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Angle rigid formations with discrete dynamics

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Abstract Most angle rigid formations have been achieved via continuous dynamics, while practical control implementation is usually discrete, and the extension from continuous dynamics to discrete dynamics is nontrivial. Motivated by this, this paper investigates the planar angle rigid formation control problem for a class of discrete-time multi-agent systems. Firstly, a discrete formation control law is derived directly from the continuous formation controller to achieve the desired triangular angle rigid formation. However, the selection of the sampling period to guarantee system stability depends on the agents' desired angles and interaction topology, for which the sampling period cannot be arbitrarily chosen, and the directly derived control law is not fully distributed. To address this limitation, a novel formation algorithm is designed by using each agent's locally available information, i.e., triangle degree. Under this algorithm, the formation avoids the sampling period constraint while achieving the angle rigid formation in a fully distributed manner. Furthermore, the algorithm is extended to translational maneuvering control under a leader-following framework. Two simulation examples are selected to verify the effectiveness of the designed discrete formation control strategies.

Keywords angle rigid formations, discrete-time multi-agent systems, formation control, distributed control, stability analysis

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1 Introduction

With the rapid advancement of science and technology, the control of a single system is hard to meet the growing needs of various industrial scenarios, for which the research on multi-agent systems has received widespread attention. Among the numerous control research on multi-agent systems, formation control has occupied a significant proportion because of its irreplaceable practicality and superiority, such as satellite formation communication [1, 2], drone formation flight [3, 4], robot formation collaboration [5, 6], etc. From the perspective of geometric variables specifying a formation shape, the existing formation control methods mainly include formations specified by agents' absolute position constraints [7, 8], inter-agent relative position constraints [9, 10], inter-agent distance constraints [11, 12], inter-agent bearing constraints [13–15], and inter-agent angle constraints [16–20]. In [7], a control strategy based on absolute position constraints is proposed to achieve the formation control of vehicles. A distributed control algorithm based on relative position constraints is designed in [9], which enables unicycles to achieve specified formations. Based on the distance constraints between agents, the formation control law designed in [12] ensures the formation stability of multi-agent systems. A bearing-based formation control law is proposed in [13], which almost globally achieves bearing rigid formation stabilization. Based on the angle rigidity theory, distributed control laws are designed in [16, 17] to ensure that multi-agent systems achieve a desired angle rigid formation. In [19], based on the angle rigidity theory, maneuvering formations of leader-following multi-agent systems are realized in 2D and 3D cases, respectively. Different from several other formation methods, the angle rigid formation method possesses the angle-preserving characteristics of multi-agent formations, consisting of formations' translation, rotation, and scaling motions, which has attracted much attention in recent years.

Until now, many results have been achieved on the rigid formation control of multi-agent systems, among which a considerable number of results are for multi-agent systems described by continuous-time dynamics [21–24]. In [21], distance rigid formations in 3D space are investigated for continuous multi-agent systems. Based on bearing-only measurements between neighbors, a continuous control strategy is designed in [23] to achieve a desired bearing rigid

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formation. However, with the development of digital signal processing and the application of microcomputers, almost all controllers in engineering practice are implemented in a discrete manner. Motivated by this, a few achievements in rigid formation control described by discrete dynamics have emerged [25, 26]. For example, a displacement rigid formation control strategy is proposed in [25], which achieves the formation goal for leader-following multi-agent systems with discrete dynamics. Indeed, the control design for discrete systems is different from that of continuous systems. In particular, for those discrete multi-agent controllers directly derived from continuous controllers, many of them are not distributed since their stability conditions are usually coupled with global graph information [27, 28]. Hence, discrete control design without using global graphic information is still a challenging problem. In recent years, there have been some relevant studies on distributed angle rigid formations for continuous-time multi-agent systems [16–20], but the results on distributed angle rigid formations for discrete-time multi-agent systems have not been adequately investigated yet. Therefore, it is also crucial to investigate the distributed angle rigid formation control of multi-agent systems governed by discrete-time dynamics.

In many practical scenarios, controlled objects are often required to dynamically track target motions after approaching the desired targets, which has driven research interest in formation maneuvering. Based on rigidity theory, numerous results on achieving formation maneuvering have emerged, where the specified factors of the desired formation include relative positions [29], distances [30], bearings [31], and angles among agents [19]. Using relative position constraints, translational formation maneuvering is achieved in [29]. Joint translational and rotational formation maneuvering is realized based on distance constraints in [30]. For bearing rigid formations, joint translational and scaling maneuvering is achieved in [31]. Distinct from these three types of formations, angle-based constraints uniquely enable simultaneous translation, rotation, and scaling maneuvering [19], thus being more likely to meet the requirements for complex maneuvering in practical applications. However, to the authors' knowledge, no sufficient results exist on angle-constrained formation maneuvering for discrete-time multi-agent systems, making it necessary to explore it.

Based on the above analysis, this paper aims to investigate the formation control problem of multi-agents with discrete-time dynamics using angle rigidity theory. The corresponding results are extended to angle rigid translational formation maneuvering for discrete multi-agent systems. The contributions of this paper can be summarized in the following three aspects.

(1) Discrete dynamics. Different from the angle rigid formations with continuous dynamics [16, 19, 20], the agents' dynamics in this paper are discrete. Through theoretical analysis, we find that the convergence of the closed-loop discrete multi-agent systems cannot be trivially guaranteed by directly extending the corresponding continuous-time control strategy alone, as it is strictly constrained by the selection of the sampling period. However, an appropriate sampling period is related to the angle constraints of all desired triangles, which is centralized information in large-scale practical formations (see Theorem 1 and Remark 3). To address this, a novel distributed algorithm is proposed to ensure that the selection of the sampling period is independent of the centralized information.

(2) Distributed algorithms. Different from many traditional discrete formation algorithms that rely on agents' global interaction graph information and that are not fully distributed, we design a fully distributed and discrete angle rigid formation algorithm based on the locally available information, namely, the number of agents' associated neighbors.

(3) Translational maneuvering. To achieve translational formation maneuvering, a discrete angle rigid maneuvering control strategy with an estimated velocity vector is proposed. This strategy ensures that the formation moves at the desired velocity while maintaining the agents' configurations to satisfy the desired angle constraints.

The remainder of this paper is outlined as follows. Section 2 provides the necessary preliminaries. Section 3 discusses angle rigid formation control of discrete multi-agent systems. Section 4 analyzes translational formation maneuvering of discrete multi-agent systems based on angle rigidity theory. In Section 5, simulations are conducted to validate the effectiveness of the proposed algorithms.

2 Preliminaries

2.1 Notations

Multi-agent systems composed of $n \geqslant 3$ agents on a 2D plane are considered. By regarding n agents as n nodes, the set of nodes is denoted as $\mathcal{V} = \{1, 2, \ldots, n\}$. Let $R(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ denote the 2D rotation matrix, where $\theta \in \mathbb{R}$ is the rotation angle. Let $\otimes$ denote the Kronecker product. Let I_n and 1_n denote the $n \times n$ identity matrix and $n \times 1$ column vector with all elements being 1, respectively.

