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Witnessing nonlocality in quantum networks of continuous-variable systems by generalized quasiprobability functions

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Abstract Gaussian measurements cannot be used to witness nonlocality in Gaussian states as well as the network nonlocality in networks of continuous-variable (CV) systems. Thus special non-Gaussian measurements have to be utilized. In the present paper, we first consider the scenario of two-input-arbitrary-output (including infinite-output), and propose a kind of nonlinear Bell-type inequality that is applicable to quantum networks of both finite and infinite dimensional systems. Violation of the inequality will witness the network nonlocality. This inequality allows us to propose a method of the supremum strategy for detecting network nonlocality in CV systems with source states being any multipartite multi-mode Gaussian states according to the configurations of the networks by utilizing non-Gaussian measurements based on generalized quasiprobability functions. The nonlinear Bell-type inequalities for CV networks, which depend solely on the generalized quasiprobability functions of Gaussian states, are straightforward to construct. As illustrations, we propose the corresponding nonlinear Bell-type inequalities for any chain, star, tree-shaped and cyclic networks in CV systems with source states being $(1+1)$-mode Gaussian states. The examples show that this method works well. Particularly, a thorough discussion is given for the entanglement swapping network. Our study provides a strong signature for the network nonlocality nature of CV systems and leads to precise recipes for its experimental verification.

Keywords continuous-variable system, network nonlocality, Bell-type inequality, Gaussian states, generalized quasiprobability function

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1 Introduction

Bell nonlocality has emerged as a topic of significant interest over the past few decades, with its implications recognized in various domains of quantum information processing [1–3], quantum communication [4–6], and quantum key distribution [7,8]. To detect whether a quantum state is nonlocal, Bell inequalities were proposed [9–11]. In [11], Bell derived inequalities that any theory based on local hidden variables must satisfy. Consequently, the violation of these Bell inequalities is indicative of Bell nonlocality.

Recently, a generalization of the concept of Bell nonlocality was proposed to address the phenomenon of nonlocality in quantum networks [12], which we call network nonlocality in the present paper, to distinguish it from the usual nonlocality. The nonlocality within various quantum network configurations has been extensively explored and studied, including chain networks [12, 13], star networks [14, 15], tree-shaped networks [16–18], and cyclic networks [19, 20], by establishing corresponding nonlinear Bell-type inequalities. Research has demonstrated that network nonlocality plays an indispensable role across multiple domains, including secure communication, distributed quantum computing, cryptography, quantum key distribution, random number generation, and quantum sensing [21–25]. Detecting nonlocality in quantum networks becomes a basic task in this field. The current approach is to establish nonlinear Bell-type inequalities; the violation of any one of these inequalities will witness the nonlocality in the involved quantum networks. However, the majority of studies on network nonlocality have focused on finite-dimensional systems, and most known nonlinear Bell-type inequalities are confined to the two-input-two-output scenario.

The continuous-variable (CV) systems are also fundamentally important from theoretical and experimental views. Specifically, Gaussian states have become important objects in the study of CV quantum information theory due

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to their ease of production and manipulation in laboratories, as well as their elegant mathematical structures. The Bell nonlocality of Gaussian states has been the subject of intensive study. It is noteworthy that Gaussian states cannot exhibit nonlocality through Gaussian measurements. This is because they inherently have a positive-definite Wigner function, which allows for the construction of a hidden-variable model [26]. Therefore, the set of Gaussian nonlocal states is defined as the intersection of the set of Gaussian states and the set of nonlocal states. Additionally, it is essential to employ special non-Gaussian measurements that are close to Gaussian measurements or can be implemented in the laboratory to detect the nonlocality inherent in Gaussian states. Banaszek and Wódkiewicz [27] first demonstrated the Bell nonlocality of the two-mode squeezed vacuum state using the Wigner functions. They highlighted that there is no direct correlation between the negativity of the Wigner function and Bell nonlocality. Following their initial work, several studies have expanded upon the concepts introduced in [27]. Some of these studies considered more phase space displacements, as referenced in [28], while others introduced new observables that allow for Bell inequality violations for Gaussian states, as seen in [29]. These advancements contribute to a deeper understanding of the conditions under which Gaussian states can be shown to exhibit nonlocal properties, emphasizing the role of non-Gaussian measurements in detecting such nonlocality.

The CV quantum networks have garnered increasing attention due to their high transmission efficiency, low channel loss, and ease of implementation [30–38]. Moreover, experimental advancements have been at the forefront of developing the hybrid quantum optical paradigm [39–41], with a significant interest from the scientific community in studying hybrid networks that combine CV and discrete variable (DV) systems [42–46]. Given the practical advantages of Gaussian quantum information processing [47,48], such as deterministic entanglement generation [49], it is both natural and crucial to delve into the study of quantum correlations within CV quantum networks with source states being Gaussian states. Currently, almost all known studies concerning networks in CV systems, such as on the topics of entanglement swapping [30,50,51], entanglement transmission [52], entanglement distillation [53], and hybrid quantum communication network [43], assume the source states to be $(1+1)$-mode Gaussian states. Recently, Chakrabarty et al. [54] detected the nonlocality of an entanglement swapping network in CV systems via pseudospin measurements. However, compared with DV systems, research on network nonlocality in CV systems remains relatively limited. One of the main challenges is the lack of nonlinear Bell-type inequalities applicable to infinite-dimensional networks with infinitely many measurement outcomes. Another challenge is that Gaussian measurements are not able to witness the nonlocality in Gaussian states, and thus one has to find and utilize suitable non-Gaussian measurements that can be realized in laboratory. The third issue is that, when these non-Gaussian measurements are performed, whether the measurement scheme is sufficient to detect the network nonlocality contained in the networks of CV systems.

