

• Supplementary File •

ISS analysis for asynchronous impulsive stochastic interconnected systems

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Appendix A Notations

Let $p_1 \vee p_2 = \max\{p_1, p_2\}$, $p_1 \wedge p_2 = \min\{p_1, p_2\}$ and $\mathbb{E}(\cdot)$ denote the mathematical expectation. Define $\mathbb{L} = \{1, 2, \dots, L\}$, $\mathbb{Z}_+ = \{1, 2, \dots\}$, $\mathbb{R}_{\geq t_0} = [t_0, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. $\|\cdot\|$ is the Euclidean norm of a vector or the trace norm of matrix. Let U^{ζ_u} be the set of all measurable and locally essentially bounded function $y_u : \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^{\zeta_u}$ with norm $\|y_u\|_U = \text{esssup}_{t \geq t_0} |y_u(t)|$. Denote $\|y\|_t = \text{esssup}_{t_0 \leq s \leq t} |y(s)|$, and $\|y_u\|_{[s,t]} = \text{esssup}_{\tilde{t} \in [s,t]} |y_u(\tilde{t})|$. $\mathcal{K} = \{f : [0, \bar{f}] \rightarrow \mathbb{R}^+ | f \text{ is a strictly increasing continuous function with } f(0) = 0\}$, $\mathcal{K}_\infty = \{f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ | f \in \mathcal{K} \text{ and } \lim_{t \rightarrow \infty} f(t) = \infty\}$, $\mathcal{VK}_\infty = \{f \in \mathcal{K}_\infty | f \text{ is convex}\}$, $\mathcal{CK}_\infty = \{f \in \mathcal{K}_\infty | f \text{ is concave}\}$, $\mathcal{L} = \{f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ | f \text{ is strictly decreasing continuous function with } \lim_{t \rightarrow \infty} f(t) = 0\}$, $\mathcal{KL} = \{\bar{f} : [0, \bar{r}] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ | \bar{f}(\cdot, s) \in \mathcal{K}, \bar{f}(r, \cdot) \in \mathcal{L}, \text{ for any } r \in [0, \bar{r}], s \in \mathbb{R}_+\}$. Digraph \mathcal{G} is said to be strongly connected if, for any pair of distinct nodes, there exists a directed path from one to the other.

Appendix B Literature review

The concept of input-to-state stability (ISS), proposed in [1], ensures that the solution's norm asymptotically converges to an origin-centered region with radius scaling proportionally to the input signal's magnitude, independent of the initial state. Subsequent research has significantly expanded ISS theory across diverse dynamical systems, such as impulsive systems [2], switched systems [3], stochastic systems [4], discrete-time systems [5], interconnected systems [6], and delayed systems [7]. Particularly for stochastic systems, ISS analysis becomes crucial since stochastic disturbances inherently exist in many physical implementations [8, 9]. Recent research has successfully generalized deterministic ISS theory to stochastic frameworks, establishing dedicated ISS criteria for such systems. For example, the work in [10] investigated stochastic ISS for switched stochastic nonlinear systems.

Moreover, impulsive systems, characterized by continuous-time evolution punctuated by instantaneous state jumps, constitute a fundamental class of hybrid systems with broad applications in power networks, biological processes, and networked control [11–13]. The stability analysis of such systems is particularly challenging yet crucial, as the coexistence of continuous flows and discrete resets may induce complex dynamic behaviors [14, 15]. The ISS analysis of impulsive systems therefore presents particular theoretical challenges, especially when examining the interplay between these two components. Of particular significance are two archetypal scenarios: (i) systems where the continuous dynamics are stabilizing but impulses are destabilizing, and (ii) systems where stabilizing discrete jumps compensate for destabilizing continuous dynamics. Recent advances [2, 7, 16] have established ISS criteria for impulsive systems, while extensions to stochastic impulsive systems [4, 17–19] further account for stochastic disturbances.

On the other hand, interconnected systems, comprising multiple dynamically coupled subsystems through specific interaction topologies, have attracted considerable research attention due to their prevalence in modern engineering applications such as smart power grids and networked microgrids [20–22]. When these systems exhibit hybrid dynamics featuring both continuous interactions and discrete state jumps at impulsive instants, they form impulsive interconnected systems whose stability properties have been extensively investigated [23, 24]. Then, by considering stochastic disturbances, the stability analysis of impulsive stochastic interconnected systems (ISISs) becomes fundamentally more challenging [18, 25]. Recent work has demonstrated that combining the Lyapunov method with graph-theoretic approaches can effectively establish ISS criteria for ISISs [18].

In particular, time delay, inherently present in signal transmission and processing, introduces additional complexity to the ISS analysis of dynamical systems [7, 24]. It is worth pointing out that the aforementioned ISS of impulsive systems (including interconnected systems, delayed systems, and stochastic systems) consider only synchronously impulsive effects, where all subsystems experience simultaneous impulses. However, in many biological and engineered systems, impulsive effects naturally exhibit asynchrony due to decentralized structures or heterogeneous time scales [21, 26]. For instance, spiking neurons exhibit asynchronous impulsive effects due to heterogeneous synaptic integrations and refractory dynamics [26], a feature fundamentally distinct from synchronous impulse models. Despite its practical relevance, the ISS of asynchronous ISISs remains largely unexplored, particularly for scenarios where subsystems undergo independent jumps at distinct instants. Therefore, this work establishes ISS criteria for asynchronous ISISs, comprehensively addressing both delay-free and time-delayed cases.

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To guarantee ISS of impulsive systems, various conditions have been developed for characterizing impulsive instants, including fixed dwell-time [7], average dwell-time (ADT) [24], generalized ADT [18], and average impulsive interval (AII) [17] conditions. The fixed dwell-time condition enforces both lower and upper bounds on inter-impulse durations, providing guaranteed stability at the cost of operational flexibility. To improve adaptability, the ADT condition is introduced, requiring only an ADT while allowing interval variability. Subsequent generalized ADT further enhanced applicability by permitting varying ADTs in impulsive systems. Notably, these conditions are subsystem-agnostic, enforcing identical timing constraints across all components regardless of their individual dynamics. To address this limitation, an asynchronous AII framework was developed in [22] for analyzing synchronization of complex dynamical networks based on asynchronous impulsive distributed control. Although the asynchronous AII framework in [22] is promising, note that distributed impulse differs fundamentally from the decentralized impulse considered in this paper. Moreover, it should be noted that existing results on impulsive effects under either the AII or ADT framework require the two impulsive types, namely stabilizing and destabilizing impulses, to be analyzed separately when considering ISS.

To the best of the authors' knowledge, no existing studies on the ISS of systems have considered asynchronous impulsive control or disturbance. In fact, apart from control, asynchronous impulsive disturbances have been widely studied. For instance, asynchronous impulsive noise is prevalent in both broadband powerline communications and wireless networks, including ad hoc and cellular networks [21, 27]. Motivated by discussions above, this paper investigates the ISS of asynchronous ISISs under an asynchronous AII. The obtained results are further applied to analyze the ISS of interconnected spring-mass damper systems (ISDSs).

Appendix C Motivational example

Here, we provide a simple example of two linear interconnected subsystems to illustrate that the traditional synchronously impulsive results (taking periodic impulses as an example) cannot guarantee its stability. Consider the following asynchronous impulsive stochastic differential equation

$$\begin{cases} dx_1(t) = [\epsilon_1 x_1(t) + \epsilon_2(x_2(t) - x_1(t)) + \gamma(t)]dt + \epsilon_3 x_1(t)dW(t), & t \neq t_n^1, \\ dx_2(t) = [\tilde{\epsilon}_1 x_2(t) + \tilde{\epsilon}_2(x_1(t) - x_2(t)) + \tilde{\gamma}(t)]dt + \tilde{\epsilon}_3 x_2(t)dW(t), & t \neq t_n^2, t \geq t_0, \\ x_1(t) = \vartheta x_1(t-), & t = t_n^1, \\ x_2(t) = \tilde{\vartheta} x_2(t-), & t = t_n^2, n \in \mathbb{Z}_+. \end{cases} \quad (C1)$$

Note that several pairs of coefficients can be included in regard to the heterogeneity of interconnected systems, such as ϵ_1 and $\tilde{\epsilon}_1$; ϵ_2 and $\tilde{\epsilon}_2$; ϵ_3 and $\tilde{\epsilon}_3$; ϑ and $\tilde{\vartheta}$; $\gamma(t)$ and $\tilde{\gamma}(t)$. Next, for convenience, in addition to the stochastic disturbance intensities ϵ_3 and $\tilde{\epsilon}_3$, the other pairs of coefficients are set to be consistent.

Case 1: destabilizing impulses: Destabilizing impulse, also termed impulsive disturbance, is a type of perturbation that disrupts system stability. Let $\epsilon_1 = \tilde{\epsilon}_1 = -1$, $\epsilon_2 = \tilde{\epsilon}_2 = 0.5$, $\vartheta = \tilde{\vartheta} = 1.1$, $\gamma(t) = \tilde{\gamma}(t) = \sin t$, $\epsilon_3 = 1$ and $\tilde{\epsilon}_3 = 0.1$. Moreover, let $t_n^1 = 0.2n$ and $t_n^2 = 0.1n$. Denote $x_1(t) = 1.1x_1(t-)$, $t = 0.2n$; $x_1(t) = x_1(t-)$, $t = 0.1 + 0.2(n - 1)$; $x_2(t) = 1.1x_2(t-)$, $t = 0.1n$, $n \in \mathbb{Z}_+$. Using synchronously impulsive method, define $\Phi(x, t) = x_1^2 + x_2^2$. Through calculation, from (C1), it can be obtained

$$\begin{aligned} \mathcal{L}\Phi(x(t), t) &= 2[-x_1^2(t) + 0.5(x_1(t)x_2(t) - x_1^2(t)) + x_1(t)\gamma(t)] + x_1^2(t) \\ &\quad + 2[-x_2^2(t) + 0.5(x_1(t)x_2(t) - x_2^2(t)) + x_2(t)\tilde{\gamma}(t)] + 0.01x_2^2(t) \\ &\leq -x_1^2(t) - 1.99x_2^2(t) + 2x_1(t)\gamma(t) + 2x_2(t)\tilde{\gamma}(t) \\ &\leq -0.99\Phi(x(t), t) + 100(\gamma^2(t) + \tilde{\gamma}^2(t)), \quad t \neq 0.1n, \\ \Phi(x(0.1n), 0.1n) &= x_1^2(0.1n) + x_2^2(0.1n) \leq 1.21(x_1^2(0.1n-) + x_2^2(0.1n-)) = 1.21\Phi(x(0.1n-), 0.1n-). \end{aligned}$$

According to the above analysis, from $1.21e^{-0.99 \times 0.1} > 1$, we cannot guarantee the stability of system (C1).