2.2 Triangularly angle rigid formations

In this subsection, we briefly introduce those notations related to multi-agent formations described by angle constraints. For more details, please refer to [16, Section 2]. The triple (i,j,m) is used to represent the angle constraint $\angle ijm$, which is assumed to be equivalent to constraining the angle $\angle mji$, that is, (i,j,m) and (m,j,i) are equivalent. For the node set $\mathcal{V}$, define $\mathcal{A} \subset \mathcal{V}\times\mathcal{V}\times\mathcal{V} = \{(i,j,m), i,j,m \in \mathcal{V}, i \neq j \neq m\}$ as an angle set, and let $p_i \in \mathbb{R}^2$ denote the position of the i-th agent, $i \in \mathcal{V}$. The combination of node set $\mathcal{V}$, angle set $\mathcal{A}$ and position $p = [p_1^{\mathrm{T}}, p_2^{\mathrm{T}}, \ldots, p_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ is called angularity, which is expressed as $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ [16]. If $(i,j,m) \in \mathcal{A}$, then $\{j,m\} \in \mathcal{N}_i$, $\{i,m\} \in \mathcal{N}_j$, $\{i,j\} \in \mathcal{N}_m$, where $\mathcal{N}_i$ represents the neighbor set of i. A set $\mathcal{A}$ is a triangular angle set if for each $(i,j,m) \in \mathcal{A}$, $(j,m,i) \in \mathcal{A}$ and $(m,i,j) \in \mathcal{A}$ are also satisfied. If $\mathcal{A}$ is a triangular angle set, then $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ is called a triangular angularity. Define $m_{\triangle}^{\mathcal{A}} \in \mathbb{N}^+$ as the number of triangles in $\mathbb{A}(\mathcal{V},\mathcal{A},p)$.

Given any three agents $i,j,m \in \mathcal{V}$ with $i \neq j \neq m$, the directional interior angle $\alpha_{mij} \in [0, 2\pi)$ among them is defined as [16]

$$\alpha_{mij} := \begin{cases} \arccos(b_{ij}^{\mathrm{T}} b_{im}), & b_{ij}^{\mathrm{T}} b_{im}^{\perp} \geqslant 0, \\ 2\pi - \arccos(b_{ij}^{\mathrm{T}} b_{im}), & \text{else}, \end{cases} \tag{1}$$

where angle α_{mij} refers to the angle rotated counterclockwise from ray $\overrightarrow{im}$ to ray $\overrightarrow{ij}$. The unit vector $b_{ij} := (p_j - p_i)/\|p_j - p_i\|$ is the bearing direction from agent i to agent j, and $b_{im}^{\perp} = R(\frac{\pi}{2}) b_{im} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] b_{im}$. According to [16], the angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ is said to be angle rigid if the magnitude of all possible angles formed by the nodes in $\mathcal{V}$ remain unchanged when the configuration p is subjected to a small perturbation under the given angle constraints. Define $d_i \in \mathbb{N}^+$ as the triangle degree of node i, which is the number of triangles associated with the i-th node in the triangular angularity.

For multi-agent formations with angle constraints, if the angularity of the formations is angle rigid, then the corresponding formations are called angle rigid formations. The considered formations in this paper are composed of $3m_{\triangle}^{\mathcal{A}}$ angle constraints derived from $m_{\triangle}^{\mathcal{A}}$ triangles among n agents. Thus, the desired angle constraints are expressed as follows:

$$f_{\mathcal{A}}(\alpha^*) := [\ldots\ \alpha_{mij}^*\ \ldots]^{\mathrm{T}} \in \mathbb{R}^{3m_{\triangle}^{\mathcal{A}}},\ (m,i,j) \in \mathcal{A}, \tag{2}$$

where $\alpha_{mij}^* \in [0, 2\pi)$, and $\mathcal{A}$ is the corresponding triangular angle set. Consider that the desired triangular formation of multi-agent systems is specified by angle constraints in (2), whose corresponding desired angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p^*)$ is angle rigid, and $p^* = [p_1^{*\mathrm{T}}, p_2^{*\mathrm{T}}, \ldots, p_n^{*\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ is one of the configurations satisfying angle constraints in (2). In this paper, we assume that the configuration p^* corresponding to the desired formation is generic [16], under which none of the desired triangles in $f_{\mathcal{A}}(\alpha^*)$ are 0 or π, and none of the nodes are collinear.

2.3 Angle-induced linear constraints in triangles

Since the angle constraint (1) is highly nonlinear with respect to the positions of the agents, it brings difficulties to the global stabilization of the formation task (2). In order to facilitate the control design, we transform the nonlinear angle constraint into a linear equation when the formation is triangular.

According to [20], the angle-induced linear constraint in $\triangle ijm$ is derived as follows:

$$\begin{aligned} f_i^{\triangle ijm}(\alpha, p) &= A_i^{\triangle ijm}(\alpha) p_i + A_j^{\triangle ijm}(\alpha) p_j + A_m^{\triangle ijm}(\alpha) p_m \\ &= \sin\alpha_{jmi}(p_i - p_m) - \sin\alpha_{ijm} R^{\mathrm{T}}(\alpha_{mij})(p_i - p_j) \\ &= 0, \end{aligned} \tag{3}$$

where α represents the relevant interior angles in $\triangle ijm$:

$$\begin{aligned} &A_i^{\triangle ijm}(\alpha) = (\sin\alpha_{jmi} I_2 - \sin\alpha_{ijm} R^{\mathrm{T}}(\alpha_{mij})) \in \mathbb{R}^{2\times 2}, \\ &A_j^{\triangle ijm}(\alpha) = \sin\alpha_{ijm} R^{\mathrm{T}}(\alpha_{mij}) \in \mathbb{R}^{2\times 2}, \\ &A_m^{\triangle ijm}(\alpha) = -\sin\alpha_{jmi} I_2 \in \mathbb{R}^{2\times 2}. \end{aligned}$$

Remark 1. According to [32, Lemma 3], the angle α such that Eq. (3) holds is calculated from the non-collinear $p = [p_i^{\mathrm{T}},\ p_j^{\mathrm{T}},\ p_m^{\mathrm{T}}]^{\mathrm{T}}$. Furthermore, if $f_i^{\triangle ijm}(\alpha, p') = A_i^{\triangle ijm}(\alpha) p_{i'} + A_j^{\triangle ijm}(\alpha) p_{j'} + A_m^{\triangle ijm}(\alpha) p_{m'} = 0$, where $p' = [p_{i'}^{\mathrm{T}},\ p_{j'}^{\mathrm{T}},\ p_{m'}^{\mathrm{T}}]^{\mathrm{T}}$, there must be $\triangle ijm \simeq \triangle i'j'm'$. This means that as long as $f_i^{\triangle ijm}(\alpha, p') = 0$, the triangle $\triangle i'j'm'$ formed by p' must be similar to $\triangle ijm$ formed by p in (3).

For the triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ consisting of multiple triangles, all the corresponding angle-induced linear constraints (3) are arranged into a compact form as

$$R_{\mathcal{A}}(\alpha)p = 0, \tag{4}$$

where $R_{\mathcal{A}}(\alpha)$ is specifically expressed as [20]

$$\begin{array}{c|ccccccc} & \dots & \text{Node } i & \dots & \text{Node } j & \dots & \text{Node } m & \dots \\ \hline \begin{array}{c} 1\text{st}\triangle \\ \vdots \\ \triangle ijm \\ \vdots \\ m_{\triangle}^{\mathcal{A}}\text{th} \end{array} & \multicolumn{7}{c}{\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & A_i^{\triangle ijm} & 0 & A_j^{\triangle ijm} & 0 & A_m^{\triangle ijm} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}} \end{array}. \tag{5}$$

In the above matrix, each row is determined by the corresponding triangle that constitutes the desired configuration, and each column is determined by the corresponding node that constitutes the triangle. From [16, 20], it can be seen that the maximum rank of $R_{\mathcal{A}}(\alpha)$ is $2n - 4$.

3 Angle rigid formation control for discrete-time multi-agent systems

The single-integrator discrete-time multi-agent systems are considered as

$$p_i(k+1) = p_i(k) + T \cdot u_i(k),\ i \in \mathcal{V}, \tag{6}$$

where the position $p_i[kT] \in \mathbb{R}^2$ of i-th agent at sampling instant kT is simplified as $p_i(k)$, $T \in \mathbb{R}$ is the sampling period, and the control input $u_i[kT] \in \mathbb{R}^2$ of i-th agent at sampling instant kT is simplified as $u_i(k)$.