Addressing these issues mentioned above, the purpose of the present paper is to propose an approach for detecting the nonlocality of networks in CV systems with source states being Gaussian states. To do this, we first establish a new nonlinear Bell-type inequality for general networks of finite or infinite dimensional systems, in the scenarios where all parties have two inputs but the number of outputs is not limited (which may be infinite). Secondly, we utilize this inequality and the non-Gaussian measurement scheme based on the generalized quasiprobability functions, which is physically feasible under ideal conditions, to develop a general method of the supremum strategy for detecting network nonlocality in CV quantum networks. To address the third issue, several classes of networks are discussed to illustrate that the measurements based on generalized quasiprobability functions are effective in detecting the network nonlocality of CV systems. Our work establishes a fundamental connection between Bell nonlocality and quantum networks in CV systems. This scheme not only deepens the understanding of quantum nonlocality, but also highlights its potential application prospects in various quantum communication protocols.

This paper is organized as follows. In Section 2, we establish a nonlinear Bell-type inequality for a general network, which is applicable to scenarios where all parties have two inputs and arbitrary outputs. In Section 3, we review some notions and notations concerning Gaussian states and generalized quasiprobability functions. In addition, by applying the inequality established in Section 2, we propose a general approach, that is, the supremum strategy, to detect network nonlocality in CV systems with any Gaussian states as source states. As illustrations of how to apply the supremum strategy, Section 4 derives a Bell-type inequality for arbitrary chain networks with each source state being a $(1+1)$-mode Gaussian state. Particularly, the entanglement swapping network is discussed in more detail. The corresponding nonlinear inequalities are established, respectively, for the star networks, the determinate f-forked tree-shaped networks, and the cyclic networks in Sections 5–7. A short conclusion and discussion are provided in Section 8.

2 A nonlinear Bell-type inequality for quantum networks of any dimensional systems

There are no known nonlinear Bell-type inequalities for quantum networks that are valid for networks of CV systems with infinitely many measurement outcomes. So, to discuss the network nonlocality of CV systems, we need to develop nonlinear Bell-type inequalities applicable to CV systems. This is the purpose of the present section.

Consider a finite-size network $\Xi(y,z)$ consisting of y parties (nodes) $A_1, A_2, \cdots, A_y$, and z independent sources $S_1, S_2, \cdots, S_z$ that distribute hidden variables $\lambda_1, \lambda_2, \cdots, \lambda_z$, respectively. Here, for any $j = 1, 2, \cdots, z$, S_j usually emits a multipartite quantum source state ρ_j, with the number of parties determined by the number of recipient nodes. Note that each ρ_j is at least a bipartite state. The number $w = \max\{\text{the number of parties involved } \rho_j : j = 1, 2, \cdots, z\}$ is called the depth of the network. Each party A_i $(i = 1, 2, \cdots, y)$ receives hidden variables $\Lambda_i = \{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_{e_i}}\}$ from the corresponding sources $\mathbf{S}_i = \{S_{i_1}, S_{i_2}, \cdots, S_{i_{e_i}}\}$, where $\mathbf{S}_1, \mathbf{S}_2, \cdots, \mathbf{S}_y$ satisfy $\bigcup_{i=1}^{y} \mathbf{S}_j = \{S_1, S_2, \cdots, S_z\}$. The network locality of $\Xi(y,z)$ suggests a joint conditional probability distribution of the measurement outcomes as

$$P(\mathbf{a}|\mathbf{x}) = \int \cdots \int d\lambda_1 \cdots d\lambda_z \prod_{j=1}^{z} \mu_j(\lambda_j) \prod_{i=1}^{y} P(a_i|x_i, \Lambda_i), \tag{1}$$

where $\mathbf{a} = (a_1, a_2, \cdots, a_y)$ denotes the parties' outputs, $\mathbf{x} = (x_1, x_2, \cdots, x_y)$ denotes their inputs. Additionally, $\mu_j(\lambda_j)$ is the probability distribution of λ_j with the normalization condition $\int d\mu_j(\lambda_j) = 1$ [55–57]. Otherwise, $\Xi(y,z)$ is network nonlocal. It is clear that $\Xi(y,z)$ is network local if all its source states are separable.

For a finite-size quantum network $\Xi(y,z)$ in systems of any dimensions, recall that, a subset $\{B_1, B_2, \cdots, B_k\} \subset \{A_i\}_{i=1}^{y}$ of k elements is an independent subset if any two elements of $\{B_1, B_2, \cdots, B_k\}$ do not share source of the network. In the case that such an independent subset of k parties exists, we say that the network $\Xi(y,z)$ has k independent parties [58]. It is clear that, if $\Xi(y,z)$ has k independent parties, then it has l independent parties for any positive integer $l \leqslant k$.

The following is our main result in this section which establishes a nonlinear Bell-type inequality for general quantum networks of any depth.

Theorem 1. Let $\Xi(y,z)$ be a finite-size network with k independent parties. Suppose that the inputs of all parties are dichotomic, while no restriction is placed on the number of possible measurement outcomes (which may be infinite when the system is infinite dimensional). If $\Xi(y,z)$ is network local, then for any local measurements $\{M_{x_i} : x_i \in \{0,1\}\}_{i=1}^{y}$ with countable outcomes, the following inequality holds:

$$\mathcal{B} = |\mathcal{I}|^{\frac{1}{k}} + |\mathcal{J}|^{\frac{1}{k}} \leqslant 1, \tag{2}$$

where

$$\mathcal{I} = \frac{1}{2^k} \left\langle \prod_{i_s \in \mathcal{K}} (M_{x_{i_s}=0} + M_{x_{i_s}=1}) \prod_{j \in \bar{\mathcal{K}}} M_{x_j=0} \right\rangle, \tag{3}$$

$$\mathcal{J} = \frac{1}{2^k} \left\langle \prod_{i_s \in \mathcal{K}} (M_{x_{i_s}=0} - M_{x_{i_s}=1}) \prod_{j \in \bar{\mathcal{K}}} M_{x_j=1} \right\rangle, \tag{4}$$

$\mathcal{K} = \{i_1, i_2, \cdots, i_k\}$ is any fixed index set so that A_{i_1}, A_{i_2}, $\cdots$, A_{i_k} are k independent parties and $\bar{\mathcal{K}} = \{1, 2, \cdots, y\} \backslash \mathcal{K}$.