When using asynchronous impulsive idea in this paper, define $\Phi(x, t) = e^{\rho_1(t)}x_1^2 + e^{\rho_2(t)}x_2^2$, where

$$\rho_1(t) = \begin{cases} 0.9530(t - 0.2n), & t \in [0.2n, 0.2(n + 1)), \\ 0, & t = 0.2n, \end{cases} \quad \rho_2(t) = \begin{cases} 1.9060(t - 0.1n), & t \in [0.1n, 0.1(n + 1)), \\ 0, & t = 0.1n. \end{cases}$$

Through calculation, from (C1), it can be obtained

$$\begin{aligned} \mathbb{E}\mathcal{L}\Phi(x(t), t) &= 0.9530e^{\rho_1(t)}\mathbb{E}x_1^2(t) + 1.9060e^{\rho_2(t)}\mathbb{E}x_2^2(t) \\ &\quad + 2e^{\rho_1(t)}\mathbb{E}[-x_1^2(t) + 0.5(x_1(t)x_2(t) - x_1^2(t)) + x_1(t)\gamma(t)] + e^{\rho_1(t)}\mathbb{E}x_1^2(t) \\ &\quad + 2e^{\rho_2(t)}\mathbb{E}[-x_2^2(t) + 0.5(x_1(t)x_2(t) - x_2^2(t)) + x_2(t)\tilde{\gamma}(t)] + 0.01e^{\rho_2(t)}\mathbb{E}x_2^2(t) \\ &\leq -0.0470e^{\rho_1(t)}\mathbb{E}x_1^2(t) - 0.0840e^{\rho_2(t)}\mathbb{E}x_2^2(t) + 2.42\mathbb{E}x_1(t)\gamma(t) + 2.42\mathbb{E}x_2(t)\tilde{\gamma}(t) \\ &\leq -0.03\mathbb{E}\Phi(x(t), t) + 87(\gamma^2(t) + \tilde{\gamma}^2(t)), \quad t \neq 0.1n, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}e^{\rho_1(0.2n)}x_1^2(0.2n) &= 1.21\mathbb{E}x_1^2(0.2n-) = e^{0.1906}\mathbb{E}x_1^2(0.2n-) = e^{\rho_1(0.2n-)}\mathbb{E}x_1^2(0.2n-), \\ \mathbb{E}e^{\rho_1(0.2n+0.1)}x_1^2(0.2n + 0.1) &= \mathbb{E}e^{\rho_1((0.2n+0.1)-)}x_1^2((0.2n + 0.1)-), \end{aligned}$$

$$\mathbb{E}e^{\rho_2(0.1n)}x_2^2(0.1n) = 1.21\mathbb{E}x_2^2(0.1n-) = e^{0.1906}\mathbb{E}x_2^2(0.1n-) = e^{\rho_2(0.1n-)}\mathbb{E}x_2^2(0.1n-),$$

by using the continuity of $\mathbb{E}e^{\rho_1(t)}x_1^2(t)$ on $[0.2n, 0.2(n+1))$, thus $\mathbb{E}\Phi(x(0.1n), 0.1n) \leq \mathbb{E}\Phi(x(0.1n-), 0.1n-)$. Based on the above analysis, it is not difficult to see that system (C1) is ISS.

Case 2: stabilizing impulses: Stabilizing impulse, also called as impulsive control, is a type of jump that stabilizes systems. Let $\epsilon_1 = \tilde{\epsilon}_1 = 0.5, \vartheta = \tilde{\vartheta} = 0.9, \epsilon_3 = 0.1, \tilde{\epsilon}_3 = 1$, and other parameters are consistent with **Case 1**. Denote $x_1(t) = 0.9x_1(t-), t = 0.2n; x_1(t) = x_1(t-), t = 0.1 + 0.2(n-1); x_2(t) = 0.9x_2(t-), t = 0.1n, n \in \mathbb{Z}_+$. Using synchronously impulsive method, define $\Phi(x, t) = x_1^2 + x_2^2$. Through calculation, from (C1), we have

$$\begin{aligned} \mathcal{L}\Phi(x(t), t) &= 2[0.5x_1^2(t) + 0.5(x_1(t)x_2(t) - x_1^2(t)) + x_1(t)\gamma(t)] + 0.01x_1^2(t) \\ &\quad + 2[0.5x_2^2(t) + 0.5(x_1(t)x_2(t) - x_2^2(t)) + x_2(t)\tilde{\gamma}(t)] + x_2^2(t) \\ &\leq 1.01x_1^2(t) + 2x_2^2(t) + 2x_1(t)\gamma(t) + 2x_2(t)\tilde{\gamma}(t) \\ &\leq 2.01\Phi(x(t), t) + 100(\gamma^2(t) + \tilde{\gamma}^2(t)), \quad t \neq 0.1n, \\ \Phi(x(0.2n), 0.2n) &= x_1^2(0.2n) + x_2^2(0.2n) = 0.81(x_1^2(0.2n-) + x_2^2(0.2n-)) \\ &= 0.81\Phi(x(0.2n-), 0.2n-), \\ \mathbb{E}\Phi(x(0.2n + 0.1), 0.2n + 0.1) &= \mathbb{E}x_1^2(0.2n + 0.1) + \mathbb{E}x_2^2(0.2n + 0.1) \\ &= \mathbb{E}x_1^2((0.2n + 0.1)-) + 0.81\mathbb{E}x_2^2((0.2n + 0.1)-) \\ &\leq \mathbb{E}\Phi(x((0.2n + 0.1)-), (0.2n + 0.1)-), \end{aligned}$$

by using the continuity of $\mathbb{E}x_1^2(t)$ on $[0.2n, 0.2(n+1))$. Hence, from $0.81e^{2.01*0.2} > 1$, we cannot guarantee the stability of system (C1).

When using asynchronous impulsive idea, define $\Phi(x, t) = e^{\rho_1(t)}x_1^2 + e^{\rho_2(t)}x_2^2$, where

$$\begin{aligned} \rho_1(t) &= \begin{cases} \rho_1(0.2n) - 1.0535(t - 0.2n), & t \in [0.2n, 0.2(n+1)), \\ 0.2107, & t = 0.2n, \end{cases} \\ \rho_2(t) &= \begin{cases} \rho_2(0.1n) - 2.1070(t - 0.1n), & t \in [0.1n, 0.1(n+1)), \\ 0.2107, & t = 0.1n. \end{cases} \end{aligned}$$

Through calculation, we obtain

$$\begin{aligned} \mathbb{E}\mathcal{L}\Phi(x(t), t) &= -1.0535e^{\rho_1(t)}\mathbb{E}x_1^2(t) - 2.1070e^{\rho_2(t)}\mathbb{E}x_2^2(t) \\ &\quad + 2e^{\rho_1(t)}\mathbb{E}[0.5x_1^2(t) + 0.5(x_1(t)x_2(t) - x_1^2(t)) + x_1(t)\gamma(t)] + 0.01e^{\rho_1(t)}\mathbb{E}x_1^2(t) \\ &\quad + 2e^{\rho_2(t)}\mathbb{E}[0.5x_2^2(t) + 0.5(x_1(t)x_2(t) - x_2^2(t)) + x_2(t)\tilde{\gamma}(t)] + e^{\rho_2(t)}\mathbb{E}x_2^2(t) \\ &\leq -0.0435e^{\rho_1(t)}\mathbb{E}x_1^2(t) - 0.1070e^{\rho_2(t)}\mathbb{E}x_2^2(t) + 2.4690\mathbb{E}x_1(t)\gamma(t) + 2.4690\mathbb{E}x_2(t)\tilde{\gamma}(t) \\ &\leq -0.03\mathbb{E}\Phi(x(t), t) + 113(\gamma^2(t) + \tilde{\gamma}^2(t)), \quad t \neq 0.1n, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}e^{\rho_1(0.2n)}x_1^2(0.2n) &= 0.81e^{0.2107}\mathbb{E}x_1^2(0.2n-) = \mathbb{E}x_1^2(0.2n-) = e^{\rho_1(0.2n-)}\mathbb{E}x_1^2(0.2n-), \\ \mathbb{E}e^{\rho_1(0.2n+0.1)}x_1^2(0.2n + 0.1) &= \mathbb{E}e^{\rho_1((0.2n+0.1)-)}x_1^2((0.2n + 0.1)-), \\ \mathbb{E}e^{\rho_2(0.1n)}x_2^2(0.1n) &= 0.81e^{0.2107}\mathbb{E}x_2^2(0.1n-) = \mathbb{E}x_2^2(0.1n-) = e^{\rho_2(0.1n-)}\mathbb{E}x_2^2(0.1n-), \end{aligned}$$

by using the continuity of $\mathbb{E}e^{\rho_1(t)}x_1^2(t)$ on $[0.2n, 0.2(n+1))$, thus $\mathbb{E}\Phi(x(0.1n), 0.1n) \leq \mathbb{E}\Phi(x(0.1n-), 0.1n-)$. As to the analysis above, it is easy to see that system (C1) is ISS.

Indeed, in addition to selecting distinct coefficients for the diffusion terms (i.e., stochastic disturbance intensities ϵ_3 and $\tilde{\epsilon}_3$), one may also consider assigning different values to the drift term coefficients or the coupling strengths, such as ϵ_1 and $\tilde{\epsilon}_1; \epsilon_2$ and $\tilde{\epsilon}_2$. For example, let $\epsilon_1 = -1, \tilde{\epsilon}_1 = -1.495, \epsilon_2 = \tilde{\epsilon}_2 = 0.5, \epsilon_3 = \tilde{\epsilon}_3 = 1$; or $\epsilon_1 = \tilde{\epsilon}_1 = -1, \epsilon_2 = \frac{5+\sqrt{198}}{10}, \tilde{\epsilon}_2 = 0.5, \epsilon_3 = \tilde{\epsilon}_3 = 0.1$ for Case 1, we can obtain $\mathcal{L}\Phi(x(t), t) \leq -x_1^2(t) - 1.99x_2^2(t) + 2x_1(t)\gamma(t) + 2x_2(t)\tilde{\gamma}(t)$. Similarly, let $\epsilon_1 = 0.5, \tilde{\epsilon}_1 = 0.995, \epsilon_2 = \tilde{\epsilon}_2 = 0.5, \epsilon_3 = \tilde{\epsilon}_3 = 0.1$ or $\epsilon_1 = \tilde{\epsilon}_1 = 0.5, \epsilon_2 = 0.5, \tilde{\epsilon}_2 = \frac{5+\sqrt{198}}{10}, \epsilon_3 = \tilde{\epsilon}_3 = 0.1$ for Case 2, we can obtain $\mathcal{L}\Phi(x(t), t) = 1.01x_1^2(t) + 2x_2^2(t) + 2x_1(t)\gamma(t) + 2x_2(t)\tilde{\gamma}(t)$. Consistent with the preceding analysis, analogous results can be derived.

Furthermore, in this example, we employed identical impulsive strengths $\vartheta, \tilde{\vartheta}$ with different impulsive frequencies t_n^1, t_n^2 . In practice, one could also use identical impulsive frequencies with different impulsive strengths. For instance, letting $t_n^1 = t_n^2 = 0.1n$ and $\vartheta = 1.01, \tilde{\vartheta} = 1.1$ for Case 1, or let $t_n^1 = t_n^2 = 0.1n$ and $\vartheta = 0.7, \tilde{\vartheta} = 0.9$ for Case 2, applying our method guarantees ISS of system (5), unlike prior synchronously impulsive approaches.