Given (2), the goal of angle rigid formation control is to design a controller $u_i(k)$ for the system (6) such that

$$\lim_{k\to\infty} (\alpha_{mij}(k) - \alpha^*_{mij}) = 0,\ \forall (m, i, j) \in \mathcal{A}. \tag{7}$$

3.1 Formation controller design and closed-loop stability analysis

Define a positive semi-definite matrix $L(\alpha^*) := R^{\mathrm{T}}_{\mathcal{A}}(\alpha^*) \cdot R_{\mathcal{A}}(\alpha^*)$. Inspired by [20], a compact form of formation control law is proposed as

$$u(k) = -L(\alpha^*)p(k) = -R^{\mathrm{T}}_{\mathcal{A}}(\alpha^*)R_{\mathcal{A}}(\alpha^*)p(k), \tag{8}$$

where $u(k) = [u_1^{\mathrm{T}}(k), \dots, u_n^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^{2n}$, $p(k) = [p_1^{\mathrm{T}}(k), \dots, p_n^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^{2n}$.

The component form of the controller (8) is

$$\begin{aligned} u_i(k) = -\Bigg[&\sum_{(i,j_1,m_1)\in\bar{\mathcal{A}}} \left(A_i^{\triangle ij_1m_1}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle ij_1m_1}(\alpha^*, p(k)) + \sum_{(m_2,i,j_2)\in\bar{\mathcal{A}}} \left(A_i^{\triangle m_2ij_2}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle m_2ij_2}(\alpha^*, p(k)) \\ &+ \sum_{(j_3,m_3,i)\in\bar{\mathcal{A}}} \left(A_i^{\triangle j_3m_3i}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle j_3m_3i}(\alpha^*, p(k)) \Bigg], \end{aligned} \tag{9}$$

where $f_i^{\triangle ij_1m_1}(\alpha^*, p(k))$, $f_i^{\triangle m_2ij_2}(\alpha^*, p(k))$ and $f_i^{\triangle j_3m_3i}(\alpha^*, p(k))$ are all in accordance with (3), for example, $f_i^{\triangle j_3m_3i}(\alpha^*, p(k)) = \sin\alpha^*_{m_3ij_3}(p_{j_3}(k) - p_i(k)) - \sin\alpha^*_{j_3m_3i}R^{\mathrm{T}}(\alpha^*_{ij_3m_3})(p_{j_3}(k) - p_{m_3}(k))$, $\bar{\mathcal{A}} \subset \mathcal{A}$, $|\bar{\mathcal{A}}| = m^{\mathcal{A}}_{\triangle}$, and if $(i, j, m) \in \bar{\mathcal{A}}$, then $(j, i, m) \notin \bar{\mathcal{A}}$, $(j, m, i) \notin \bar{\mathcal{A}}$. Obviously, the designed control law (9) depends only on relative positions among neighboring agents, not absolute positions.

Theorem 1. Consider that the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is angle rigid. The control algorithm (9) will globally realize the angle rigid formation of the multi-agent system (6) if

$$D^k(\alpha^*) := [I_{2n} - T \cdot L(\alpha^*)]^k \to (1_n h^{\mathrm{T}}) \otimes I_2, \quad k \to \infty, \tag{10}$$

where h is an n-by-1 constant column vector.

Proof. Substituting controller (9) into system (6) yields

$$p(k+1) = p(k) - T \cdot L(\alpha^*)p(k) = [I_{2n} - T \cdot L(\alpha^*)]p(k) = D(\alpha^*)p(k). \tag{11}$$

Let $p_r^* = \epsilon_1 p^* + \epsilon_2 (I_n \otimes R(\frac{\pi}{2}))p^* + \epsilon_3 1_n \otimes [1 \;\; 0]^{\mathrm{T}} + \epsilon_4 1_n \otimes [0 \;\; 1]^{\mathrm{T}}$, $\epsilon_i \in \mathbb{R},\ i = 1,2,3,4$, which is the combination of the desired formation configuration's translation, rotation and scaling. The specific vector component form of p_r^* is $p_r^* = [p_{r_1}^{*\mathrm{T}}, \ldots, p_{r_n}^{*\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ with $p_{r_i}^* \in \mathbb{R}^2$, $i = 1, \ldots, n$. Subtracting p_r^* from both sides of (11) yields

$$p(k+1) - p_r^* = D(\alpha^*)p(k) - p_r^* = D(\alpha^*)(p(k) - p_r^*). \tag{12}$$

Iterating (12) yields $p(k) - p_r^* = D^k(\alpha^*)(p(0) - p_r^*)$. According to (10), it can be transformed into

$$\begin{aligned} p(k) - p_r^* \to & [(1_n h^{\mathrm{T}}) \otimes I_2](p(0) - p_r^*) \\ &= \Big[[(h^{\mathrm{T}} \otimes I_2)(p(0) - p_r^*)]^{\mathrm{T}} \;\; [(h^{\mathrm{T}} \otimes I_2)(p(0) - p_r^*)]^{\mathrm{T}} \; \ldots \; [(h^{\mathrm{T}} \otimes I_2)(p(0) - p_r^*)]^{\mathrm{T}} \Big]^{\mathrm{T}},\ k \to \infty, \end{aligned}$$

which means that $p_i(k) - p_{r_i}^* \to p_j(k) - p_{r_j}^*$, $k \to \infty$, i.e., the relative position of agents converges to the scaling, translation, and rotation of the generic configuration implemented from the desired rigid formation. This is equivalent to that $\alpha_{mij}(k) \to \alpha_{mij}^*$, $\forall (m,i,j) \in \mathcal{A}$, as $k \to \infty$.

Remark 2. Different from the previous results [16, 19, 20], Theorem 1 proposes a sufficient condition (10) for realizing the angle rigid formation of discrete multi-agent systems, which depends on the selection of sampling period T. This implies that when a discrete formation control law is directly derived from the continuous version, if the sampling period is not properly chosen, the discrete formation will be unstable.

Remark 3. For the continuous-time multi-agent system $\dot{p}_i(t) = u_i(t)$, it is known from [20] that under the control law $u(t) = -L(\alpha^*)p(t)$, the almost global convergence of the desired angle rigid formation can be achieved in a distributed manner. For the discrete-time multi-agent system (6) in this paper, as shown in Theorem 1, although the same control strategy can globally achieve the desired angle rigid formation, there are two limiting factors. First, the sampling period T must satisfy condition (10). Second, condition (10) is not distributed, because the sampling period T is related to $L(\alpha^*)$ formed by the interaction topology of all desired triangular angles. These two points are key challenges when extending continuous-time control strategies to discrete-time scenarios.

To overcome the limitations mentioned in Remark 3 and achieve the desired formation in a distributed manner, we will improve the control algorithm in the next subsection to achieve fully distributed formation control.

3.2 Distributed formation controller design and closed-loop stability analysis

A distributed formation control law is proposed:

$$\begin{aligned} u_i(k) = & - k_i \Bigg[\sum_{(i,j_1,m_1)\in\bar{\mathcal{A}}} \Big(A_i^{\triangle ij_1 m_1}(\alpha^*) \Big)^{\mathrm{T}} f_i^{\triangle ij_1 m_1}(\alpha^*, p(k)) + \sum_{(m_2,i,j_2)\in\bar{\mathcal{A}}} \Big(A_i^{\triangle m_2 i j_2}(\alpha^*) \Big)^{\mathrm{T}} f_i^{\triangle m_2 i j_2}(\alpha^*, p(k)) \\ & + \sum_{(j_3,m_3,i)\in\bar{\mathcal{A}}} \Big(A_i^{\triangle j_3 m_3 i}(\alpha^*) \Big)^{\mathrm{T}} f_i^{\triangle j_3 m_3 i}(\alpha^*, p(k)) \Bigg], \end{aligned} \tag{13}$$

where $k_i \in \mathbb{R}$ is a positive coefficient. Let $k_i = \frac{1}{3d_i \cdot T}$, and $d_i \in \mathbb{N}^+$ be the triangle degree of the i-th node in the triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$. Then, the compact form of the formation control law (13) is

$$u(k) = -\left(\frac{1}{3T} M \otimes I_2 \right) L(\alpha^*)p(k), \tag{14}$$

where $M = \mathrm{diag}\{d_1^{-1}, \ldots, d_n^{-1}\}$.