In addition, for any $i = 1, 2, \cdots, y$, with $\Upsilon_{x_i} = \{a_h : a_h \text{ is the eigenvalue of } M_{x_i}\}$, we have

$$\langle M_{x_1} M_{x_2} \cdots M_{x_y} \rangle = \sum_{a_1 \in \Upsilon_{x_1}} \cdots \sum_{a_y \in \Upsilon_{x_y}} a_1 \cdots a_y \times P(a_1, \cdots, a_y | x_1, \cdots, x_y),$$

where $P(a_1, \cdots, a_y | x_1, \cdots, x_y)$ is the same as defined in (1).

A proof of Theorem 1 is presented in Appendix A.

From Theorem 1, violation of the inequality (2) demonstrates the network nonlocality of $\Xi(y,z)$. Thus $\mathcal{I}$ and $\mathcal{J}$ are important quantities for characterizing the network nonlocality. For different network configurations, $\mathcal{I}$ and $\mathcal{J}$ have different expressions. In addition, k is another important quantity representing the network nonlocality. Roughly speaking, the larger k is, the more network nonlocality is involved by the inequality (2). Therefore, it is optimal to find the largest such k, denoted by $k_{\max}$, and the corresponding index set $\mathcal{K}_{\max}$ for the maximal set of independent parties. However, for a general network, its index set $\mathcal{K}_{\max}$ for the maximal set of independent parties and consequently, its maximal independent number $k_{\max} = \sharp(\mathcal{K}_{\max})$ (the cardinal of $\mathcal{K}_{\max}$) are difficult to be found out. Nevertheless, analytical methods to determine $k_{\max}$ exist for the chain, tree-shaped, star, cyclic networks, and even some more complex networks [13, 14].

Theorem 1 is valid for finite-size quantum networks of any depth in both DV systems and CV systems. However, in the present paper, we mainly consider its applications to CV quantum networks.

3 A general approach to detect network nonlocality in CV systems by generalized quasiprobability functions

In this section, we first briefly review some notions and notations concerning Gaussian states and the generalized quasiprobability functions. Subsequently, we propose a general approach to detect network nonlocality in CV systems, applicable to any Gaussian networks. Here, a quantum network in a CV system is called a Gaussian network if its sources emit Gaussian states.

3.1 Gaussian states

Every state ρ in an n-mode CV system with state space $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ has a characteristic function $\chi_\rho(\boldsymbol{\alpha})$ defined as

$$\chi_\rho(\boldsymbol{\alpha}) = \mathrm{Tr}(\rho V(\boldsymbol{\alpha})),$$

where $\boldsymbol{\alpha} = (x_1, y_1, \cdots, x_n, y_n)^{\mathrm{T}} \in \mathbb{R}^{2n}$, $V(\boldsymbol{\alpha}) = \exp(\mathrm{i}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{\alpha})$ is the Weyl displacement operator, $\boldsymbol{R} = (\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{2n}) = (\hat{Q}_1, \hat{P}_1, \cdots, \hat{Q}_n, \hat{P}_n)$. As usual, $\hat{Q}_g = (\hat{a}_g + \hat{a}_g^\dagger)/\sqrt{2}$ and $\hat{P}_g = -\mathrm{i}(\hat{a}_g - \hat{a}_g^\dagger)/\sqrt{2}$ $(g = 1, 2, \cdots, n)$ are, respectively, the position and the momentum operators of the gth mode, where $\hat{a}_g^\dagger$ and $\hat{a}_g$ are respectively the creation and the annihilation operators of the gth mode satisfying the canonical commutation relation:

$$[\hat{a}_g, \hat{a}_l^\dagger] = \delta_{gl} I, [\hat{a}_g^\dagger, \hat{a}_l^\dagger] = [\hat{a}_g, \hat{a}_l] = 0, \ \forall\, g, l = 1, 2, \cdots, n.$$

Particularly, ρ is called a Gaussian state if its characteristic function $\chi_\rho(\boldsymbol{\alpha})$ has the form as

$$\chi_\rho(\boldsymbol{\alpha}) = \exp\left[-\frac{1}{4}\boldsymbol{\alpha}^{\mathrm{T}}\Gamma\boldsymbol{\alpha} + \mathrm{i}\mathbf{d}^{\mathrm{T}}\boldsymbol{\alpha}\right], \tag{5}$$

where

$$\mathbf{d} = (\langle\hat{R}_1\rangle, \langle\hat{R}_2\rangle, \cdots, \langle\hat{R}_{2n}\rangle)^{\mathrm{T}} = (\mathrm{Tr}(\rho\hat{R}_1), \mathrm{Tr}(\rho\hat{R}_2), \cdots, \mathrm{Tr}(\rho\hat{R}_{2n}))^{\mathrm{T}} \in \mathbb{R}^{2n}$$

is called the mean vector of ρ and $\Gamma = (\gamma_{kl}) \in \mathcal{M}_{2n}(\mathbb{R})$, the algebra of all $2n \times 2n$ matrices over the real field $\mathbb{R}$, is the covariance matrix of ρ defined by

$$\gamma_{kl} = \mathrm{Tr}[\rho(\Delta\hat{R}_k\Delta\hat{R}_l + \Delta\hat{R}_l\Delta\hat{R}_k)],$$

in which, $\Delta\hat{R}_k = \hat{R}_k - \langle\hat{R}_k\rangle$, $\langle\hat{R}_k\rangle = \mathrm{Tr}[\rho\hat{R}_k]$ [49, 59]. Note that a matrix Γ is a covariance matrix for some Gaussian state if and only if Γ is real symmetric and satisfies the condition $\Gamma + i\Omega_n \geqslant 0$, where $\Omega_n = \underbrace{\Omega \oplus \Omega \cdots \oplus \Omega}_{n} \in \mathcal{M}_{2n}(\mathbb{R})$ with $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By (5), every Gaussian state ρ is uniquely determined by its covariance matrix Γ and mean vector $\mathbf{d}$, and thus, one can write $\rho = \rho(\Gamma, \mathbf{d})$.