Remark 1. In this example, it is evident that direct analysis methods for the overall system (such as linear matrix inequality) are inapplicable. This paper establishes auxiliary functions for each subsystem, constructing a modified theoretical framework that systematically analyzes individual subsystems before integration. This framework guarantees that the improved Lyapunov function estimates demonstrate monotonic decrease during continuous flows and non-increasing characteristics during discrete jumps, thereby rigorously deriving the final conclusions. This estimation framework, to our best knowledge, has not been reported in the existing literature on ISS.

Remark 2. Furthermore, the results of this study encompass exponential stability conclusions for system (1) under asynchronous impulsive disturbances, controls, and effects. As evidenced by Cases 1 and 2, when compared with traditional synchronously impulsive approaches, our proposed method demonstrates two significant advantages: 1) all nodes except the one with minimum-frequency impulsive disturbance can withstand higher-frequency disturbances; 2) all nodes except the one with maximum-frequency impulsive control can operate effectively with lower-frequency controls.

Remark 3. The AII condition has been applied in the stability analysis of impulsive systems [2, 22, 28]. In contrast to earlier works, this paper introduces an improved asynchronous AII condition (i.e., Assumption 1 [22]) tailored to account for asynchronous impulses, which is dependent on each individual node. Furthermore, to prevent the Zeno behavior, it is typically necessary for the impulsive interval to have a positive lower bound, denoted as $\tau_{\min}^{(u)}$ (which can be arbitrarily small). At the same time, a positive upper bound on the impulsive interval, $\tau_{\max}^{(u)}$ (which can be infinite), is often required to ensure that there are enough stabilizing impulses to sustain system stability. Assumptions 1 and 2 are compatible because they describe different aspects of the system. A node obeying Assumption 1 automatically satisfies Assumption 2, confirming their joint rationality. For example, if let $\tau_u^* = 0.8$, $N_u^{(0)} = 1.5$, then we can obtain $\tau_{\min}^{(u)} > 0$, $\tau_{\max}^{(u)} \leq 1.2$. Thus, we can choose $\tau_{\min}^{(u)} = 0.1$, $\tau_{\max}^{(u)} = 1.2$.

Remark 4. As stated in [12, 16, 18], the impulsive effect plays a significant role in many dynamic systems, as it can describe the actual evolution process exhibiting instantaneous changes. The asynchronous impulsive effects discussed in this paper naturally extend these characteristics and are capable of modeling more general systems. Compared to strategies that focus solely on impulsive control [22], asynchronous impulsive effects can represent either destabilizing or stabilizing impulses, thereby offering a more flexible and comprehensive framework for system analysis and design. Furthermore, the asynchronous nature of impulsive effects permits independent impulse activation governed by local node states, circumventing both the wasted resources and implementation constraints imposed by globally synchronous schemes.

Appendix D The proof of Theorem 1

Proof. **Step 1.** First, define an auxiliary function by

$$\rho_u(t) = \begin{cases} N_u^{(0)}|d_u|, & t = t_0, \\ \min \{N_u^{(0)}|d_u|, \rho_u(t_n^u) - \frac{d_u}{\tau_u^*}(t - t_n^u)\}, & t \in [t_n^u, t_{n+1}^u), \\ \min \{N_u^{(0)}|d_u|, \rho_u(t_n^u) + d_u\}, & t = t_n^u. \end{cases} \quad (D1)$$

It is apparent that $\rho_u(t)$ is continuous on $[t_n^u, t_{n+1}^u)$, $\lim_{t \rightarrow t_n^u} \rho_u(t)$ exists, $\dot{\rho}_u(t) \leq -\frac{d_u}{\tau_u^*}$, and $0 \leq \rho_u(t) \leq N_u^{(0)}|d_u|$. We prove to $\rho_u(t) \geq 0$. If it fails, there exists $t^* > t_0$ such that $\rho_u(t^*) < 0$. Denote $t_* = \sup\{t \in [t_0, t^*) | \rho_u(t) = N_u^{(0)}|d_u|\}$. From Assumption 1, $\rho(t^*) = \rho(t_*) - \frac{d_u}{\tau_u^*}(t^* - t_*) + d_u N_u^{(0)} \geq 0$, which leads to a contradiction. That is $\rho_u(t) \geq 0$. Define Lyapunov function as follows

$$\Phi(x, t) = \sum_{u=1}^L \beta_u \tilde{\Phi}_u(x_u, t), \quad \tilde{\Phi}_u(x_u, t) = e^{\rho_u(t)} \Phi_u(x_u, t), \quad (D2)$$

where $\beta_u > 0$ is the cofactor of the u -th diagonal element in Laplacian matrix for the strongly connected digraph $(\mathcal{G}, (\tilde{A}_{uv} \hat{H} e^{N_u^{(0)}|d_u|})_{L \times L})$.

From (2), $\alpha_u^{(1)}(|x_u|^p) \leq \tilde{\Phi}_u(x_u, t) \leq e^{N_u^{(0)}|d_u|} \alpha_u^{(2)}(|x_u|^p)$. Since $\alpha_u^{(1)} \in \mathcal{VK}_\infty$ and $\alpha_u^{(2)} \in \mathcal{CK}_\infty$, we obtain $\alpha_u^{(1)}(\mathbb{E}|x_u|^p) \leq \mathbb{E}\tilde{\Phi}_u(x_u, t) \leq \alpha_u^{(2)}(\mathbb{E}|x_u|^p) e^{N_u^{(0)}|d_u|}$. Let $\beta_* = \min_{u \in \mathbb{L}} \{\beta_u\}$, $\beta^* = \sum_{u=1}^L \beta_u e^{N_u^{(0)}|d_u|}$, $\alpha_1^{(0)}(\cdot) = \min_{u \in \mathbb{L}} \{\alpha_u^{(1)}(\cdot)\}$, $\alpha_2^{(0)}(\cdot) = \max_{u \in \mathbb{L}} \{\alpha_u^{(2)}(\cdot)\}$, thus they can be obtained that

$$\begin{aligned} \mathbb{E}\Phi(x, t) &= \sum_{u=1}^L \beta_u \mathbb{E}\tilde{\Phi}_u(x_u, t) \geq \sum_{u=1}^L \beta_u \alpha_u^{(1)}(\mathbb{E}|x_u|^p) \\ &\geq \sum_{u=1}^L \beta_u \alpha_1^{(0)}(\mathbb{E}|x_u|^p) \geq \beta_* \max_{u \in \mathbb{L}} \{\alpha_1^{(0)}(\mathbb{E}|x_u|^p)\} \geq \beta_* \alpha_1^{(0)}(L^{-(\frac{p}{2} \vee 1)} \mathbb{E}|x|^p), \end{aligned}$$

and $\mathbb{E}\Phi(x, t) \leq \sum_{u=1}^L \beta_u e^{N_u^{(0)}|d_u|} \alpha_u^{(2)}(\mathbb{E}|x_u|^p) \leq \beta^* \alpha_2^{(0)}(\mathbb{E}|x|^p)$. Denoting $\alpha_*(\cdot) = \beta_* \alpha_1^{(0)}(L^{-(\frac{p}{2} \vee 1)}(\cdot))$, $\alpha^*(\cdot) = \beta^* \alpha_2^{(0)}(\cdot)$, we get

$$\alpha_*(\mathbb{E}|x|^p) \leq \mathbb{E}\Phi(x, t) \leq \alpha^*(\mathbb{E}|x|^p). \quad (D3)$$

Step 2. From (5), there is $C_0 > 0$ such that $-\frac{d_u}{\tau_u^*} - C_u + \sum_{v=1}^L \tilde{A}_{uv} \hat{H}_v e^{N_u^{(0)}|d_u|} \leq -C_0$. Select positive constants $D_u^{(1)}$ and $D_u^{(2)}$ that satisfy

$$\sum_{u=1}^L D_u^{(1)} + \sum_{u=1}^L \frac{1}{\tau_u^*} \ln \left(1 + \sum_{u=1}^L D_u^{(2)} \right) < C_0. \quad (D4)$$

From (3), when $t \in \mathbb{R}_{\geq t_0} \setminus \mathcal{T}_u$, it is given that if

$$e^{N_u^{(0)}|d_u|} \chi_u^{(1)}(|y_u|) \leq \frac{D_u^{(1)}}{\beta_u} \mathbb{E}\Phi(x, t), \quad (D5)$$

we obtain

$$\begin{aligned}
 \mathbb{E}\mathcal{L}\tilde{\Phi}_u(x_u(t), t) &\leq \left(-\frac{d_u}{\tau_u^*} - C_u\right)\mathbb{E}\tilde{\Phi}_u(x_u(t), t) + e^{\rho_u(t)}\chi_u^{(1)}(|y_u(t)|) \\
 &\quad + \sum_{v=1}^L \tilde{A}_{uv}e^{\rho_u(t)}\mathbb{E}\tilde{H}_v(\Phi_v(x_v(t), t)) \\
 &\leq \sum_{v=1}^L \tilde{A}_{uv}\hat{H}_ve^{N_u^{(0)}|d_u|}\mathbb{E}(\Phi_v(x_v(t), t) - \Phi_u(x_u(t), t)) \\
 &\quad - C_0\mathbb{E}\tilde{\Phi}_u(x_u(t), t) + \frac{D_u^{(1)}}{\beta_u}\mathbb{E}\Phi(x(t), t).
 \end{aligned} \tag{D6}$$

From (4) and (D1), it holds that if

$$e^{N_u^{(0)}|d_u|}\chi_u^{(2)}(|y_u|) \leq \frac{D_u^{(2)}}{\beta_u}\mathbb{E}\Phi(x, t), \tag{D7}$$

we have

$$\begin{aligned}
 \mathbb{E}\tilde{\Phi}_u(x_u(t_n^u), t_n^u) &= e^{\rho_u(t_n^u)}\mathbb{E}\Phi_u\left(T_u(x_u(t_n^u-), y_u(t_n^u-), t_n^u), t_n^u\right) \\
 &\leq e^{\rho_u(t_n^u-)}\mathbb{E}\Phi_u(x_u(t_n^u-), t_n^u-) + e^{N_u^{(0)}|d_u|}\chi_u^{(2)}(|y_u(t_n^u-)|) \\
 &\leq \mathbb{E}\tilde{\Phi}_u(x_u(t_n^u-), t_n^u-) + \frac{D_u^{(2)}}{\beta_u}\mathbb{E}\Phi(x(t_n^u-), t_n^u-).
 \end{aligned} \tag{D8}$$

Let $\mathcal{T} = \bigcup_{u \in \mathbb{L}} \mathcal{T}_u \triangleq \{\tilde{t}_n\}_{n \in \mathbb{Z}_+}$, and $N_{\mathcal{T}}(t, s)$ be the number of \mathcal{T} in $(s, t]$. Denote $\tau_{max} = \max_{u \in \mathbb{L}}\{\tau_{max}^{(u)}\}$. From Assumption 2, $0 < \tilde{t}_{n+1} - \tilde{t}_n \leq \tau_{max}$, $\lim_{n \rightarrow \infty} \tilde{t}_n = \infty$, that is, there is no the Zeno behavior. Thus, we get that there is no impulse on $(\tilde{t}_n, \tilde{t}_{n+1})$ and at least a node experiences an impulse at \tilde{t}_n .