Now, we show the properties of matrix $L(\alpha^*)$ based on the previously investigated D_{ff} in [33]. Let $L[i,j]$ denote the submatrix of $L(\alpha^*)$ with its rows from $2i-1$ to $2i$ and columns from $2j-1$ to $2j$.

Lemma 1 ([33])**.** The specific expression of matrix $L(\alpha^*)$ satisfies

$$L(\alpha^*) = \begin{array}{c} \\ \text{Rows 1 to 2} \\ \vdots \\ \text{Rows } 2i-1 \text{ to } 2i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \dots & \text{Columns } 2i-1 \text{ to } 2i & \dots & \text{Columns } 2j-1 \text{ to } 2j & \dots \end{array} \\ \begin{bmatrix} \dots & L[1,i] & \dots & L[1,j] & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & L[i,i] & \dots & L[i,j] & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \end{array}, \tag{15}$$

where $L[i,i] = \sum_{(i,j_2,m_2)\in\mathcal{A}} (A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_i^{\triangle i_e j_e m_e}$ with $i \neq j_2 \neq m_2$, $\{j_2, m_2\} \subset \mathcal{V}$, $\{i, j_2, m_2\} = \{i_e, j_e, m_e\}$, $(i_e, j_e, m_e) \in \bar{\mathcal{A}}$, and $L[i,j] = \sum_{(i,j,m_1)\in\mathcal{A}} (A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_j^{\triangle i_e j_e m_e}$ with $i \neq j \neq m_1$, $m_1 \in \mathcal{V}$, $\{i, j, m_1\} = \{i_e, j_e, m_e\}$, $(i_e, j_e, m_e) \in \bar{\mathcal{A}}$.

For $L[i,i]$ and $L[i,j]$, there are $(A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_i^{\triangle i_e j_e m_e} = \beta_1^{\triangle i_e, j_e, m_e} I_2$ with $\beta_1^{\triangle i_e, j_e, m_e} \in (0,1]$, $i \in \{i_e, j_e, m_e\}$, and $(A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_j^{\triangle i_e j_e m_e} = \left[\begin{smallmatrix} \beta_2^{\triangle i_e, j_e, m_e} & \beta_3^{\triangle i_e, j_e, m_e} \\ -\beta_3^{\triangle i_e, j_e, m_e} & \beta_2^{\triangle i_e, j_e, m_e} \end{smallmatrix}\right]$ with $|\beta_2^{\triangle i_e, j_e, m_e}| < 1$, $|\beta_3^{\triangle i_e, j_e, m_e}| < 1$ and $|\beta_2^{\triangle i_e, j_e, m_e}| + |\beta_3^{\triangle i_e, j_e, m_e}| < \sqrt{2}$.

Proof. According to the definition of $R_{\mathcal{A}}(\alpha)$, we can easily get (15).

First, we prove the expression of $(A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_i^{\triangle i_e j_e m_e}$. When $i = i_e$, from (3), it is deduced that

$$\left(A_{i_e}^{\triangle i_e j_e m_e}\right)^{\mathrm{T}} A_{i_e}^{\triangle i_e j_e m_e} = \left(\sin^2\alpha_{j_e m_e i_e} + \sin^2\alpha_{i_e j_e m_e} - \rho_1\right) I_2,$$

where $\rho_1 := 2\sin\alpha_{j_e m_e i_e} \sin\alpha_{i_e j_e m_e} \cos\alpha_{m_e i_e j_e}$. From the product-to-sum trigonometric formulas, ρ_1 can be further derived as

$$\begin{aligned} \rho_1 &= \sin\alpha_{j_e m_e i_e} \cdot 2\sin\alpha_{i_e j_e m_e} \cos\alpha_{m_e i_e j_e} = \sin\alpha_{j_e m_e i_e}[\sin(\alpha_{i_e j_e m_e} + \alpha_{m_e i_e j_e}) + \sin(\alpha_{i_e j_e m_e} - \alpha_{m_e i_e j_e})] \\ &= \sin^2\alpha_{j_e m_e i_e} + \sin(\alpha_{i_e j_e m_e} + \alpha_{m_e i_e j_e})\sin(\alpha_{i_e j_e m_e} - \alpha_{m_e i_e j_e}) \\ &= \sin^2\alpha_{j_e m_e i_e} + \sin^2\alpha_{i_e j_e m_e} - \sin^2\alpha_{m_e i_e j_e}. \end{aligned}$$

Obviously, $(A_{i_e}^{\triangle i_e j_e m_e})^{\mathrm{T}} A_{i_e}^{\triangle i_e j_e m_e} = \sin^2\alpha_{m_e i_e j_e} I_2$, where $\sin^2\alpha_{m_e i_e j_e} \in (0,1]$. Accordingly, when $i = j_e$ and $i = m_e$, it follows that $(A_{j_e}^{\triangle i_e j_e m_e})^{\mathrm{T}} A_{j_e}^{\triangle i_e j_e m_e} = \sin^2\alpha_{i_e j_e m_e} I_2$ and $(A_{m_e}^{\triangle i_e j_e m_e})^{\mathrm{T}} A_{m_e}^{\triangle i_e j_e m_e} = \sin^2\alpha_{j_e m_e i_e} I_2$. Therefore, it can be concluded that $(A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_i^{\triangle i_e j_e m_e} = \beta_1^{\triangle i_e j_e m_e} I_2$ holds, and the value of β_1 is 1 if and only if the interior angle with i as vertex is $\pi/2$, $i \in \{i_e, j_e, m_e\}$.

Next, we prove the expression of $(A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_j^{\triangle i_e j_e m_e}$. When $i = i_e$ and $j = j_e$, from (3), it is deduced that

$$(A_{i_e}^{\triangle i_e j_e m_e})^{\mathrm{T}} A_{j_e}^{\triangle i_e j_e m_e} = \sin\alpha_{i_e j_e m_e} \cdot \Omega,$$

where $\Omega = \left[\begin{smallmatrix} -\cos\alpha_{j_e m_e i_e}\sin\alpha_{m_e i_e j_e} & -\sin\alpha_{j_e m_e i_e}\sin\alpha_{m_e i_e j_e} \\ \sin\alpha_{j_e m_e i_e}\sin\alpha_{m_e i_e j_e} & -\cos\alpha_{j_e m_e i_e}\sin\alpha_{m_e i_e j_e} \end{smallmatrix}\right]$. When $i = i_e$, $j = m_e$, and $i = j_e$, $j = m_e$, the corresponding results can be derived. Therefore, it can be concluded that $(A_i^{\triangle i_e j_e m_e})^{\mathrm{T}} A_j^{\triangle i_e j_e m_e} = \left[\begin{smallmatrix} \beta_2^{\triangle i_e, j_e, m_e} & \beta_3^{\triangle i_e, j_e, m_e} \\ -\beta_3^{\triangle i_e, j_e, m_e} & \beta_2^{\triangle i_e, j_e, m_e} \end{smallmatrix}\right]$ holds. We present the following result.

Theorem 2. If the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is angle rigid, then the distributed control algorithm (13) will guarantee that $p(k)$ converges globally to $\beta_1 p^* + \beta_2(I_n \otimes R(\frac{\pi}{2}))p^* + \beta_3 1_n \otimes [1 \quad 0]^{\mathrm{T}} + \beta_4 1_n \otimes [0 \quad 1]^{\mathrm{T}}$, $\beta_i \in \mathbb{R}$, $i = 1,2,3,4$, and for all initial values $p(0)$, formation control goal (7) is realized almost globally.