Besides the characteristic function, quantum states can also be represented by the Wigner function [49]. The Wigner function of an n-mode state ρ is defined as the symplectic Fourier transform of its characteristic function, that is,

$$W_\rho(\boldsymbol{\alpha}) = \frac{1}{(2\pi)^{2n}} \int \mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\eta}} \chi_\rho(\boldsymbol{\eta}) d^{2n}\boldsymbol{\eta},$$

where $\boldsymbol{\alpha} = (x_1, y_1, \cdots, x_n, y_n)^{\mathrm{T}} \in \mathbb{R}^{2n}$.

3.2 The generalized quasiprobability function

For a single-mode quantum state ρ in a CV system, its generalized quasiprobability function is

$$Q_\rho^{(1)}(\alpha; s) = \frac{2}{\pi(1-s)}\mathrm{Tr}[\rho\hat{\Pi}(\alpha; s)] = \frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}\left(\frac{s+1}{s-1}\right)^n \langle\alpha, n|\rho|\alpha, n\rangle, \tag{6}$$

where $\hat{\Pi}(\alpha; s) = \sum_{n=0}^{\infty}(\frac{s+1}{s-1})^n|\alpha, n\rangle\langle\alpha, n|$ with $|\alpha, n\rangle = D(\alpha)|n\rangle$. Here, as usual, $\{|n\rangle\}_{n=0}^{\infty}$ is the Fock basis and $D(\alpha) = \exp(\alpha\hat{a}^\dagger + \alpha^*\hat{a})$ for each $\alpha \in \mathbb{C}$ [60, 61]. Since the eigenvalues of $\hat{\Pi}(\alpha; s)$ are unbounded when $s > 0$, we always assume $s \leqslant 0$ in this paper unless otherwise specified. It should be noted that some literature refers to the generalized quasiprobability function $Q_\rho^{(1)}(\alpha; s)$ as the s-parameterized quasiprobability function.

Generally, for any n-mode Gaussian state ρ, its generalized quasiprobability function is defined as

$$Q_\rho^{(n)}(\boldsymbol{\alpha}; s) = \frac{2^n}{\pi^n(1-s)^n}\mathrm{Tr}[\rho(\otimes_{i=1}^n\hat{\Pi}(\alpha_i; s))],$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{C}^n$.

Then, for any bipartite $(n_1 + n_2)$-mode Gaussian state ρ, its generalized quasiprobability function is

$$Q_\rho^{(n_1,n_2)}(\boldsymbol{\alpha}, \boldsymbol{\beta}; s) = \frac{2^{n_1+n_2}}{\pi^{n_1+n_2}(1-s)^{n_1+n_2}}\mathrm{Tr}[\rho(\otimes_{i=1}^{n_1}\hat{\Pi}(\alpha_i; s) \otimes_{j=1}^{n_2} \hat{\Pi}(\beta_j; s))], \tag{7}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{n_1}) \in \mathbb{C}^{n_1}$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_{n_2}) \in \mathbb{C}^{n_2}$. In addition, the marginal distribution of $Q_\rho^{(n_1,n_2)}(\boldsymbol{\alpha}, \boldsymbol{\beta}; s)$ is

$$Q_\rho^{(n_1,\square)}(\boldsymbol{\alpha}; s) = \int Q_\rho^{(n_1+n_2)}(\boldsymbol{\alpha}, \boldsymbol{\beta}; s)d\boldsymbol{\beta}, \tag{8}$$

and $Q_\rho^{(\square,n_2)}(\boldsymbol{\beta}; s)$ is defined similarly.

For any r-partite $(n_1 + n_2 + \cdots + n_r)$-mode Gaussian state ρ, its generalized quasiprobability function can be defined similarly.

Particularly, if $\rho(\Gamma_\rho, 0)$ is a (1+1)-mode Gaussian state with zero mean, its Wigner function can be represented as

$$W_\rho(\alpha, \beta) = \frac{4}{\pi^2\sqrt{\det\Gamma_\rho}}\exp\left\{-\frac{1}{2}X^{\mathrm{T}}\Gamma_\rho^{-1}X\right\}, \tag{9}$$

where X^{T} is the real row vector $[\alpha + \alpha^*, -\mathrm{i}(\alpha - \alpha^*), \beta + \beta^*, -\mathrm{i}(\beta - \beta^*)]$ and $\alpha, \beta \in \mathbb{C}$. Then the generalized quasiprobability function of ρ is

$$Q_\rho^{(1,1)}(\alpha, \beta; s) = \frac{1}{4s^2\pi^2}\int d^4Y W_\rho(Y)\exp\left\{-\frac{1}{2}(Y - X)^{\mathrm{T}}|s|^{-1}(Y - X)\right\}. \tag{10}$$

Using the properties of Gaussian convolution and replacing Γ_ρ by $\Gamma_\rho^s = \Gamma_\rho + |s|I_4$ in (9), one sees that the Wigner function of ρ becomes the generalized quasiprobability function $Q_\rho^{(1,1)}(\alpha, \beta; s)$, where I_4 is the 4×4 unit matrix [29, 62].

Below, we list the generalized quasiprobability functions as well as their marginal distributions of two common types of $(1+1)$-mode Gaussian states, namely the EPR state and the squeezed thermal state (STS), which are used repeatedly in this paper.

Example 1. The $(1 + 1)$-mode EPR state is a pure Gaussian state of the form $\rho^r = |\psi(r)\rangle\langle\psi(r)|$, where

$$|\psi(r)\rangle = \mathrm{sech}r\sum_{n=0}^{\infty}\tanh^n r|n, n\rangle, \tag{11}$$

and $r \geqslant 0$ is the squeezing parameter. For a non-positive real number s, the generalized quasiprobability function of the EPR state ρ^r is given by

$$Q_{\rho^r}^{(1,1)}(\alpha, \beta; s) = \frac{4}{\pi^2 R(s)}\exp\left\{-\frac{2}{R(s)}[S(s)(|\alpha|^2 + |\beta|^2) - \sinh 2r(\alpha\beta + \alpha^*\beta^*)]\right\}, \tag{12}$$

and its marginal single-mode distribution is

$$Q_{\rho^r}^{(1,\square)}(\alpha;s)=\frac{2}{\pi S(s)}\exp\left(-\frac{2|\alpha|^2}{S(s)}\right), \tag{13}$$

where $R(s)=s^2-2s\cosh(2r)+1$, $S(s)=\cosh(2r)-s$, and $\alpha,\beta\in\mathbb{C}$ [63]. Obviously, ρ^r is $(1+1)$-mode entangled if only and if $r>0$.