From Theorem 2.2 in [20], we obtain that

$$\sum_{u,v=1}^L \beta_u \tilde{A}_{uv} \hat{H}_v e^{N_u^{(0)}|d_u|} (\Phi_v(x_v(t), t) - \Phi_u(x_u(t), t)) = 0. \tag{D9}$$

Denote the upper-right Dini derivative $D^+\hat{\Phi}(t) = \overline{\lim}_{s \rightarrow 0^+} \frac{\hat{\Phi}(t+s) - \hat{\Phi}(t)}{s}$. From (D2), (D6), (D9), and $D^+\mathbb{E}\Phi(x(t), t) = \mathbb{E}\mathcal{L}\Phi(x(t), t)$, when $t \in \mathbb{R}_{\geq t_0} \setminus \mathcal{T}$, it can be derived that

$$D^+\mathbb{E}\Phi(x(t), t) \leq -\left(C_0 - \sum_{u=1}^L D_u^{(1)}\right)\mathbb{E}\Phi(x(t), t) \tag{D10}$$

if (D5) holds. From (D8) and the continuity of $\mathbb{E}\tilde{\Phi}_u(x_u(t), t)$ on $[t_n^u, t_{n+1}^u)$, the following formula can be derived

$$\mathbb{E}\tilde{\Phi}_u(x_u(\tilde{t}_n), \tilde{t}_n) \begin{cases} = \mathbb{E}\tilde{\Phi}_u(x_u(\tilde{t}_n-), \tilde{t}_n-), & \text{if } \tilde{t}_n \notin \mathcal{T}_u, \\ \leq \frac{D_u^{(2)}}{\beta_u}\mathbb{E}\Phi(x(\tilde{t}_n-), \tilde{t}_n-) + \mathbb{E}\tilde{\Phi}_u(x_u(\tilde{t}_n-), \tilde{t}_n-), & \text{if } \tilde{t}_n \in \mathcal{T}_u, \end{cases}$$

which implies

$$\mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) \leq \left(1 + \sum_{u=1}^L D_u^{(2)}\right)\mathbb{E}\Phi(x(\tilde{t}_n-), \tilde{t}_n-) \tag{D11}$$

if (D7) holds.

From Assumption 1, it follows that $N_u(t, s) \leq \frac{t-s}{\tau_u^*} + N_u^{(0)}$, which implies

$$\sum_{u=1}^L N_u(t, s) \leq \sum_{u=1}^L \frac{1}{\tau_u^*}(t-s) + \sum_{u=1}^L N_u^{(0)}. \tag{D12}$$

From (D10) and (D11), we obtain

$$\mathbb{E}\Phi(x(t), t) \leq \mathbb{E}\Phi(x(s), s)e^{-(C_0 - \sum_{u=1}^L D_u^{(1)})(t-s) + \ln(1 + \sum_{u=1}^L D_u^{(2)})N_{\mathcal{T}}(t,s)}, \quad t \geq s \geq t_0 \tag{D13}$$

if (D5) and (D7) hold. Let

$$\tilde{C}_0 = C_0 - \sum_{u=1}^L D_u^{(1)} - \sum_{u=1}^L \frac{1}{\tau_u^*} \ln\left(1 + \sum_{u=1}^L D_u^{(2)}\right), \quad \tilde{C}_1 = \sum_{u=1}^L N_u^{(0)} \ln\left(1 + \sum_{u=1}^L D_u^{(2)}\right),$$

$$\chi(\cdot) = \max_{u \in \mathbb{L}} \left\{ \frac{\beta_u e^{N_u^{(0)}|d_u|}}{D_u^{(1)}} \chi_u^{(1)}(\cdot), \frac{\beta_u e^{N_u^{(0)}|d_u|}}{D_u^{(2)}} \chi_u^{(2)}(\cdot) \right\},$$

and then from (D4), (D5), (D7), (D12), (D13), and $N_{\mathcal{T}}(t, s) \leq \sum_{u=1}^L N_u(t, s)$, we deserve

$$\mathbb{E}\Phi(x(t), t) \leq \mathbb{E}\Phi(x(s), s) e^{\tilde{C}_1 - \tilde{C}_0(t-s)}, \quad t \geq s \geq t_0 \tag{D14}$$

if $\chi(\|y\|_t) \leq \mathbb{E}\Phi(x(t), t)$.

Step 3. Due to the right-continuity of $\mathbb{E}\Phi(x(t), t)$ for $t \in \mathbb{R}_{\geq t_0}$, decompose the interval (t_0, ∞) into subintervals as $(t_0, \infty) = (\varepsilon_1, \tilde{\varepsilon}_1) \cup [\tilde{\varepsilon}_1, \varepsilon_2] \cup (\varepsilon_2, \tilde{\varepsilon}_2) \cup [\tilde{\varepsilon}_2, \varepsilon_3] \cup \dots$, such that

$$\begin{aligned} \mathbb{E}\Phi(x(t), t) &\geq \chi(\|y\|_t), \quad t \in [\varepsilon_i, \tilde{\varepsilon}_i], \\ \mathbb{E}\Phi(x(t), t) &\leq \chi(\|y\|_t), \quad t \in [\tilde{\varepsilon}_i, \varepsilon_{i+1}], \quad i \in \mathbb{Z}_+, \end{aligned}$$

where $t_0 = \varepsilon_1 \leq \tilde{\varepsilon}_1$. Then, either this decompose is finite (the last subinterval is infinite) or infinite (all subintervals are finite). When $\varepsilon_1 < \tilde{\varepsilon}_1$, from (D14), we have

$$\mathbb{E}\Phi(x(t), t) \leq \mathbb{E}\Phi(x(t_0), t_0) e^{\tilde{C}_1 - \tilde{C}_0(t-t_0)}, \tag{D15}$$

when $t \in [t_0, \tilde{\varepsilon}_1]$. From (D3) and (D15), we can obtain

$$\mathbb{E}|x(t)|^p \leq \tilde{f}(\mathbb{E}|x(t_0)|^p, t - t_0), \quad t \in [t_0, \tilde{\varepsilon}_1], \tag{D16}$$

where $\tilde{f}(r, s) = \alpha_*^{-1} \circ \alpha^*(r) e^{\tilde{C}_1 - \tilde{C}_0 s} \in \mathcal{K}\mathcal{L}$.

If $\tilde{\varepsilon}_1 = \infty$, then (D16) holds for any $t > t_0$. By choosing $\gamma(s) = 0$, the conclusion of Theorem 1 holds. Therefore, we consider the case when $\tilde{\varepsilon}_1 < \infty$. If $t_0 = \tilde{\varepsilon}_1$, (D15) and (D16) can be ignored, and we directly analyze the case in $[\tilde{\varepsilon}_1, \varepsilon_2]$. For $[\tilde{\varepsilon}_i, \varepsilon_{i+1}]$, $i \in \mathbb{Z}_+$, we deserve

$$\mathbb{E}\Phi(x(t), t) \leq \chi(\|y\|_t). \tag{D17}$$

Since $\mathbb{E}\Phi(x(t), t)$ is continuous on $\mathbb{R}_{\geq t_0} \setminus \mathcal{T}$, the following equations can be obtained

$$\mathbb{E}\Phi(x(\varepsilon_{i+1}), \varepsilon_{i+1}) = \chi(\|y\|_{\varepsilon_{i+1}}), \quad \text{if } \varepsilon_{i+1} \notin \mathcal{T}, \tag{D18}$$

$$\begin{aligned} \mathbb{E}\Phi(x(\varepsilon_{i+1}), \varepsilon_{i+1}) &\leq \mathbb{E}\Phi(x(\varepsilon_{i+1}), \varepsilon_{i+1}) + \sum_{u=1}^L \beta_u e^{N_u^{(0)}|d_u|} \chi_u^{(2)}(|y_u(\varepsilon_{i+1}) - |) \\ &\leq \left(1 + \sum_{u=1}^L D_u^{(2)} \right) \chi(\|y\|_{\varepsilon_{i+1}}), \quad \text{if } \varepsilon_{i+1} \in \mathcal{T}. \end{aligned} \tag{D19}$$

From (D17) - (D19), we can derive when $t \in [\tilde{\varepsilon}_i, \varepsilon_{i+1}]$

$$\mathbb{E}\Phi(x(t), t) \leq \left(1 + \sum_{u=1}^L D_u^{(2)} \right) \chi(\|y\|_t). \tag{D20}$$

For $(\varepsilon_i, \tilde{\varepsilon}_i)$, $i = 2, 3, \dots$, similar to (D15), from (D14), the following equation can be obtained when $t \in (\varepsilon_i, \tilde{\varepsilon}_i)$

$$\begin{aligned} \mathbb{E}\Phi(x(t), t) &\leq \mathbb{E}\Phi(x(\varepsilon_i), \varepsilon_i) e^{\tilde{C}_1 - \tilde{C}_0(t-\varepsilon_i)} \leq e^{\tilde{C}_1} \left(1 + \sum_{u=1}^L D_u^{(2)} \right) \chi(\|y\|_{\varepsilon_i}) \\ &\leq e^{\tilde{C}_1} \left(1 + \sum_{u=1}^L D_u^{(2)} \right) \chi(\|y\|_t). \end{aligned} \tag{D21}$$

From (D3), (D20) and (D21), it is attainable to be acquired

$$\mathbb{E}|x(t)|^p \leq \gamma(\|y\|_t), \quad t \geq \tilde{\varepsilon}_1, \tag{D22}$$

where $\gamma(\cdot) = \alpha_*^{-1} \circ \alpha^* \left(1 + \sum_{u=1}^L D_u^{(2)} \right) \chi(\cdot) \in \mathcal{K}_\infty$. From (D10) and (D22), it can be ascertained

$$\mathbb{E}|x(t)|^p \leq \tilde{f}(\mathbb{E}|x(t_0)|^p, t - t_0) + \gamma(\|y\|_t), \quad t \geq t_0, \tag{D23}$$

which implies that system (1) is ISS.

Step 4. For any $\lambda \in (0, 1)$, define $\bar{f}(r, s) = \frac{1}{\lambda} \tilde{f}(r, s) \in \mathcal{K}\mathcal{L}$, $\bar{\gamma}(s) = \frac{1}{\lambda} \gamma(s) \in \mathcal{K}_\infty$. By Chebyshev's inequality, from (D23), it can be acquired

$$\wp \{ |x(t)|^p \geq \bar{f}(\mathbb{E}|x(t_0)|^p, t - t_0) + \bar{\gamma}(\|y\|_t) \} \leq \frac{\mathbb{E}|x(t)|^p}{\bar{f}(\mathbb{E}|x(t_0)|^p, t - t_0) + \bar{\gamma}(\|y\|_t)} \leq \lambda, \quad t \geq t_0,$$

that is,

$$\wp \{ |x(t)|^p \geq \bar{f}(\mathbb{E}|x(t_0)|^p, t - t_0) + \bar{\gamma}(\|y\|_t) \} \geq 1 - \lambda, \quad t \geq t_0,$$

as thus system (1) is stochastic ISS, where \wp is the probability measure.