Proof. Substituting controller (13) into system (6) yields

$$p(k+1) = p(k) - \left(\frac{1}{3}M \otimes I_2\right) L(\alpha^*) p(k) = \left[I_{2n} - \left(\frac{1}{3}M \otimes I_2\right) L(\alpha^*)\right] p(k). \tag{16}$$

For $(\frac{1}{3}M \otimes I_2)L(\alpha^*)$, the matrix M is positive definite and the matrix L is positive semi-definite. It can be inferred from [34, Theorem 10.31] that all eigenvalues λ of $(\frac{1}{3}M \otimes I_2)L(\alpha^*)$ are non-negative, i.e., $\lambda \geqslant 0$.

According to the Gershgorin circle theorem and the structural characteristics of the matrix $L(\alpha^*)$, all eigenvalues λ of $(\frac{1}{3}M \otimes I_2)L(\alpha^*)$ are located in the union of the following $2n$ disks:

$$\left|\lambda - \frac{\sum_{j_1,m_1} \beta_1^{\triangle i_e j_e m_e}}{3d_i}\right| \leqslant \frac{\sum_{j_1,m_1}\left(|\beta_2^{\triangle i_e j_e m_e}| + |\beta_3^{\triangle i_e j_e m_e}|\right)}{3d_i},\ i \in \mathcal{V}, \tag{17}$$

where $\{i, j_1, m_1\} = \{i_e, j_e, m_e\}$, $(i_e, j_e, m_e) \in \bar{\mathcal{A}}$, $\frac{\sum_{j_1,m_1}\beta_1^{\triangle i_e j_e m_e}}{3d_i}$ represents the diagonal elements from row $2i-1$ to row $2i$ in matrix $(\frac{1}{3}M\otimes I_2)L(\alpha^*)$, and $\frac{\sum_{j_1,m_1}|\beta_2^{\triangle i_e j_e m_e}|}{3d_i}$ and $\frac{\sum_{j_1,m_1}|\beta_3^{\triangle i_e j_e m_e}|}{3d_i}$ both represent the absolute values of the non-diagonal elements from row $2i-1$ to row $2i$ in matrix $(\frac{1}{3}M\otimes I_2)L(\alpha^*)$. According to Lemma 1, we know $0 < \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_e j_e m_e}}{3d_i} \leqslant \frac{d_i}{3d_i} = \frac{1}{3}$ and $0 < \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_e j_e m_e}|+|\beta_3^{\triangle i_e j_e m_e}|)}{3d_i} < \frac{2\sqrt{2}d_i}{3d_i} = \frac{2\sqrt{2}}{3}$, so we have $\lambda < \frac{1+2\sqrt{2}}{3} < 2$.

Therefore, the eigenvalue $\bar{\lambda}$ of $I_{2n} - (\frac{1}{3}M\otimes I_2)L(\alpha^*)$ in the closed-loop system (16) satisfies: $\bar{\lambda} \in (-1, 1]$, that is, all eigenvalues will not go outside the unit circle.

Since $(\frac{1}{3}M\otimes I_2)L(\alpha^*)$ is a square matrix with eigenvalue $\lambda \geqslant 0$, there exists a nonsingular matrix $Q \in \mathbb{R}^{2n\times 2n}$ whose first four columns are $p^*, (I_n\otimes R(\frac{\pi}{2}))p^*, 1_n\otimes[1\ \ 0]^{\mathrm{T}}, 1_n\otimes[0\ \ 1]^{\mathrm{T}}$, such that $C := Q^{-1}(\frac{1}{3}M\otimes I_2)L(\alpha^*)Q = \begin{bmatrix} 0_{4\times 4} & 0 \\ 0 & \Lambda\end{bmatrix}$, where Λ is a Jordan matrix with positive eigenvalues on the main diagonal.

Let $p(k) = Qz(k)$, then

$$z(k+1) = Q^{-1}p(k+1) = Q^{-1}\left[I_{2n} - \left(\frac{1}{3}M\otimes I_2\right)L(\alpha^*)\right]Qz(k) = [I_{2n} - C]z(k) = \begin{bmatrix} I_4 & 0 \\ 0 & I_{2n-4}-\Lambda\end{bmatrix} z(k). \tag{18}$$

Since all eigenvalues of I_4 are 1, one has that all eigenvalues of $I_{2n-4} - \Lambda$ are within $(-1,\ 1)$.

Let $z(k) = [z_1(k),\ z_2(k),\ z_3(k),\ z_4(k),\ \bar{z}^{\mathrm{T}}(k)]^{\mathrm{T}}$, $z_i(k) \in \mathbb{R}$, $i = 1,\ldots,4$, $\bar{z}(k) \in \mathbb{R}^{2n-4}$. According to (18), one has

$$\begin{cases} z_i(k+1) = z_i(k),\ i = 1,\ldots,4, \\ \bar{z}(k+1) = (I_{2n-4} - \Lambda)\bar{z}(k), \end{cases} \tag{19}$$

which can be further deduced

$$\begin{cases} z_i(k) = z_i(0), & i = 1,\ldots,4, \\ \bar{z}(k) \to 0, & k \to \infty. \end{cases} \tag{20}$$

Let $Q = [p^*,\ (I_n\otimes R(\frac{\pi}{2}))p^*,\ 1_n\otimes[1\ \ 0]^{\mathrm{T}},\ 1_n\otimes[0\ \ 1]^{\mathrm{T}},\ \tilde{Q}]$, $\tilde{Q} \in \mathbb{R}^{2n\times 2n-4}$, then

$$p(k) = Qz(k) \to z_1(0)p^* + z_2(0)\left(I_n\otimes R\left(\frac{\pi}{2}\right)\right)p^* + z_3(0)1_n\otimes[1\ \ 0]^{\mathrm{T}} + z_4(0)1_n\otimes[0\ \ 1]^{\mathrm{T}},\ k\to\infty, \tag{21}$$

which indicates $z_i(0) = \beta_i,\ i = 1,\ldots,4$.

Assume that the first two rows of Q^{-1} are q_1 and q_2, then $z_1(0) = q_1p(0)$, $z_2(0) = q_2p(0)$. Because q_1 and q_2 in the nonsingular matrix Q^{-1} are linearly independent, for almost all initial states $p(0)$, $z_1(0)$ and $z_2(0)$ are not zero at the same time, for which the desired angle rigid formation is almost globally achieved.

Remark 4. Eq. (21) reflects that the closed-loop multi-agent system converges to the desired angle rigid formation, where $z_1(0)p^* + z_2(0)(I_n\otimes R(\frac{\pi}{2}))p^*$ represents the desired configuration after scaling and rotation (in which the desired angle constraints are satisfied), and $z_3(0)1_n\otimes[1\ \ 0]^{\mathrm{T}} + z_4(0)1_n\otimes[0\ \ 1]^{\mathrm{T}}$ represents the translation of the configuration. The reason for ensuring that the constant coefficients $z_1(0)$ and $z_2(0)$ are not zero simultaneously is that if both are zero, the first two terms will vanish, leading to $p(k) \to z_3(0)1_n\otimes[1\ \ 0]^{\mathrm{T}} + z_4(0)1_n\otimes[0\ \ 1]^{\mathrm{T}}$. This means that all agents will eventually converge to the same position, whose drawback is that the triangular angle constraints between neighboring agents will no longer exist. In this case, the angle rigid formation will degenerate into a common point, corresponding to the 'almost' part of the stability conclusion. Therefore, it is necessary to analyze the case when $z_1(0)$ and $z_2(0)$ are not zero simultaneously.

According to the above proof, one has the following result on the convergence.

Proposition 1. Under the distributed control law (13) and the desired triangular angularity $\mathbb{A}^*(\mathcal{V},\mathcal{A},p^*)$, if the initial states $p(0)$ satisfy $q_1p(0) \neq 0$ and $q_2p(0) \neq 0$, then the multi-agent formation globally converges to the desired formation.