Example 2. The STS is a $(1+1)$-mode mixed Gaussian state defined as a special unitary transform of some $(1+1)$-mode thermal state, that is,

$$\rho(v_1,v_2,r)=S_{12}(r)(\rho_{T_1}\otimes\rho_{T_2})S_{12}^{\dagger}(r), \tag{14}$$

where $S_{12}(r)=\exp[r(\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}-\hat{a}_1\hat{a}_2)]$ is a two-mode squeeze operator with the squeeze factor $r>0$ and

$$\rho_{T_j}=\frac{1}{\bar{n}_j+1}\sum_{n=0}^{\infty}\left(\frac{\bar{n}_j}{\bar{n}_j+1}\right)^n|n\rangle\langle n|,\ j=1,2$$

is the density operator of the jth thermal field with $\bar{n}_j$ its average number of thermal photons. The covariance matrix for the $(1+1)$-mode STS ρ is

$$\Gamma_\rho=\begin{pmatrix} a & 0 & c & 0\\ 0 & a & 0 & d\\ c & 0 & b & 0\\ 0 & d & 0 & b\end{pmatrix},$$

where the matrix elements are given by $a=v_1\cosh^2(r)+v_2\sinh^2(r)$, $b=v_1\sinh^2(r)+v_2\cosh^2(r)$, $c=-d=\frac{1}{2}(v_1+v_2)\sinh(2r)$ and $v_j=2\bar{n}_j+1$ $(j=1,2)$. Thus a $(1+1)$-mode STS $\rho(v_1,v_2,r)$ is uniquely determined by parameters v_1,v_2,r. Recall that, $\rho(v_1,v_2,r)$ is separable if and only if $\cosh^2(r)\leqslant\frac{(v_1+1)(v_2+1)}{2(v_1+v_2)}$ [64]. In the case $v_1=v_2=v$, we refer to $\rho(v,v,r)$ as the $(1+1)$-mode symmetric STS.

For a non-positive real number s, the generalized quasiprobability function of the $(1+1)$-mode STS $\rho(v_1,v_2,r)$ is given by

$$Q_{\rho(v_1,v_2,r)}^{(1,1)}(\alpha,\beta;s)=\frac{4}{\pi^2A}\exp\Big\{-\frac{2}{A}[A_1|\alpha|^2+A_2|\beta|^2-\frac{(v_1+v_2)\sinh(2r)}{2}(\alpha\beta+\alpha^*\beta^*)]\Big\}, \tag{15}$$

and its marginal single-mode distributions are

$$Q_{\rho(v_1,v_2,r)}^{(1,\square)}(\alpha;s)=\frac{2}{\pi A_2}\exp\left\{-\frac{2}{A_2}|\alpha|^2\right\}, \tag{16}$$

and

$$Q_{\rho(v_1,v_2,r)}^{(\square,1)}(\beta;s)=\frac{2}{\pi A_1}\exp\left\{-\frac{2}{A_1}|\beta|^2\right\}, \tag{17}$$

where $A=-2\cosh^2(r)s(v_1+v_2)+(v_1+s)(v_2+s)$, $A_1=(v_1+v_2)\cosh^2(r)-v_1-s$, $A_2=(v_1+v_2)\cosh^2(r)-v_2-s$, and $\alpha,\beta\in\mathbb{C}$.

In the sequel, without causing confusion, let $Q_\rho^{(n_1)}(\boldsymbol{\alpha};s)=Q_\rho^{(n_1,\square)}(\boldsymbol{\alpha},\boldsymbol{\beta};s)$ and $Q_\rho^{(n_2)}(\boldsymbol{\beta};s)=Q_\rho^{(\square,n_2)}(\boldsymbol{\alpha},\boldsymbol{\beta};s)$ with $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_{n_1})\in\mathbb{C}^{n_1}$ and $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_{n_2})\in\mathbb{C}^{n_2}$.

3.3 A general approach to detect network nonlocality in CV systems

Though our approach is valid for any Gaussian networks with arbitrary depth, for the sake of simplicity, we mainly present our approach for the most common class of finite-size Gaussian networks $\Xi(y,z)$, that is, those networks with depth 2, or equivalently, those networks of which all sources emit bipartite multi-mode Gaussian states. Also, we assume that all measurement operators are of the form

$$M_{x_i}(\boldsymbol{\alpha}_{x_i}^{\lambda_{i_1}},\boldsymbol{\alpha}_{x_i}^{\lambda_{i_2}},\cdots,\boldsymbol{\alpha}_{x_i}^{\lambda_{i_{e_i}}};s)=\hat{O}^{(n_{i_1})}(\boldsymbol{\alpha}_{x_i}^{\lambda_{i_1}};s)\otimes\hat{O}^{(n_{i_2})}(\boldsymbol{\alpha}_{x_i}^{\lambda_{i_2}};s)\otimes\cdots\otimes\hat{O}^{(n_{i_{e_i}})}(\boldsymbol{\alpha}_{x_i}^{\lambda_{i_{e_i}}};s),\quad x_i\in\{0,1\}, \tag{18}$$

