda}_c$ and the NN observer weight $\hat{\Lambda}$. Figs. C1-C5 indicate that the proposed predefined-time adaptive NN optimal output feedback control scheme can guarantee the systems predefined-time stable within $T_c = 5.8097$ s and does not exceed the control input saturation bound $\bar{u} = 0.5$.

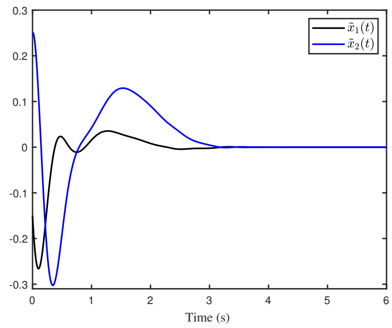

Figure C1 Observer errors $\tilde{x}_1$ and $\tilde{x}_2$.

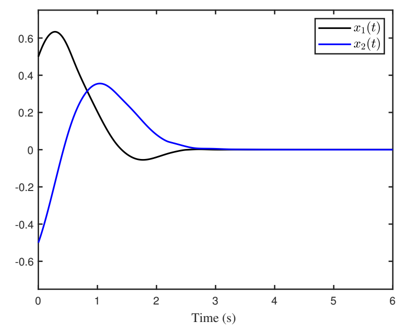

Figure C2 The curves of the system state $x_1(t)$ and $x_2(t)$.

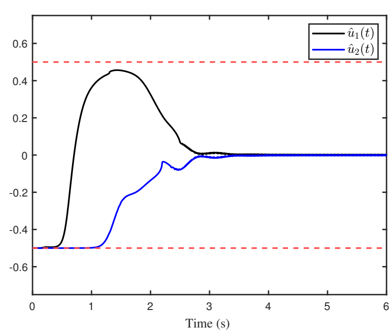

Figure C3 The curves of controllers $\hat{u}_1(t)$ and $\hat{u}_2(t)$

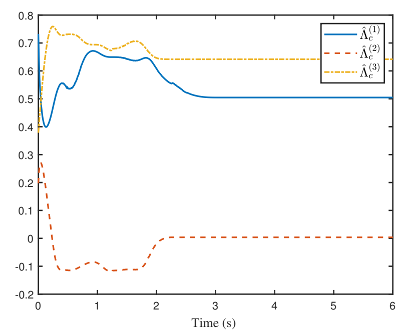

Figure C4 The curves of the critic weights $\hat{\Lambda}_c$.

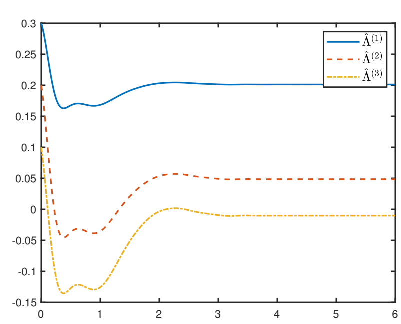

Figure C5 The curves of the NN observer weights $\hat{\Lambda}$.

To illustrate the fast convergence of the presented predefined-time optimal control scheme, we compare with optimal NN control method [2]. Choose system initial conditions are the same as the above simulation. The comparison results are shown by Fig. C6. From Fig. C6, one knows that with almost the same control efforts, the system state vector $x_1(t)$ and $x_2(t)$ in [2] convergence time is 10 s, while the convergence time in this paper is 4 s.

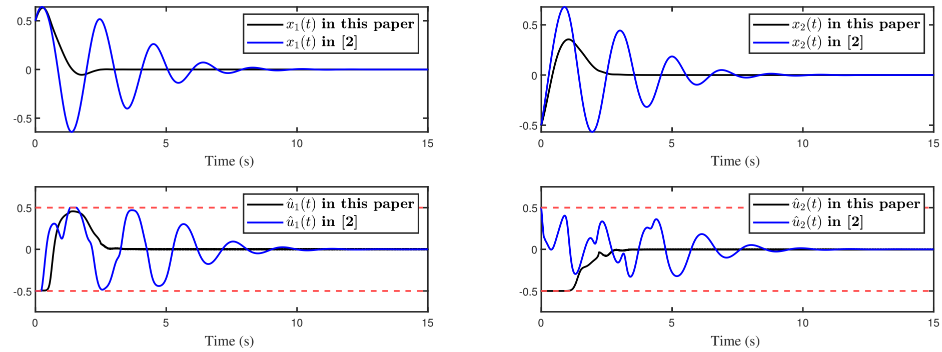

Figure C6 Comparison of the system state $x_1(t)$, $x_2(t)$ and controllers $\hat{u}_1(t)$, $\hat{u}_2(t)$.

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