Remark 5. In earlier studies on the ISS of synchronously impulsive interconnected systems, stabilizing and destabilizing impulses were treated separately [18, 23, 24]. This means that all nodes' impulses could only be classified as either stabilizing or destabilizing, which limits the ability to address mixed impulsive effects (where some nodes are stabilizing impulses while others are destabilizing impulses). This approach presents a significant limitation. However, this paper overcomes that limitation. As demonstrated in Theorem 1, under the conditions established here, system (1) remains ISS and stochastic ISS for all impulsive sequences $\mathcal{T}_u \in \mathcal{T}_u^*$ that satisfy condition (5), regardless of whether impulses on nodes are stabilizing or destabilizing.

Remark 6. Condition (5) plays a pivotal role in Theorem 1, revealing that the ISS of systems is governed by the net balance between the stabilizing and destabilizing factors. The term $\frac{d_u}{\tau_u^*} + C_u$ represents the net stability contribution from the impulsive and continuous dynamics. A larger positive value of this sum strongly promotes ISS. This can be achieved by tuning the parameters in an intuitive way: increasing the stabilizing impulse strength ($d_u > 0$) or frequency (decreasing τ_u^*) when the continuous decay rate $C_u < 0$; conversely, decreasing the destabilizing impulse strength $d_u < 0$ or frequency (increasing τ_u^*) when $C_u > 0$. The coupling term $\sum_{v=1}^L \tilde{A}_{uv} \hat{H}_v e^{N_u^{(0)} |d_u|}$ acts as a destabilizing perturbation. A smaller coupling strength is always beneficial for satisfying the condition and achieving ISS. In essence, the parameters affect ISS by shifting this balance, which aligns well with practical control intuition.

Appendix E The proof of Theorem 2

Proof. **Step 1.** Similar to **Step 1** in the proof of Theorem 1, using $\rho_u(t)$, $\tilde{\Phi}_u(x_u, t)$, $\Phi(x, t)$ and from (2), (D3) holds true.

From (8), there exist $C_0 > 0$ and $q > 1$, such that $-\frac{d_u}{\tau_u^*} - C_u^{(1)} + \sum_{v=1}^L \tilde{A}_{uv} \hat{H}_v e^{N_u^{(0)} |d_u|} < -C_0$, $\tilde{C}_0 \triangleq C_0 - q \max_{u \in \mathbb{L}} \{C_u^{(2)} e^{N_u^{(0)} |d_u|}\} > 0$. From (7), when $t \in \mathbb{R}_{\geq t_0} \setminus \mathcal{T}_u$, we are able to extract

$$\begin{aligned} \mathbb{E} \mathcal{L} \tilde{\Phi}_u((x_u)_t, t) &\leq \sum_{v=1}^L \tilde{A}_{uv} \hat{H}_v e^{N_u^{(0)} |d_u|} \mathbb{E}(\Phi_v(x_v(t), t) - \Phi_u(x_u(t), t)) \\ &\quad + C_u^{(2)} e^{N_u^{(0)} |d_u|} \mathbb{E} \tilde{\Phi}_u(x_u(t - \delta(t)), t - \delta(t)) \\ &\quad + e^{N_u^{(0)} |d_u|} \chi_u^{(1)}(|y_u(t)|) - C_0 \mathbb{E} \tilde{\Phi}_u(x_u(t), t). \end{aligned} \quad (\text{E1})$$

From (D2), (D9) and (E1), it can be derived

$$\begin{aligned} \mathbb{E} \mathcal{L} \Phi(x_t, t) &\leq \max_{u \in \mathbb{L}} \{C_u^{(2)} e^{N_u^{(0)} |d_u|}\} \mathbb{E} \Phi(x(t - \delta(t)), t - \delta(t)) \\ &\quad - C_0 \mathbb{E} \Phi(x(t), t) + \chi(|y(t)|), \quad t \in [\tilde{t}_n, \tilde{t}_{n+1}), \end{aligned} \quad (\text{E2})$$

where $\chi(\cdot) = \sum_{u=1}^L \beta_u e^{N_u^{(0)} |d_u|} \max\{\chi_u^{(1)}(\cdot), \chi_u^{(2)}(\cdot)\}$. From (E2), we are able to acquire

$$\mathbb{E} \mathcal{L} \Phi(x_t, t) \leq -\tilde{C}_0 \mathbb{E} \Phi(x(t), t) + \chi(|y(t)|), \quad t \in [\tilde{t}_n, \tilde{t}_{n+1}), \quad (\text{E3})$$

whenever $\mathbb{E} \Phi(x(t + \theta), t + \theta) \leq q \mathbb{E} \Phi(x(t), t)$, $\theta \in [-\tilde{\delta}, 0]$. From (4) and (D1), we can derive

$$\mathbb{E} \tilde{\Phi}_u(x_u(t_n^u), t_n^u) \leq \mathbb{E} \tilde{\Phi}_u(x_u(t_n^u -), t_n^u -) + e^{N_u^{(0)} |d_u|} \chi_u^{(2)}(|y_u(t_n^u -)|).$$

Combining this with (D2), it is feasible to obtain

$$\mathbb{E} \Phi(x(\tilde{t}_n), \tilde{t}_n) \leq \mathbb{E} \Phi(x(\tilde{t}_n -), \tilde{t}_n -) + \chi(|y(\tilde{t}_n -)|), \quad \tilde{t}_n \in \mathcal{T}. \quad (\text{E4})$$

Step 2. For any $t \in [\tilde{t}_n, \tilde{t}_{n+1})$, $n \in \mathbb{Z}_+$, there are two scenarios to consider:

- ① $\sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E} \Phi(x(s + \theta), s + \theta) \leq q \mathbb{E} \Phi(x(s), s)$ holds for any $s \in [\tilde{t}_n, t]$.
- ② $\sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E} \Phi(x(s + \theta), s + \theta) \leq q \mathbb{E} \Phi(x(s), s)$ does not hold for some $s \in [\tilde{t}_n, t]$.

For ①, from (E3) and $D^+ \mathbb{E} \Phi(x_t, t) = \mathbb{E} \mathcal{L} \Phi(x_t, t)$, we get

$$\begin{aligned} \mathbb{E} \Phi(x(t), t) &\leq \mathbb{E} \Phi(x(\tilde{t}_n), \tilde{t}_n) e^{-\tilde{C}_0(t - \tilde{t}_n)} + \int_{\tilde{t}_n}^t e^{-\tilde{C}_0(t-s)} \chi(|y(s)|) ds \\ &\leq \mathbb{E} \Phi(x(\tilde{t}_n), \tilde{t}_n) e^{-\tilde{C}_0(t - \tilde{t}_n)} + \frac{1}{\tilde{C}_0} \chi(\|y\|_{[\tilde{t}_n, t]}), \quad t \in [\tilde{t}_n, \tilde{t}_{n+1}). \end{aligned} \quad (\text{E5})$$

For ②, denote $t_{(0)} = \sup \{s \in [\tilde{t}_n, t] : \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E} \Phi(x(s + \theta), s + \theta) > q \mathbb{E} \Phi(x(s), s)\}$, which implies $t_{(0)} \leq t$. From the continuity of $\mathbb{E} \Phi(x(t), t)$ on $[\tilde{t}_n, \tilde{t}_{n+1})$, it can be extracted $\sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E} \Phi(x(t_{(0)} + \theta), t_{(0)} + \theta) = q \mathbb{E} \Phi(x(t_{(0)}), t_{(0)})$. When $t_{(0)} < t$, for all $s \in [t_{(0)}, t]$, $\sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E} \Phi(x(s + \theta), s + \theta) \leq q \mathbb{E} \Phi(x(s), s)$, thus

$$\mathbb{E} \Phi(x(t), t) \leq \mathbb{E} \Phi(x(t_{(0)}), t_{(0)}) e^{-\tilde{C}_0(t - t_{(0)})} + \int_{t_{(0)}}^t e^{-\tilde{C}_0(t-s)} \chi(|y(s)|) ds$$

$$\begin{aligned} &\leq q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(t_{(0)} + \theta), t_{(0)} + \theta) + \frac{1}{\bar{C}_0} \chi(\|y\|_{[\tilde{t}_n, t]}) \\ &\leq q^{-1} \sup_{s \in [\tilde{t}_n, t]} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(s + \theta), s + \theta) + \frac{1}{\bar{C}_0} \chi(\|y\|_{[\tilde{t}_n, t]}). \end{aligned} \quad (\text{E6})$$

When $t_{(0)} = t$, it can be determined

$$\begin{aligned} \mathbb{E}\Phi(x(t), t) &= \mathbb{E}\Phi(x(t_{(0)}), t_{(0)}) = q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(t_{(0)} + \theta), t_{(0)} + \theta) \\ &\leq q^{-1} \sup_{s \in [\tilde{t}_n, t]} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(s + \theta), s + \theta). \end{aligned} \quad (\text{E7})$$

From (E5), (E6) and (E7), it is feasible to obtain

$$\begin{aligned} \mathbb{E}\Phi(x(t), t) &\leq \max \left\{ q^{-1} \sup_{s \in [\tilde{t}_n, t]} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(s + \theta), s + \theta), \mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) e^{-\bar{C}_0(t - \tilde{t}_n)} \right\} \\ &\quad + \frac{1}{\bar{C}_0} \chi(\|y\|_{[\tilde{t}_n, t]}), \quad t \in [\tilde{t}_n, \tilde{t}_{n+1}), \quad n \in \mathbb{Z}^+. \end{aligned} \quad (\text{E8})$$

Step 3. From (E8), if

$$\mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) < q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(\tilde{t}_n + \theta), \tilde{t}_n + \theta), \quad (\text{E9})$$

then for $t \in [\tilde{t}_n, \tilde{t}_{n+1})$, we can derive

$$\mathbb{E}\Phi(x(t), t) \leq q^{-1} \sup_{\theta \in [-\tau_{max} - \delta, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta) + \frac{1}{\bar{C}_0} \chi(\|y\|_t). \quad (\text{E10})$$

If $\mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) \geq q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(\tilde{t}_n + \theta), \tilde{t}_n + \theta)$, then for enough large Δ , consider the following two cases

- (1) For any $t \in [\tilde{t}_n - \Delta, \tilde{t}_n)$, $\mathbb{E}\Phi(x(t), t) \geq q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)$ holds.
- (2) For some $t \in [\tilde{t}_n - \Delta, \tilde{t}_n)$, $\mathbb{E}\Phi(x(t), t) \geq q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)$ does not hold.