$i = 1, 2, \cdots, y$, where $\Lambda_i = \{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_{e_i}}\}$ denotes the set of local variables associated with the sources $\mathbf{S}_i = \{S_{i_1}, S_{i_2}, \cdots, S_{i_{e_i}}\}$ that connect to party A_i. For each $j = 1, 2, \cdots, e_i$, $\boldsymbol{\alpha}_{x_i}^{\lambda_{i_j}} = (\alpha_{x_i,1}^{\lambda_{i_j}}, \alpha_{x_i,2}^{\lambda_{i_j}}, \cdots, \alpha_{x_i,n_{i_j}}^{\lambda_{i_j}}) \in \mathbb{C}^{n_{i_j}}$. Thus each source S_{i_j} emits an n_{i_j}-mode Gaussian state to A_i, and the party A_i shares an $n_i(= \sum_{j=1}^{e_i} n_{i_j})$-mode e_i-partite Gaussian state. In addition,

$$\hat{O}^{(n_{i_j})}(\boldsymbol{\alpha}_{x_i}^{\lambda_{i_j}}; s) = \begin{cases} (1-s) \otimes_{t=1}^{n_{i_j}} \hat{\Pi}(\alpha_{x_i,t}^{\lambda_{i_j}}; s) + sI, & \text{if } -1 < s \leqslant 0, \\ 2 \otimes_{t=1}^{n_{i_j}} \hat{\Pi}(\alpha_{x_i,t}^{\lambda_{i_j}}; s) - I, & \text{if } s \leqslant -1 \end{cases} \tag{19}$$

is a Hermitian operator parameterized by a non-positive real number s and some arbitrary complex variables $\alpha_{x_i,1}^{\lambda_{i_j}}, \alpha_{x_i,2}^{\lambda_{i_j}}, \cdots, \alpha_{x_i,n_{i_j}}^{\lambda_{i_j}}$ [62]. As usual, I stands for the identity operator. The possible measurement outcomes of $\hat{O}^{(n_{i_j})}(\boldsymbol{\alpha}_{x_i}^{\lambda_{i_j}}; s)$ are given by its eigenvalues

$$\xi_m = \begin{cases} (1-s)(\frac{s+1}{s-1})^m + s, & \text{if } -1 < s \leqslant 0, \\ 2(\frac{s+1}{s-1})^m - 1, & \text{if } s \leqslant -1, \end{cases}$$

where $m = 0, 1, 2, \cdots$. Clearly, for any non-positive s, $|\xi_m| \leqslant 1$.

Under ideal conditions, our measurement scheme is straightforward to implement [63, 65]. As illustrated in the entanglement swapping network of Figure 1(a), two independent sources emit $(1+1)$-mode entangled states $\rho^{A_1A_2}$ and $\rho^{A_2A_3}$. For each entangled pair, the corresponding parties perform local measurements using unbalanced homodyne detection, as depicted in Figure 1(b). In this setup, each signal mode is first mixed with a coherent state at a beam splitter (BS) with transmissivity T, and the output is measured using photon-number-resolving detector (P_N). In particular, when $s = -1$, the measurement only requires an on-off detector (P_{OF}) to distinguish between the presence and absence of photons.

For Gaussian state $\rho^{A_1A_2}$, coherent fields $|\tau_1\rangle$ and $|\xi_1\rangle$ enter through the other input ports of each BS. For high transmissivity $T \to 1$ and strong coherent fields $\tau_1, \xi_1 \to \infty$, the BSs of A_1 and $A_2^{(1)}$ can be described by the displacement operators $D(\alpha_{x_1}^{\lambda_1})$ and $D(\alpha_{x_2}^{\lambda_1})$, respectively, where $\alpha_{x_1}^{\lambda_1} = \tau_1\sqrt{\frac{1-T}{T}}$ and $\alpha_{x_2}^{\lambda_1} = \xi_1\sqrt{\frac{1-T}{T}}$ [65]. For a fixed parameter s, perfect photon number detectors $P_N^{A_1}(P_{OF}^{A_1})$ and $P_N^{A_{2(1)}}(P_{OF}^{A_{2(1)}})$ are used for measurement. Each measurement yields a pair of results $(n_1, n_{2(1)})$. By performing multiple measurements and collecting statistics, the joint probability distribution $P(n_1, n_{2(1)})$ is obtained, leading to the following expression:

$$\begin{aligned}
&Q_{\rho^{A_1A_2}}^{(1,1)}(\alpha_{x_1}^{\lambda_1}, \alpha_{x_2}^{\lambda_1}; s) \\
&= \frac{4}{\pi^2(1-s)^2}\mathrm{Tr}[\rho(\hat{\Pi}(\alpha_{x_1}^{\lambda_1}; s) \otimes \hat{\Pi}(\alpha_{x_2}^{\lambda_1}; s))] \\
&= \frac{4}{\pi^2(1-s)^2}\sum_{n_1, n_{2(1)}}\left(\frac{s+1}{s-1}\right)^{n_1+n_{2(1)}}\langle n_1, n_{2(1)}|[D^\dagger(\alpha_{x_1}^{\lambda_1}) \otimes D^\dagger(\alpha_{x_2}^{\lambda_1})]\rho^{A_1A_2}[D(\alpha_{x_1}^{\lambda_1}) \otimes D(\alpha_{x_2}^{\lambda_1})]|n_1, n_{2(1)}\rangle \\
&= \frac{4}{\pi^2(1-s)^2}\sum_{n_1, n_{2(1)}}\left(\frac{s+1}{s-1}\right)^{n_1+n_{2(1)}}P(n_1, n_{2(1)}).
\end{aligned}$$

The same procedure can be applied to the Gaussian state $\rho^{A_2A_3}$ to obtain $Q_{\rho^{A_2A_3}}^{(1,1)}(\alpha_{x_2}^{\lambda_2}, \alpha_{x_3}^{\lambda_2}; s)$.