4 Translational formation maneuvering for discrete-time multi-agent systems

The multi-agent system (6) is divided into 2 leaders and $n-2$ followers, where followers can only obtain local measurements relative to their neighbors. Assuming that nodes $i = 1, 2$ represent leaders, the set of leaders is denoted as $\mathcal{V}_l = \{1, 2\}$, and the set of followers is denoted as $\mathcal{V}_f = \{3,\ldots,n\}$. The position $p(k)$ can be divided into $p(k) = [p_l(k)^{\mathrm{T}},\ p_f(k)^{\mathrm{T}}]^{\mathrm{T}}$, $p_l(k) = [p_1^{\mathrm{T}}(k),\ p_2^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^4$, $p_f(k) = [p_3^{\mathrm{T}}(k),\ldots,p_n^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^{2(n-2)}$. The leaders' movement dynamics are as follows:

$$p_l(k) = s(k)[I_2\otimes\phi(\theta(k))]p_l(0) + 1_2\otimes\mu(k), \tag{22}$$

where $s(k) \in \mathbb{R}$, $\phi(\theta(k)) \in SO(2)$ and $\mu(k) \in \mathbb{R}^2$ denote scaling, rotation and translation maneuver parameters respectively, $\theta(k) \in \mathbb{R}^2$. The dynamics of the followers still maintain the single integrator discrete multi-agent form as in (6)

$$p_i(k+1) = p_i(k) + T \cdot u_{fi}(k),\ i \in \mathcal{V}_f. \tag{23}$$

Consider the case where the leaders move at a constant velocity, and assume that the two leaders will not collide. Partition $R_{\mathcal{A}}(\alpha)$ into $R_{\mathcal{A}}(\alpha) = [R_{\mathcal{A}_l}(\alpha),\ R_{\mathcal{A}_f}(\alpha)]$ with $R_{\mathcal{A}_l}(\alpha) \in \mathbb{R}^{2m\times 4}$, $R_{\mathcal{A}_f}(\alpha) \in \mathbb{R}^{2m\times(2n-4)}$. Correspondingly, the matrix $L(\alpha) := R_{\mathcal{A}}^{\mathrm{T}}(\alpha) \cdot R_{\mathcal{A}}(\alpha) = \left[\begin{smallmatrix} L_{ll}(\alpha) & L_{lf}(\alpha) \\ L_{fl}(\alpha) & L_{ff}(\alpha) \end{smallmatrix}\right] \in \mathbb{R}^{2n\times 2n}$, where $L_{ll}(\alpha) = R_{\mathcal{A}_l}^{\mathrm{T}}(\alpha)R_{\mathcal{A}_l}(\alpha)$, $L_{lf}(\alpha) = R_{\mathcal{A}_l}^{\mathrm{T}}(\alpha)R_{\mathcal{A}_f}(\alpha)$, $L_{fl}(\alpha) = R_{\mathcal{A}_f}^{\mathrm{T}}(\alpha)R_{\mathcal{A}_l}(\alpha)$, $L_{ff}(\alpha) = R_{\mathcal{A}_f}^{\mathrm{T}}(\alpha)R_{\mathcal{A}_f}(\alpha)$. To facilitate the subsequent investigation, the following lemma is provided.

Lemma 2 ([19]). If $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is angle rigid, then $L_{fl}(\alpha^*)p_l^* + L_{ff}(\alpha^*)p_f^* = 0$ with $L_{ff}(\alpha^*)$ is positive definite.

According to Lemma 2, the desired position and velocity of the followers are derived as

$$\begin{cases} p_f^*(k) = -\, L_{ff}^{-1}(\alpha^*)L_{fl}(\alpha^*)p_l(k), \\ v_f^*(k) = -\, L_{ff}^{-1}(\alpha^*)L_{fl}(\alpha^*)v_l(k). \end{cases} \tag{24}$$

Define $\tilde{p}_f(k) := p_f(k) - p_f^*(k)$ as the position error. The angle rigid formation control goal (7) is transformed into designing a maneuvering formation controller for the system (23) such that

$$\lim_{k\to\infty} \tilde{p}_f(k) = 0,\ \ \lim_{k\to\infty} (u_f(k) - v_f^*(k)) = 0. \tag{25}$$

The proposed distributed formation maneuvering control law for followers (23) is

$$\begin{cases} \hat{v}_{fi}(k+1) = \hat{v}_{fi}(k) - \dfrac{1}{4d_i}\Bigg[\displaystyle\sum_{(i,j_1,m_1)\in\bar{\mathcal{A}}} \left(A_i^{\triangle ij_1m_1}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle ij_1m_1}(\alpha^*, \hat{v}(k)) \\ \qquad + \displaystyle\sum_{(m_2,i,j_2)\in\bar{\mathcal{A}}} \left(A_i^{\triangle m_2ij_2}(\alpha^*)\right)^{\mathrm{T}} \cdot f_i^{\triangle m_2ij_2}(\alpha^*, \hat{v}(k)) \\ \qquad + \displaystyle\sum_{(j_3,m_3,i)\in\bar{\mathcal{A}}} \left(A_i^{\triangle j_3m_3i}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle j_3m_3i}(\alpha^*, \hat{v}(k))\Bigg], \\ u_{fi}(k) = -\, k_i\Bigg[\displaystyle\sum_{(i,j_1,m_1)\in\bar{\mathcal{A}}} \left(A_i^{\triangle ij_1m_1}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle ij_1m_1}(\alpha^*, p(k)) \\ \qquad + \displaystyle\sum_{(m_2,i,j_2)\in\bar{\mathcal{A}}} \left(A_i^{\triangle m_2ij_2}(\alpha^*)\right)^{\mathrm{T}} \cdot f_i^{\triangle m_2ij_2}(\alpha^*, p(k)) \\ \qquad + \displaystyle\sum_{(j_3,m_3,i)\in\bar{\mathcal{A}}} \left(A_i^{\triangle j_3m_3i}(\alpha^*)\right)^{\mathrm{T}} f_i^{\triangle j_3m_3i}(\alpha^*, p(k))\Bigg] + \hat{v}_{fi}(k), \end{cases} \tag{26}$$

where $\hat{v}(k) = [v_l^{\mathrm{T}}(k)\ \hat{v}_f^{\mathrm{T}}(k)]^{\mathrm{T}}$, $\hat{v}_f(k) = [\hat{v}_{f3}^{\mathrm{T}}(k), \ldots, \hat{v}_{fn}^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^{2(n-2)}$ is the stacked vector of the followers' estimated velocities.

The compact form of the above control law is

$$\begin{cases} \hat{v}_f(k+1) = \hat{v}_f(k) - \left(\dfrac{1}{4}\bar{M} \otimes I_2\right)\Big[L_{fl}(\alpha^*)v_l(k) + L_{ff}(\alpha^*)\hat{v}_f(k)\Big], \\ u_f(k) = -\left(\dfrac{1}{3T}\bar{M} \otimes I_2\right)\Big[L_{fl}(\alpha^*)p_l(k) + L_{ff}(\alpha^*)p_f(k)\Big] + \hat{v}_f(k), \end{cases} \tag{27}$$

where $\bar{M} = \mathrm{diag}\{d_3^{-1}, \ldots, d_n^{-1}\}$.

Theorem 3. If the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is angle rigid and the two leaders move at constant velocity, then under the distributed control algorithm (26), the angle rigid formation control goal (7) of the leader-following multi-agent systems is realized globally.