For the case (1), there exists an integer $n_\Delta \geq 0$ such that $\tilde{t}_n - \Delta \in [\tilde{t}_{n-n_\Delta-1}, \tilde{t}_{n-n_\Delta})$. From (E4) and (E5), it is possible to achieve

$$\begin{aligned} \mathbb{E}\Phi(x(\tilde{t}_{n-n_\Delta}), \tilde{t}_{n-n_\Delta}) &\leq \mathbb{E}\Phi(x(\tilde{t}_{n-n_\Delta-}), \tilde{t}_{n-n_\Delta-}) + \chi(|y(\tilde{t}_{n-n_\Delta-})|) \\ &\leq \mathbb{E}\Phi(x(\tilde{t}_n - \Delta), \tilde{t}_n - \Delta) e^{-\bar{C}_0(\tilde{t}_{n-n_\Delta-} - (\tilde{t}_n - \Delta))} \\ &\quad + \frac{1}{\bar{C}_0} \chi(\|y\|_{\tilde{t}_{n-n_\Delta-}}) + \chi(|y(\tilde{t}_{n-n_\Delta-})|), \\ \mathbb{E}\Phi(x(\tilde{t}_{n-n_\Delta+1}), \tilde{t}_{n-n_\Delta+1}) &\leq \mathbb{E}\Phi(x(\tilde{t}_{n-n_\Delta+1-}), \tilde{t}_{n-n_\Delta+1-}) + \chi(|y(\tilde{t}_{n-n_\Delta+1-})|) \\ &\leq \mathbb{E}\Phi(x(\tilde{t}_{n-n_\Delta}), \tilde{t}_{n-n_\Delta}) e^{-\bar{C}_0(\tilde{t}_{n-n_\Delta+1-} - \tilde{t}_{n-n_\Delta})} \\ &\quad + \frac{1}{\bar{C}_0} \chi(\|y\|_{\tilde{t}_{n-n_\Delta+1-}}) + \chi(|y(\tilde{t}_{n-n_\Delta+1-})|), \\ &\quad \vdots \\ \mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) &\leq \mathbb{E}\Phi(x(\tilde{t}_n-), \tilde{t}_n-) + \chi(|y(\tilde{t}_n-)|) \\ &\leq \mathbb{E}\Phi(x(\tilde{t}_{n-1}), \tilde{t}_{n-1}) e^{-\bar{C}_0(\tilde{t}_n - \tilde{t}_{n-1})} + \frac{1}{\bar{C}_0} \chi(\|y\|_{\tilde{t}_n}) + \chi(|y(\tilde{t}_n-)|), \end{aligned}$$

which imply

$$\mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) \leq \left(\frac{1}{\bar{C}_0} + 1 \right) N_{\mathcal{T}}(\tilde{t}_n, \tilde{t}_n - \Delta) \chi(\|y\|_{\tilde{t}_n}) + \mathbb{E}\Phi(x(\tilde{t}_n - \Delta), \tilde{t}_n - \Delta) e^{-\bar{C}_0 \Delta}. \quad (\text{E11})$$

Combining (D12) with (E11), it is attainable to be acquired

$$\mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) \leq e^{-\bar{C}_0 \Delta} \mathbb{E}\Phi(x(\tilde{t}_n - \Delta), \tilde{t}_n - \Delta) + \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)} \right) \left(\frac{1}{\bar{C}_0} + 1 \right) \chi(\|y\|_{\tilde{t}_n}). \quad (\text{E12})$$

For the case (2), denote $t_{(1)} = \sup \{ t \in [\tilde{t}_n - \Delta, \tilde{t}_n) : \mathbb{E}\Phi(x(t), t) < q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta) \}$. If $t_{(1)} = \tilde{t}_n$, then from (E4) it can be extracted

$$\mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) \leq \mathbb{E}\Phi(x(\tilde{t}_n-), \tilde{t}_n-) + \chi(|y(\tilde{t}_n-)|) \leq q^{-1} \sup_{\theta \in [-\delta, 0]} \mathbb{E}\Phi(x(\tilde{t}_n + \theta), \tilde{t}_n + \theta) + \chi(\|y\|_{\tilde{t}_n}). \quad (\text{E13})$$

If $t_{(1)} < \tilde{t}_n$ and $t_{(1)} \notin \mathcal{T}$, then

$$\mathbb{E}\Phi(x(t_{(1)}), t_{(1)}) = q^{-1} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_{(1)} + \theta), t_{(1)} + \theta)$$

and

$$\mathbb{E}\Phi(x(t), t) \geq q^{-1} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta), \quad t \in [t_{(1)}, \tilde{t}_n].$$

Similar to (E11), it can be acquired

$$\begin{aligned} \mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) &\leq \left(\frac{1}{\tilde{C}_0} + 1\right) N_{\mathcal{T}}(\tilde{t}_n, t_{(1)}) \chi(\|y\|_{\tilde{t}_n}) + \mathbb{E}\Phi(x(t_{(1)}), t_{(1)}) e^{-\tilde{C}_0(\tilde{t}_n - t_{(1)})} \\ &\leq \left(\frac{1}{\tilde{C}_0} + 1\right) N_{\mathcal{T}}(\tilde{t}_n, \tilde{t}_n - \Delta) \chi(\|y\|_{\tilde{t}_n}) + q^{-1} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_{(1)} + \theta), t_{(1)} + \theta) \\ &\leq \left(\frac{1}{\tilde{C}_0} + 1\right) \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)}\right) \chi(\|y\|_{\tilde{t}_n}) + q^{-1} \sup_{\theta \in [-\tilde{\delta} - \Delta, 0]} \mathbb{E}\Phi(x(\tilde{t}_n + \theta), \tilde{t}_n + \theta). \end{aligned} \quad (\text{E14})$$

If $t_{(1)} < \tilde{t}_n$ and $t_{(1)} \in \mathcal{T}$, then $\mathbb{E}\Phi(x(t_{(1)} -), t_{(1)} -) \leq q^{-1} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_{(1)} + \theta), t_{(1)} + \theta)$ and $\mathbb{E}\Phi(x(t + \theta), t + \theta) \geq q^{-1} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)$, $t \in [t_{(1)}, \tilde{t}_n]$. Similar to (E11), it can be ascertained

$$\begin{aligned} \mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) &\leq \left(\frac{1}{\tilde{C}_0} + 1\right) N_{\mathcal{T}}(\tilde{t}_n, t_{(1)}) \chi(\|y\|_{\tilde{t}_n}) + \mathbb{E}\Phi(x(t_{(1)}), t_{(1)}) e^{-\tilde{C}_0(\tilde{t}_n - t_{(1)})} \\ &\leq \left[\left(\frac{1}{\tilde{C}_0} + 1\right) N_{\mathcal{T}}(\tilde{t}_n, t_{(1)}) + 1\right] \chi(\|y\|_{\tilde{t}_n}) + \mathbb{E}\Phi(x(t_{(1)} -), t_{(1)} -) e^{-\tilde{C}_0(\tilde{t}_n - t_{(1)})} \\ &\leq \left[\left(\frac{1}{\tilde{C}_0} + 1\right) N_{\mathcal{T}}(\tilde{t}_n, t_{(1)}) + 1\right] \chi(\|y\|_{\tilde{t}_n}) + q^{-1} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_{(1)} + \theta), t_{(1)} + \theta) \\ &\leq \left[\left(\frac{1}{\tilde{C}_0} + 1\right) \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)}\right) + 1\right] \chi(\|y\|_{\tilde{t}_n}) + q^{-1} \sup_{\theta \in [-\tilde{\delta} - \Delta, 0]} \mathbb{E}\Phi(x(\tilde{t}_n + \theta), \tilde{t}_n + \theta). \end{aligned} \quad (\text{E15})$$

From (E9), (E12), (E13), (E14) and (E15), it can be derived

$$\begin{aligned} \mathbb{E}\Phi(x(\tilde{t}_n), \tilde{t}_n) &\leq \max\{q^{-1}, e^{-\tilde{C}_0\Delta}\} \sup_{\theta \in [-\tilde{\delta} - \Delta, 0]} \mathbb{E}\Phi(x(\tilde{t}_n + \theta), \tilde{t}_n + \theta) \\ &\quad + \left[\left(\frac{1}{\tilde{C}_0} + 1\right) \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)}\right) + 1\right] \chi(\|y\|_{\tilde{t}_n}), \end{aligned} \quad (\text{E16})$$

for any $n \in \mathbb{Z}_+$. From (E8), (E10) and (E16), it is possible to achieve for $t \geq t_1$

$$\begin{aligned} \mathbb{E}\Phi(x(t), t) &\leq \max\{q^{-1}, e^{-\tilde{C}_0\Delta}\} \sup_{\theta \in [-\Delta - \tau_{max} - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta) \\ &\quad + \left[\left(\frac{1}{\tilde{C}_0} + 1\right) \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)}\right) + 1\right] \chi(\|y\|_{\tilde{t}_n}). \end{aligned} \quad (\text{E17})$$

Step 4. Here, when $t \in [t_0, t_1]$, analyze whether $\mathbb{E}\Phi(x(t), t)$ satisfies the form of (E17). Completing initial conditions, it is necessary to assign values to $x(t)$ for $t \in [t_0 - \Delta - \tau_{max} - \tilde{\delta}, t_0 - \tilde{\delta}]$. Set $\tilde{q} = \max\{q^{-1}, e^{-\tilde{C}_0\Delta}\}$, which satisfies $0 < \tilde{q} < 1$. Select appropriate $x(t)$, $t \in [t_0 - \Delta - \tau_{max} - \tilde{\delta}, t_0 - \tilde{\delta}]$ such that

$$\mathbb{E}\Phi(x(t_0), t_0) \leq \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_0 + \theta), t_0 + \theta). \quad (\text{E18})$$

If $\mathbb{E}\Phi(x(t_0), t_0) \leq \tilde{q} \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_0 + \theta), t_0 + \theta)$, then choose $x(t) = x(t_0 - \tilde{\delta})$, $t \in [t_0 - \Delta - \tau_{max} - \tilde{\delta}, t_0 - \tilde{\delta}]$. Otherwise, choose suitable $x(t)$, $t \in [t_0 - \Delta - \tau_{max} - \tilde{\delta}, t_0 - \tilde{\delta}]$ such that $x(t) = x(t_0 - \Delta - \tilde{\delta})$, $t \in [t_0 - \Delta - \tau_{max} - \tilde{\delta}, t_0 - \Delta - \tilde{\delta}]$, $\mathbb{E}\Phi(x(t_0), t_0) = \tilde{q} \mathbb{E}\Phi(x(t_0 - \Delta - \tilde{\delta}), t_0 - \Delta - \tilde{\delta})$, and $x(t)$ is continuous on $[t_0 - \Delta - \tau_{max} - \tilde{\delta}, t_0]$. Then we have (E18) and

$$\tilde{q}^{-1} \sup_{t_0 - \tilde{\delta} \leq s \leq t_0} \mathbb{E}\Phi(x(s), s) \geq \sup_{t_0 - \Delta - \tau_{max} - \tilde{\delta} \leq s \leq t_0} \mathbb{E}\Phi(x(s), s). \quad (\text{E19})$$

From (E18), for $\hat{t} \in (t_0, t_1)$, there are two scenarios to consider

- 1) $\mathbb{E}\Phi(x(t), t) \leq \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)$ for any $t \in (t_0, \hat{t}]$ holds.
- 2) $\mathbb{E}\Phi(x(t), t) \leq \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)$ for some $t \in (t_0, \hat{t}]$ dose not hold.