Consider a finite-size CV network $\Xi(y, z)$ where each source S_j $(j = 1, 2, \cdots, z)$ emits arbitrary bipartite multi-mode Gaussian state ρ_j. Then the whole network state is a Gaussian state $\rho = \otimes_{j=1}^{z}\rho_j$. More clearly, let $\Xi(y, z; \rho)$ denote the network $\Xi(y, z)$ with network state ρ. Each party A_i performs two measurements of the form given in (18) for any $i = 1, 2, \cdots, y$. If $\Xi(y, z; \rho)$ has k independent parties, then, according to Theorem 1, $\Xi(y, z; \rho)$ is network local implies that the nonlinear Bell-type inequality

$$\mathcal{B}_\rho^\Xi(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s) = |\mathcal{I}|^{\frac{1}{k}} + |\mathcal{J}|^{\frac{1}{k}} \leqslant 1 \tag{20}$$

holds for any complex variables $\boldsymbol{\alpha}_t = (\boldsymbol{\alpha}_{x_1=t}^{\lambda_{1_1}}, \cdots, \boldsymbol{\alpha}_{x_1=t}^{\lambda_{1_{e_1}}}, \cdots, \boldsymbol{\alpha}_{x_y=t}^{\lambda_{y_1}}, \cdots, \boldsymbol{\alpha}_{x_y=t}^{\lambda_{y_{e_y}}})$ with $t \in \{0, 1\}$ and real $s \leqslant 0$. Furthermore, Eqs. (7) and (8) and Eqs.(18) and (19) allow us to transform the nonlinear inequality in (20) into an expression involving the generalized quasiprobability functions of Gaussian states.

Also note that, by (7) and (8), for any fixed $s \leqslant 0$, the generalized quasiprobability function $Q_\sigma^{(n)}(\boldsymbol{\alpha}, \boldsymbol{\beta}; s)$ and their marginal distribution $Q_\sigma^{(n_1)}(\boldsymbol{\alpha}; s)$ of any $(n_1 + n_2)$-mode Gaussian state σ are bounded functions, where

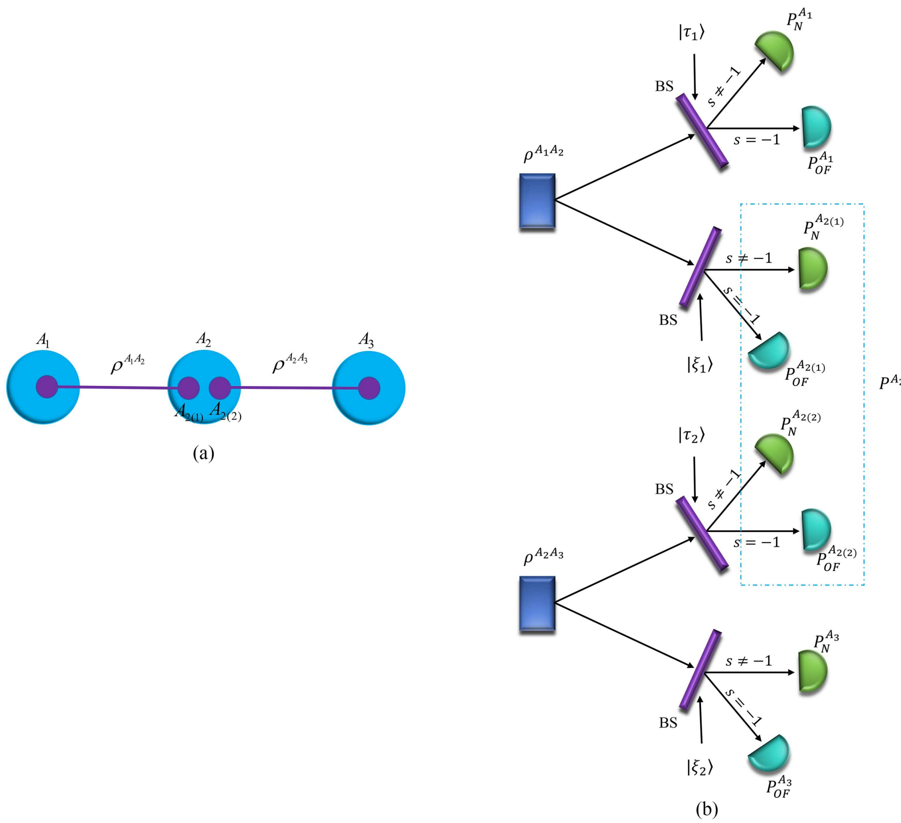

Figure 1 (Color online) The optical setup for the measurement strategy in entanglement swapping. (a) An entanglement swapping network with parties A_1, A_2, A_3 and independent $(1+1)$-mode source states $\rho^{A_1A_2}$, $\rho^{A_2A_3}$. (b) Optical measurement setup: each local measurement process involves mixing the input field with the coherent state at a high-transmittance beam splitter (BS), with detection subsequently carried out by a photon number detector (P_N). Notably, for $s=-1$, only an on-off detector (P_{OF}) is required to register the presence of photons.

$n = n_1 + n_2$, $\boldsymbol{\alpha} \in \mathbb{C}^{n_1}$ and $\boldsymbol{\beta} \in \mathbb{C}^{n_2}$. Then, for every fixed $s \leqslant 0$, $\mathcal{B}_\rho^\Xi(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s)$ in (20) is a bounded function of $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$. By Theorem 1, the network $\Xi(y, z; \rho)$ is network local which will imply

$$B^\Xi(s,\rho) = \sup_{\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1} \mathcal{B}_\rho^\Xi(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s) \leqslant 1 \tag{21}$$

holds for all $s \leqslant 0$. Therefore, we have proven the following.

Theorem 2. The Gaussian network $\Xi(y, z; \rho)$ is network nonlocal if $B^\Xi(s,\rho) > 1$ for some $s \leqslant 0$.

We refer to this general approach described above as the supremum strategy.

Note that $B^\Xi(s,\rho)$ can be calculated by (7) and (8). Although it is difficult to write out the analytic formula of $B^\Xi(s,\rho)$ in general, it may still be utilized as a tool to detect the network nonlocality in $\Xi(y, z; \rho)$ by numerical method.