Proof. Substituting controller (26) into system (23) yields

$$p_f(k+1) = p_f(k) - \left(\frac{1}{3}\bar{M} \otimes I_2\right)\Big[L_{fl}(\alpha^*)p_l(k) + L_{ff}(\alpha^*)p_f(k)\Big] + T \cdot \hat{v}_f(k). \tag{28}$$

According to (24), we can get

$$p_f^*(k+1) = -L_{ff}^{-1}(\alpha^*)L_{fl}(\alpha^*)p_l(k+1).$$

Since the leaders move at a constant velocity, this makes $\frac{p_l(k+1)-p_l(k)}{T} = v_l(k) = v_l^*(k)$, so

$$p_f^*(k+1) = -L_{ff}^{-1}(\alpha^*)L_{fl}(\alpha^*)\Big[p_l(k) + T \cdot v_l(k)\Big] = p_f^*(k) + T \cdot v_f^*(k). \tag{29}$$

Subtracting (29) from (28) yields

$$\tilde{p}_f(k+1) = \left[I_{2n-4} - \left(\frac{1}{3}\bar{M} \otimes I_2\right)L_{ff}(\alpha^*)\right]\tilde{p}_f(k) + T \cdot (\hat{v}_f(k) - v_f^*(k)). \tag{30}$$

In addition, $v_f^*(k+1) = -L_{ff}^{-1}(\alpha^*)L_{fl}(\alpha^*)v_l(k+1) = -L_{ff}^{-1}(\alpha^*)L_{fl}(\alpha^*)v_l(k) = v_f^*(k)$. Therefore,

$$\hat{v}_f(k+1) - v_f^*(k+1) = \left(I_{2n-4} - \left(\frac{1}{4}\bar{M} \otimes I_2\right)L_{ff}(\alpha^*)\right)[\hat{v}_f(k) - v_f^*(k)]. \tag{31}$$

Combining (30) and (31) yields

$$\begin{bmatrix} \tilde{p}_f(k+1) \\ \hat{v}_f(k+1) - v_f^*(k+1) \end{bmatrix} = \Phi \begin{bmatrix} \tilde{p}_f(k) \\ \hat{v}_f(k) - v_f^*(k) \end{bmatrix}, \tag{32}$$

where

$$\Phi = \begin{bmatrix} I_{2n-4} - (\frac{1}{3}\bar{M} \otimes I_2)L_{ff}(\alpha^*) & T \cdot I_{2n-4} \\ 0 & I_{2n-4} - (\frac{1}{4}\bar{M} \otimes I_2)L_{ff}(\alpha^*) \end{bmatrix}.$$

For $(\frac{1}{3}\bar{M} \otimes I_2)L_{ff}(\alpha^*)$, the matrices M and $L_{ff}(\alpha^*)$ are both positive definite, so all eigenvalues λ_p of $(\frac{1}{3}\bar{M} \otimes I_2)L_{ff}(\alpha^*)$ are positive according to [33, Theorem 10.31], i.e., $\lambda_p > 0$.

According to the Gershgorin circle theorem and the structural characteristics of the matrix $L_{ff}(\alpha^*)$, all eigenvalues λ_p of $(\frac{1}{3}\bar{M} \otimes I_2)L_{ff}(\alpha^*)$ are located in the union of the following $2(n-2)$ disks:

$$\left|\lambda_p - \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_e j_e m_e}}{3d_i}\right| \leqslant \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_e j_e m_e}| + |\beta_3^{\triangle i_e j_e m_e}|)}{3d_i},\ i \in \mathcal{V}_f. \tag{33}$$

According to Lemma 1, we know $0 < \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_e j_e m_e}}{3d_i} \leqslant \frac{d_i}{3d_i} = \frac{1}{3}$ and $0 < \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_e j_e m_e}|+|\beta_3^{\triangle i_e j_e m_e}|)}{3d_i} < \frac{2\sqrt{2}d_i}{3d_i} = \frac{2\sqrt{2}}{3}$, so we have $0 < \lambda_p < \frac{1+2\sqrt{2}}{3} < 2$.

Similarly, all eigenvalues $\bar{\lambda}_p$ of $(\frac{1}{4}\bar{M} \otimes I_2)L_{ff}(\alpha^*)$ are positive and lie in the union of the following $2(n-2)$ disks:

$$\left|\bar{\lambda}_p - \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_e j_e m_e}}{4d_i}\right| \leqslant \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_e j_e m_e}| + |\beta_3^{\triangle i_e j_e m_e}|)}{4d_i},\ i \in \mathcal{V}_f. \tag{34}$$

According to Lemma 1, we know $0 < \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_e j_e m_e}}{4d_i} < \frac{d_i}{4d_i} = \frac{1}{4}$ and $0 < \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_e j_e m_e}|+|\beta_3^{\triangle i_e j_e m_e}|)}{4d_i} < \frac{2\sqrt{2}d_i}{4d_i} = \frac{\sqrt{2}}{2}$, so we have $0 < \bar{\lambda}_p < \frac{1}{4} + \frac{\sqrt{2}}{2} < 2$.

Therefore, the eigenvalue $\tilde{\lambda}$ of the closed-loop system (32) satisfies: $\tilde{\lambda} \in (-1, 1)$, which means $\tilde{p}_f(k) \to 0,\ k \to \infty$, and $\hat{v}_f(k) - v_f^*(k) \to 0,\ k \to \infty$, that is, $p_f(k) \to p_f^*(k),\ \hat{v}_f(k) \to v_f^*(k),\ k \to \infty$. Therefore, $u_f(k) \to v_f^*(k),\ k \to \infty$, it is obvious that the desired angle rigid formation is realized globally.

Remark 5. In reality, due to the existence of obstacles, formation maneuvering with the capability of translation, rotation, and scaling is favorable. However, the research on the maneuvering control of discrete multi-agent systems based on angle rigidity theory is still insufficient. To this end, a discrete and distributed maneuvering formation control algorithm (26) is designed in this section, which ensures that the formation moves as a whole at the desired velocity and the configuration of the followers satisfies the desired angle constraint (2). It is worth mentioning that rotational and scaling maneuvering is usually not continuously needed, but needed only at some piecewise time period, for which the above-designed algorithm can be used to achieve formation maneuvering with desired translational velocity, orientation, and scale [19].

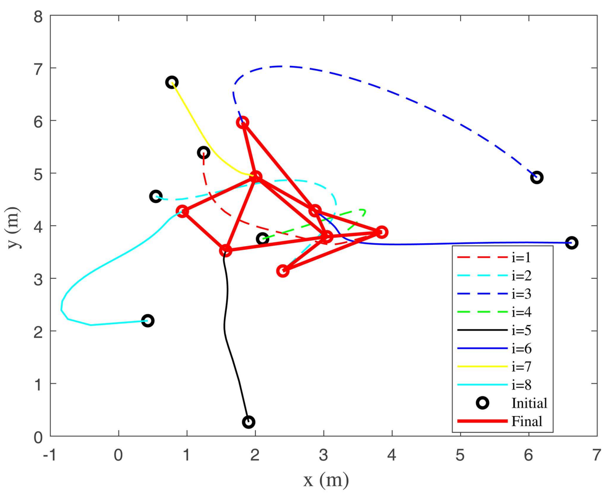

Figure 1 (Color online) The formation trajectories of agents.

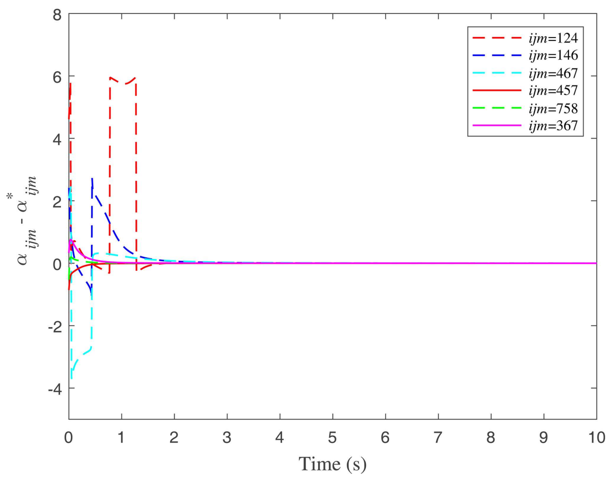

Figure 2 (Color online) The angle errors of the formation.

5 Illustrative examples

In this section, we provide two examples to verify the effectiveness of the proposed distributed formation control algorithm and distributed maneuvering control algorithm.

Example 1. The first example aims to validate angle rigid formation control for discrete-time multi-agent systems, where Theorem 2 will be verified based on the distributed control law (13).