For 1), we can derive

$$\mathbb{E}\Phi(x(t), t) \leq \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta), \quad t \in [t_0, t_1]. \quad (\text{E20})$$

For 2), denote $t_{(2)} = \sup \{t \in (t_0, \hat{t}) : \mathbb{E}\Phi(x(t), t) \leq \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)\}$. Since $\mathbb{E}\Phi(x(t), t)$ is continuous on $[t_0, t_1)$, we have $t_0 \leq t_{(2)} \leq \hat{t}$. If $t_{(2)} < \hat{t}$, then $\mathbb{E}\Phi(x(t_{(2)}), t_{(2)}) = \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_{(2)} + \theta), t_{(2)} + \theta)$, and $q\mathbb{E}\Phi(x(t), t) \geq \tilde{q}^{-1}\mathbb{E}\Phi(x(t), t) \geq \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta) \geq \sup_{\theta \in [-\tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta)$, $t \in [t_{(2)}, \hat{t}]$. From (E3), we can ascertain

$$\begin{aligned} \mathbb{E}\Phi(x(\hat{t}), \hat{t}) &\leq \mathbb{E}\Phi(x(t_{(2)}), t_{(2)})e^{-\tilde{C}_0(\hat{t}-t_{(2)})} + \frac{1}{\tilde{C}_0}\chi(\|y\|_{\hat{t}}) \\ &\leq \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t_{(2)} + \theta), t_{(2)} + \theta) + \frac{1}{\tilde{C}_0}\chi(\|y\|_{\hat{t}}) \\ &\leq \tilde{q} \sup_{t \in [t_0, \hat{t}]} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta) + \frac{1}{\tilde{C}_0}\chi(\|y\|_{\hat{t}}) \\ &\leq \tilde{q} \sup_{\theta \in [-\Delta - \tau_{max} - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(\hat{t} + \theta), \hat{t} + \theta) + \frac{1}{\tilde{C}_0}\chi(\|y\|_{\hat{t}}). \end{aligned} \quad (\text{E21})$$

If $t_{(2)} = \hat{t}$, then

$$\mathbb{E}\Phi(x(\hat{t}), \hat{t}) = \tilde{q} \sup_{\theta \in [-\Delta - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(\hat{t} + \theta), \hat{t} + \theta). \quad (\text{E22})$$

From (E18), (E20), (E21) and (E22), it is achievable to be obtained

$$\mathbb{E}\Phi(x(t), t) \leq \tilde{q} \sup_{\theta \in [-\Delta - \tau_{max} - \tilde{\delta}, 0]} \mathbb{E}\Phi(x(t + \theta), t + \theta) + \frac{1}{\tilde{C}_0}\chi(\|y\|_t), \quad t \in [t_0, t_1]. \quad (\text{E23})$$

From (E17) and (E23), Lemma 1 in [29], it is possible to be derived

$$\begin{aligned} \mathbb{E}\Phi(x(t), t) &\leq e^{\frac{\ln \tilde{q}}{\Delta + \tau_{max} + \tilde{\delta}}(t-t_0)} \sup_{t_0 - (\Delta + \tau_{max} + \tilde{\delta}) \leq s \leq t_0} \mathbb{E}\Phi(x(s), s) \\ &\quad + \frac{1}{(1-\tilde{q})^2} \left[\left(\frac{1}{\tilde{C}_0} + 1 \right) \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)} \right) + 1 \right] \chi(\|y\|_t), \quad t \geq t_0. \end{aligned} \quad (\text{E24})$$

Let $\tilde{f}(r, s) = \alpha_*^{-1} \circ \alpha^*(r) \tilde{q}^{-1} e^{\frac{\ln \tilde{q}}{\Delta + \tau_{max} + \tilde{\delta}}s} \in \mathcal{K}\mathcal{L}$, $\gamma(\cdot) = \alpha_*^{-1} \circ \frac{1}{(1-\tilde{q})^2} \left[\left(\frac{1}{\tilde{C}_0} + 1 \right) \left(\sum_{u=1}^L \frac{\Delta}{\tau_u^*} + \sum_{u=1}^L N_u^{(0)} \right) + 1 \right] \chi(\cdot) \in \mathcal{K}\infty$. From (E19) and (E24), it can be acquired

$$\mathbb{E}|x(t)|^p \leq \tilde{f} \left(\sup_{t_0 - \tilde{\delta} \leq s \leq t_0} \mathbb{E}|x(s)|^p, t - t_0 \right) + \gamma(\|y\|_t), \quad t \geq t_0,$$

which implies that system (6) is ISS.

Remark 7. By comparing the conditions of Theorems 1 and 2, it can be observed that when the model transitions from delay-free case to time-delayed case, ISS can still be guaranteed by modifying condition (5) to condition (8). Specifically, this involves adding a term that accounts for time delay coefficient to the original condition (5). In fact, when time delay coefficient in condition (8) is set to zero (i.e., time delay is not considered), condition (8) reduces to condition (5). However, although the transition from condition (5) to condition (8) may seem straightforward, the proofs are fundamentally different and not merely an extension. Building on some of the foundations established in the proof of Theorem 1, we employ the Lyapunov-Razumikhin approach and Lemma 1 in [29] to establish ISS criterion for time-delayed case. Note that system (6) is simple time-varying bounded delay equation rather than a general functional differential equation. Given the additional complexity introduced by stochastic dynamics and asynchronous impulses, the Lyapunov-Razumikhin method offers a more tractable approach than the Lyapunov-Krasovskii method for constructing Lyapunov function and conducting ISS analysis.

Remark 8. The application of Lemma 1 in [29] at the proof of Theorem 2 introduces a technique of extending initial values, whereas existing approaches typically focus on estimations from time t_1 onward, without addressing the interval $[t_0 - \tilde{\delta}, t_1]$ (where $\tilde{\delta}$ denotes the delay). When $t_1 - t_0$ and the delay are sufficiently small, estimating from t_1 onward may be feasible. However, when either $t_1 - t_0$ or the delay is large, although such an approach does not compromise the ISS property, it leads to more conservative conclusions compared to those established in this work.

Appendix F An application to ISDSs

The dynamics of ISDSs with synchronous impulses has been investigated in [13, 18]. Different from [13, 18], this section consider ISDSs with asynchronous impulses. Firstly, ISDSs (see [18]) are governed by

$$\varrho_u^{(0)} \dot{x}_u^{(1)}(t) + \varrho_u^{(1)} \dot{x}_u^{(1)}(t) + \varrho_u^{(2)} x_u^{(1)}(t) + \varrho_u^{(3)} x_u^{(1)}(t - \delta(t)) - \sum_{v=1}^L \check{A}_{uv} x_v^{(1)}(t) = \check{y}_u(t), \quad u \in \mathbb{L}, \quad (\text{F1})$$

where $\varrho_u^{(0)} > 0$ is the mass/inertia coefficient, $\varrho_u^{(1)} > 0$ is the damping coefficient, $\varrho_u^{(2)} > 0$ is the stiffness coefficient, and $\varrho_u^{(3)} \geq 0$ quantifies the delayed stiffness effect. The coupling strength $\check{A}_{uv} \geq 0$ defines interactions between nodes u and v , and the term $\check{y}_u(t) = y_u(t) + \varrho_u^{(4)} x_u^{(1)}(t) \dot{W}_u(t)$ represents external inputs. Let $x_u^{(2)}(t) = \dot{x}_u^{(1)}(t) + \varrho_u^{(5)} x_u^{(1)}(t)$, $\varrho_u^{(5)} > 0$, and then system (F1) can be rewritten as

$$\begin{cases} dx_u^{(1)}(t) = [x_u^{(2)}(t) - \varrho_u^{(5)} x_u^{(1)}(t)] dt, \\ dx_u^{(2)}(t) = \left[\left(\varrho_u^{(5)} - \frac{\varrho_u^{(1)}}{\varrho_u^{(0)}} \right) x_u^{(2)}(t) + \left(\frac{\varrho_u^{(1)} \varrho_u^{(5)} - \varrho_u^{(2)}}{\varrho_u^{(0)}} - (\varrho_u^{(5)})^2 \right) x_u^{(1)}(t) - \frac{\varrho_u^{(3)}}{\varrho_u^{(0)}} x_u^{(1)}(t - \delta(t)) \right. \\ \left. + \frac{1}{\varrho_u^{(0)}} \sum_{v=1}^L \check{A}_{uv} x_v^{(1)}(t) + \frac{y_u(t)}{\varrho_u^{(0)}} \right] dt + \frac{\varrho_u^{(4)}}{\varrho_u^{(0)}} x_u^{(1)}(t) dW_u(t), \quad u \in \mathbb{L}. \end{cases} \quad (\text{F2})$$

Let $x_u = (x_u^{(1)}, x_u^{(2)})^\top$, $F_u(x_u(t), x_u(t - \delta(t)), y_u(t), t) = (x_u^{(2)}(t) - \varrho_u^{(5)} x_u^{(1)}(t), (\varrho_u^{(5)} - \frac{\varrho_u^{(1)}}{\varrho_u^{(0)}}) x_u^{(2)}(t) + \frac{y_u(t)}{\varrho_u^{(0)}} + (\frac{\varrho_u^{(1)} \varrho_u^{(5)} - \varrho_u^{(2)}}{\varrho_u^{(0)}} - (\varrho_u^{(5)})^2) x_u^{(1)}(t) - \frac{\varrho_u^{(3)}}{\varrho_u^{(0)}} x_u^{(1)}(t - \delta(t)))^\top$, $G_u(x_u(t), t) = (\frac{\varrho_u^{(4)}}{\varrho_u^{(0)}} x_u^{(1)}(t), 0)^\top$, $H_{uv}(x_v(t)) = (x_v^{(1)}(t), 0)^\top$, $A_{uv} = \frac{\check{A}_{uv}}{\varrho_u^{(0)}}$. Then, considering asynchronous impulsive effects into system (F2), we have

$$\begin{cases} dx_u(t) = \left[F_u(x_u(t), x_u(t - \delta(t)), y_u(t), t) + \sum_{v=1}^L A_{uv} H_{uv}(x_v(t)) \right] dt \\ \quad + G_u(x_u(t), t) dW_u(t), \quad t \in \mathbb{R}_{\geq t_0} \setminus \mathcal{T}_u, \\ x_u(t) = \vartheta_u x_u(t-) + y_u(t-), \quad t \in \mathcal{T}_u, \quad u \in \mathbb{L}, \end{cases} \quad (\text{F3})$$

where $\vartheta_u > 0$ and $x_u + y_u = (x_u^{(1)} + y_u, x_u^{(2)} + y_u)^\top$.

Algorithm F1 Select parameters for system (F3) with stabilizing impulses based on Theorem 2.

- 1: Initialize system coefficients, including $\varrho_u^{(0)}, \varrho_u^{(1)}, \varrho_u^{(2)}, \varrho_u^{(3)}, \varrho_u^{(4)}, \varrho_u^{(5)}, \check{A}_{uv}, y_u(t), \delta(t), L$;
 - 2: Define a node Lyapunov function $\Phi_u(x_u, t)$ satisfying (2);
 - 3: Calculate the parameters in (7), including $C_u^{(1)}, C_u^{(2)}, \check{A}_{uv}, \check{H}_v(\cdot), \chi_u^{(1)}(\cdot)$;
 - 4: Choose the parameters with regard to stabilizing impulses $\vartheta_u, N_u^{(0)}, \tau_u^*$;
 - 5: Calculate the parameters in (4), including $d_u, \chi_u^{(2)}(\cdot)$;
 - 6: Calculate the parameter \check{C}_u in (8);
 - 7: Check the condition $\check{C}_u > 0$ by adjusting the parameters $\vartheta_u, N_u^{(0)}, \tau_u^*$;
-

Let $\Phi_u(x_u, t) = |x_u|^2$. Then, the parameter values in Definitions 1 and 2 are determined as follows through computation: $\alpha_u^{(1)}(|x_u|^2) = |x_u|^2 = \alpha_u^{(2)}(|x_u|^2)$, $\check{A}_{uv} = \frac{\check{A}_{uv}}{\varrho_u^{(0)}}$, $\check{H}_v(t) = t$, $\check{H}_v = 1$, $\chi_u^{(1)}(|y_u|) = \frac{100}{(\varrho_u^{(0)})^2} |y_u|^2$, $C_u^{(1)} = 2 \min \left\{ \varrho_u^{(5)} - \frac{1}{2} \left| \frac{\varrho_u^{(1)} \varrho_u^{(5)} - \varrho_u^{(2)}}{\varrho_u^{(0)}} - (\varrho_u^{(5)})^2 + 1 \right|, \frac{\varrho_u^{(1)}}{\varrho_u^{(0)}} - \varrho_u^{(5)} - \frac{1}{2} \left| \frac{\varrho_u^{(1)} \varrho_u^{(5)} - \varrho_u^{(2)}}{\varrho_u^{(0)}} - (\varrho_u^{(5)})^2 + 1 \right| - \frac{\varrho_u^{(3)}}{2\varrho_u^{(0)}} - \frac{1}{200} \right\} - (\frac{\varrho_u^{(4)}}{\varrho_u^{(0)}})^2 - \sum_{v=1}^L \check{A}_{uv}$, $d_u = -\ln(\vartheta_u^2 + 0.01)$, $C_u^{(2)} = \frac{\varrho_u^{(3)}}{\varrho_u^{(0)}}$, $\chi_u^{(2)}(|y_u|) = (1 + 100\vartheta_u^2) |y_u|^2$.

Next, let $L = 15$. Select parameters $\delta(t) = 0.5|\sin t|$, $y_u(t) = \frac{u}{1+u} |\sin t|$, $\varrho_u^{(0)} = \varrho_u^{(2)} = 8$, $\varrho_u^{(1)} = 9.6$, $\varrho_u^{(3)} = \varrho_u^{(4)} = 0.8$, $\varrho_u^{(5)} = 0.57$, $N_u^{(0)} = 1.5$, $\tau_u^* = 0.73$, $u \in \mathbb{L}$, $t_0 = 0$, $\vartheta_u = 1.2$, $u \in \{1, 2, \dots, 10\}$ (destabilizing impulses), $\vartheta_u = 0.8$, $u \in \{11, 12, \dots, 15\}$ (stabilizing impulses), and the nonzero components of matrix $(\check{A}_{uv})_{15 \times 15}$ are provided $\check{A}_{12} = \check{A}_{31} = \check{A}_{41} = \check{A}_{25} = \check{A}_{63} = \check{A}_{54} = \check{A}_{64} = \check{A}_{57} = \check{A}_{5,10} = \check{A}_{69} = \check{A}_{86} = \check{A}_{95} = \check{A}_{97} = \check{A}_{12,4} = \check{A}_{12,10} = \check{A}_{10,6} = \check{A}_{8,10} = \check{A}_{13,8} = \check{A}_{15,13} = \check{A}_{12,15} = \check{A}_{12,9} = \check{A}_{14,9} = \check{A}_{7,11} = \check{A}_{11,14} = \check{A}_{14,12} = \check{A}_{15,10} = 0.08$. By calculating, we obtain $\min\{\check{C}_u\} = 0.0049 > 0$. Thus, all conditions in Theorem 2 hold. From Theorem 2, system (F3) is ISS. Moreover, the impulsive sequence \mathcal{T}_u and the mean square trajectory of system (F3) are shown in Figures F1 and F2.

To demonstrate the advantages of the asynchronous AII, an example is provided below, using stabilizing impulses (i.e., impulsive control) as an illustration. Here, determine the selection of stabilizing impulses parameters according to Algorithm F1. Reselect parameters $\varrho_u^{(3)} = 4$, $\varrho_u^{(4)} = 0.8$, $u \in \mathbb{L}$; $\varrho_u^{(4)} = 8$, $\tau_u^* = 0.24$, $u \in \{1, 2, \dots, 10\}$; $\varrho_u^{(4)} = 1.6$, $\tau_u^* = 0.59$, $u \in \{11, 12, \dots, 15\}$, and the values of other parameters remain unchanged. By calculating, we obtain $\min\{\check{C}_u\} = 0.0106 > 0$. Moreover, the impulsive sequence \mathcal{T}_u and the mean square trajectory of system (F3) are shown in Figures F3 and F4.

From Figure F5, it is observed that a larger value of τ_u^* or a smaller value of ϑ_u makes it easier to achieve ISS, as this corresponds to a lower frequency of destabilizing impulses or weaker impulsive intensity. Conversely, a smaller τ_u^* , meaning a higher frequency of destabilizing impulses (where the impulses directly act on the system state and are also input-dependent), results in a larger upper bound for ISS. On the other hand, from Figure F6, it is observed that a smaller τ_u^* or a smaller ϑ_u makes it easier to achieve ISS, as this corresponds to a higher frequency of stabilizing impulses or stronger impulsive intensity. Conversely, a smaller τ_u^* or a larger ϑ_u , meaning a higher frequency of stabilizing impulses or weaker impulsive intensity, results in a larger upper bound for ISS.

Remark 9. The impulsive sequence \mathcal{T}_u in Figures F1 and F3 clearly demonstrates the asynchrony of impulsive effects. Unlike [13, 18, 24, 28, 30], which assumes a uniform impulsive scheme, Figure F1 introduces mixed impulsive effects: nodes $1, 2, \dots, 10$ are

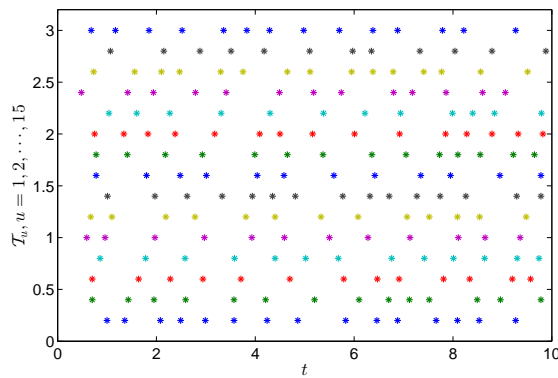


Figure F1 The impulsive sequence \mathcal{T}_u .

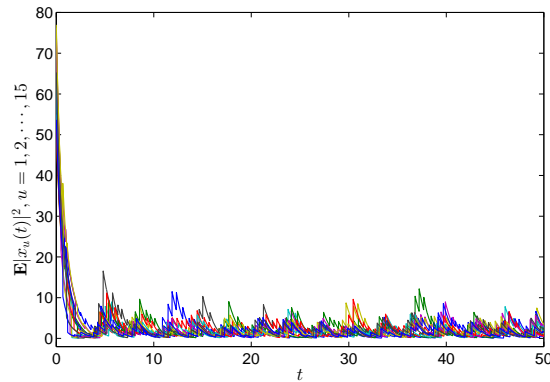


Figure F2 The mean square trajectory of system (F3).

subject to destabilizing impulses, while nodes 11, 12, \dots , 15 are controlled by stabilizing impulses. Furthermore, Figure F3 illustrates how asynchronous stabilizing impulses can effectively address the heterogeneity of interconnected systems. Specifically, when the noise disturbance varies ($\varrho_u^{(4)} = 8$ for $u = 1, 2, \dots, 10$ and $\varrho_u^{(4)} = 1.6$ for $u = 11, 12, \dots, 15$), a uniform impulsive strategy is unnecessary. Instead, nodes 11, 12, \dots , 15 require a significantly lower impulsive frequency compared to nodes 1, 2, \dots , 10, demonstrating the flexibility and adaptability of asynchronous control.

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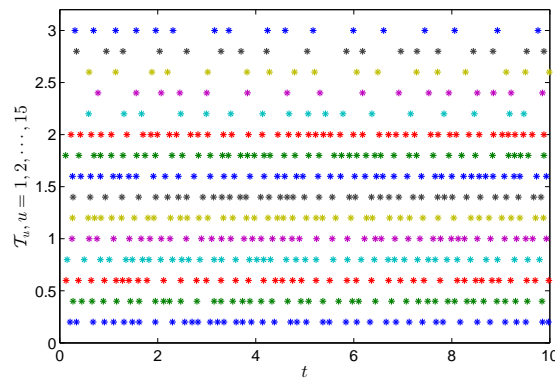


Figure F3 The impulsive sequence \mathcal{T}_u .

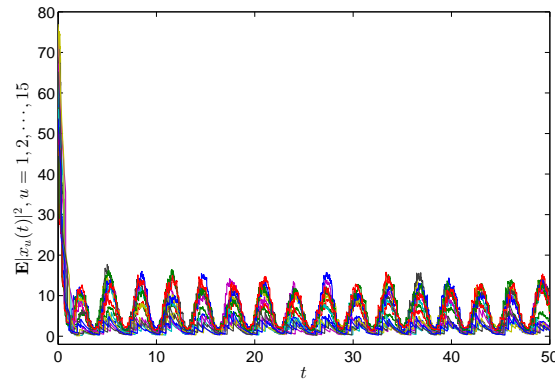


Figure F4 The mean square trajectory of system (F3).

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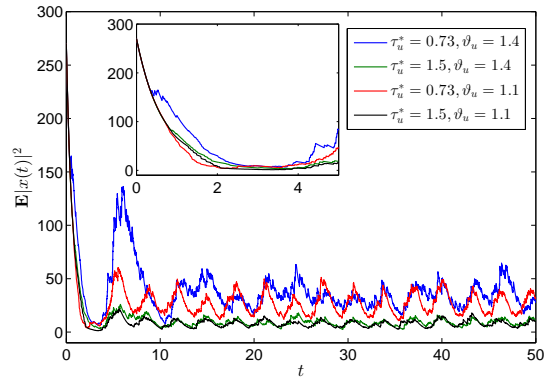


Figure F5 The mean square trajectory of system (F3) under destabilizing impulses.

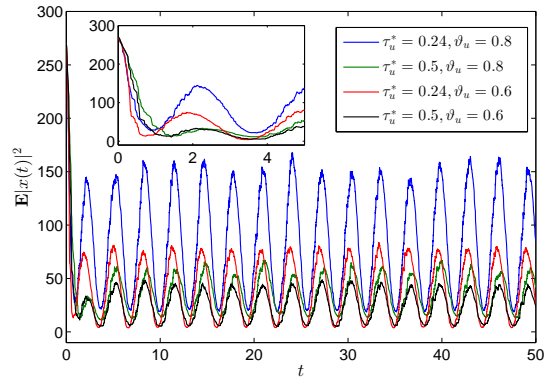


Figure F6 The mean square trajectory of system (F3) under stabilizing impulses.