The considered multi-agent systems consist of 8 agents, and a set of agents' positions constituting the desired angle rigid formation is given as $p_1^* = [-5, 1.5]^{\mathrm{T}}$, $p_2^* = [-1.3, 4.2]^{\mathrm{T}}$, $p_3^* = [1.6, -3.4]^{\mathrm{T}}$, $p_4^* = [-2.8, 2.1]^{\mathrm{T}}$, $p_5^* = [1.2, 3.5]^{\mathrm{T}}$, $p_6^* = [-2.1, 0.8]^{\mathrm{T}}$, $p_7^* = [0.6, -0.6]^{\mathrm{T}}$, $p_8^* = [3.3, 1.7]^{\mathrm{T}}$. Set the sampling period $T = 0.01$ s. In addition, six desired formation triangles are formed, namely $\triangle 124$, $\triangle 164$, $\triangle 367$, $\triangle 457$, $\triangle 467$, $\triangle 578$. The initial positions $p_i(0)$ of the agents are randomly given to verify whether the multi-agent systems can globally achieve the desired angle rigid formation control goal.

Under the distributed formation control law (13) (whose compact expression in Example 1 is $u(k) = -(\frac{1}{3T}M \otimes I_2)L(\alpha^*)p(k)$ with $M = \mathrm{diag}\{\frac{1}{2}, 1, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\}$), the formation trajectories of the multi-agent system (6) are shown in Figure 1, and the angle errors in the formation convergence process are shown in Figure 2.

Figure 1 clearly reflects the desired triangular formation configuration shown by the red solid line when the trajectories converge to the desired formation. Figure 2 clearly shows that under the distributed formation control law (13), the angle errors converge to 0 within $t = 3$ s, which means that the trajectories of the agents globally converge to the desired angle rigid formation within $t = 3$ s.

Example 2. The second example aims to validate angle rigid maneuvering control for discrete-time multi-agent systems, where Theorem 3 will be verified based on the distributed maneuvering control law (26).

The considered multi-agent systems consist of 2 leaders and 6 followers, and a set of agents' positions constituting the desired angle rigid formation is given as $p_1^* = [-5, 1.5]^{\mathrm{T}}$, $p_2^* = [-1.3, 1.5]^{\mathrm{T}}$, $p_3^* = [1.6, -2]^{\mathrm{T}}$, $p_4^* = [-2.8, 2.3]^{\mathrm{T}}$, $p_5^* = [1.4, 3.5]^{\mathrm{T}}$, $p_6^* = [-2.1, 1]^{\mathrm{T}}$, $p_7^* = [0.6, -0.2]^{\mathrm{T}}$, $p_8^* = [3.2, 1.7]^{\mathrm{T}}$. Set the sampling period $T = 0.01$ s. Similar to Example 1, six desired formation triangles are formed, namely $\triangle 124$, $\triangle 164$, $\triangle 367$, $\triangle 457$, $\triangle 467$, $\triangle 578$. The agents' initial positions are $p_1(0) = [-10, -16]^{\mathrm{T}}$, $p_2(0) = [10, -16]^{\mathrm{T}}$, $p_3(0) = [14.4, -21.6]^{\mathrm{T}}$, $p_4(0) = [6.1, -21]^{\mathrm{T}}$, $p_5(0) = [-13, -33.3]^{\mathrm{T}}$, $p_6(0) = [1, -30.5]^{\mathrm{T}}$, $p_7(0) = [-7.2, -20.3]^{\mathrm{T}}$, $p_8(0) = [15, -34.8]^{\mathrm{T}}$. Suppose that two leaders can autonomously maneuver through a 2D environment with obstacles, and the initial estimated velocities of six followers are $\hat{v}_{f3}(0) = [0.1, -0.1]^{\mathrm{T}}$, $\hat{v}_{f4}(0) = [0.2, 0.1]^{\mathrm{T}}$, $\hat{v}_{f5}(0) = [-0.1, -0.1]^{\mathrm{T}}$, $\hat{v}_{f6}(0) = [0.4, -0.3]^{\mathrm{T}}$, $\hat{v}_{f7}(0) = [0.5, 0.4]^{\mathrm{T}}$, $\hat{v}_{f8}(0) = [0.3, -0.2]^{\mathrm{T}}$.

Under the distributed formation control law (26) (whose compact expression in Example 2 is $u_f(k) = -(\frac{1}{3T}\bar{M} \otimes I_2)[L_{fl}(\alpha^*)p_l(k) + L_{ff}(\alpha^*)p_f(k)] + \hat{v}_f(k)$ with $\hat{v}_f(k+1) = \hat{v}_f(k) - (\frac{1}{4}\bar{M} \otimes I_2)[L_{fl}(\alpha^*)v_l(k) + L_{ff}(\alpha^*)\hat{v}_f(k)]$, where $\bar{M} = \mathrm{diag}\{1, \frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\}$), the formation maneuvering trajectories of the leader-following multi-agent systems (22) and (23) are shown in Figure 3, and the angle errors in the formation convergence process are shown in Figure 4.

Figure 3 clearly reflects that in the process of achieving the desired formation configuration shown by the red solid line, the multi-agent formation undergoes maneuvering changes, including translation, rotation, and scaling. Figure 4 clearly shows that under the distributed maneuvering control law (26), the angle errors converge to 0 within

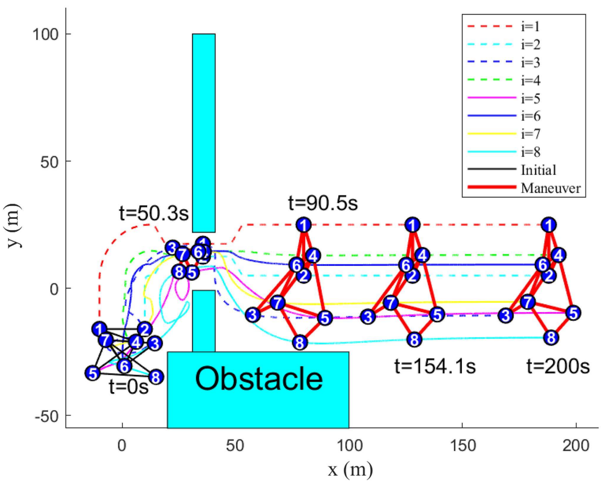

Figure 3 (Color online) The formation maneuvering trajectories of agents.

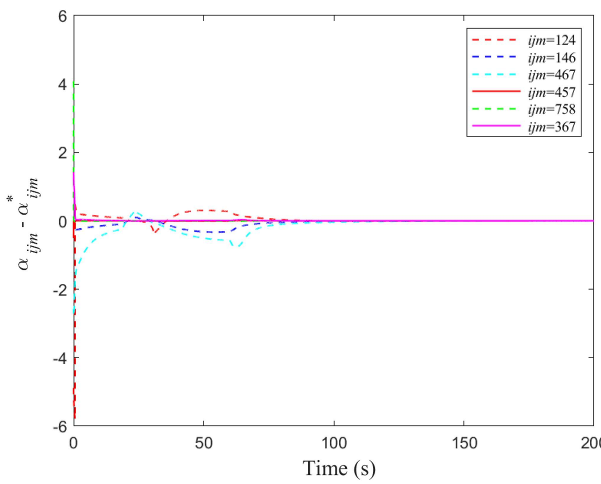

Figure 4 (Color online) The angle errors of the formation.

$t = 100$ s, which means that the trajectories of the agents globally converge to the desired angle rigid formation within $t = 100$ s.

6 Conclusion

In this paper, the angle rigid formation control problem for a class of discrete-time multi-agent systems with single-integrator dynamics has been investigated. To achieve the desired angle rigid formation, the global convergence condition of closed-loop systems has been analyzed based on a directly derived control law. To eliminate the dependence on the sampling period, the control algorithm has been refined, which enables fully distributed control of the desired formation. Furthermore, the algorithm has been extended to address the translational maneuvering problem for leader-following multi-agent systems. Future research will focus on the angle rigid formation control for discrete-time double-integrator multi-agent systems subject to time-varying disturbances.

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