In the following four sections, we apply Theorems 1 and 2 to the chain networks, the star networks, the tree-shaped networks, and the cyclic networks, where all source states are arbitrary bipartite multi-mode Gaussian states. Our results show that the measurements M_{x_i}s in (18) will work well for witnessing the network nonlocality of Gaussian networks. To avoid cumbersome presentation of the results, we in fact focus on $(1+1)$-mode Gaussian states as source states in Sections 4–7. Also, for convenience, we omit the superscript $(1+1)$ of the generalized quasiprobability function of the $(1+1)$-mode Gaussian state ρ and the superscript (1) of its marginal distribution, denoting both of them with Q_ρ. Similarly, we omit the superscript of $\hat{O}$ in (18). For any p, q, α_p^q always stands for a complex number. Moreover, we introduce some functions for the generalized quasiprobability functions of the $(1+1)$-mode Gaussian states to facilitate subsequent discussion in Sections 4–7.

Let ρ_{12} be a $(1+1)$-mode Gaussian state. We define

$$C_{\rho_{12}}^+(\alpha_1, \alpha_2, \alpha_1^{'}, \alpha_2^{'}; s) = Q_{\rho_{12}}(\alpha_1, \alpha_2; s) + Q_{\rho_{12}}(\alpha_1^{'}, \alpha_2^{'}; s), \tag{22}$$

$$C_{\rho_{12}}^-(\alpha_1, \alpha_2, \alpha_1^{'}, \alpha_2^{'}; s) = Q_{\rho_{12}}(\alpha_1, \alpha_2; s) - Q_{\rho_{12}}(\alpha_1^{'}, \alpha_2^{'}; s), \tag{23}$$

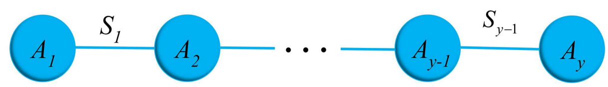

Figure 2 (Color online) A chain network $\mathcal{C}ha(y)$ consists of y parties A_1, A_2, $\cdots$, A_y and $y-1$ independent sources S_1, S_2, $\cdots$, S_{y-1}. Each source S_j emits $(1+1)$-mode Gaussian state $\rho^{A_jA_{j+1}}$.

$$D^{+}_{\rho_{12}}(\alpha_1,\alpha_2,\alpha_1^{'},\alpha_2^{'};s)=Q_{\rho_{12}}(\alpha_1;s)+Q_{\rho_{12}}(\alpha_2;s)+Q_{\rho}(\alpha_1^{'};s)+Q_{\rho}(\alpha_2^{'};s), \tag{24}$$

$$D^{-}_{\rho_{12}}(\alpha_1,\alpha_2,\alpha_1^{'},\alpha_2^{'};s)=Q_{\rho_{12}}(\alpha_1;s)+Q_{\rho_{12}}(\alpha_2;s)-Q_{\rho_{12}}(\alpha_1^{'};s)-Q_{\rho_{12}}(\alpha_2^{'};s), \tag{25}$$

$$E^{\pm}_{\rho_{12}}(\xi,\varpi,\varepsilon;\alpha_1,\alpha_2,\alpha_1^{'},\alpha_2^{'};s)=\xi C^{\pm}_{\rho_{12}}(\alpha_1,\alpha_2,\alpha_1^{'},\alpha_2^{'};s)+\varpi D^{\pm}_{\rho_{12}}(\alpha_1,\alpha_2,\alpha_1^{'},\alpha_2^{'};s)+\varepsilon, \tag{26}$$

where $Q_{\rho_{12}}(\alpha_1,\alpha_2;s)$ and $Q_{\rho_{12}}(\alpha_1^{'},\alpha_2^{'};s)$ are the generalized quasiprobability functions of the $(1+1)$-mode Gaussian state ρ_{12} defined in (7), $Q_{\rho_{12}}(\alpha_1;s)$ is the marginal function of $Q_{\rho_{12}}(\alpha_1,\alpha_2;s)$ defined in (8), that is,

$$Q_{\rho_{12}}(\alpha_1;s)=\int Q_{\rho_{12}}(\alpha_1,\alpha_2;s)d\alpha_2.$$

$Q_{\rho_{12}}(\alpha_1^{'};s)$, $Q_{\rho_{12}}(\alpha_2;s)$ and $Q_{\rho_{12}}(\alpha_2^{'};s)$ are defined similarly. In addition, let

$$\boldsymbol{\zeta}_1^{v}=\Big(\frac{\pi^2(1-s)^4}{4},\frac{\pi s(1-s)^2}{2},v\Big), \tag{27}$$

$$\boldsymbol{\zeta}_2^{v}=(\pi^2(1-s)^2,-\pi(1-s),v), \tag{28}$$

and

$$\boldsymbol{\eta}^{(l_1,l_2,l_3)}_{(\eta_1,\eta_2,\eta_3,\eta_4)}=(\alpha^{\lambda_{l_3}}_{x_{l_1}=\eta_1},\alpha^{\lambda_{l_3}}_{x_{l_2}=\eta_2},\alpha^{\lambda_{l_3}}_{x_{l_1}=\eta_3},\alpha^{\lambda_{l_3}}_{x_{l_2}=\eta_4}), \tag{29}$$

where $\eta_1,\eta_2,\eta_3,\eta_4\in\{0,1\}$.

4 Witnessing nonlocality in chain networks of CV systems

Consider an arbitrary chain network $\mathcal{C}ha(y)$ as described in Figure 2. This is a network consisting of y parties A_1, A_2, $\cdots$, A_y, and $y-1$ independent sources S_1, S_2, $\cdots$, S_{y-1}. The parties and the sources are arranged linearly such that any two neighboring parties share a common source. We assume that each S_j emits a $(1+1)$-mode Gaussian state for $j=1,2,\cdots,y-1$, and each A_i performs the measurements of the form in (18) for $i=1,2,\cdots,y$. We apply Theorem 1 to establish corresponding Bell-type inequality for arbitrary chain network $\mathcal{C}ha(y)$ in Figure 2.

For such $\mathcal{C}ha(y)$, it is clear that (1) if y is odd, then $k_{\max}=\frac{y+1}{2}$ with the index set of maximal independent parties $\mathcal{K}=\mathcal{K}_{\max}=\{1,3,\cdots,y\}$; (2) if y is even, then $k_{\max}=\frac{y}{2}$ with the index set of maximal independent parties $\mathcal{K}=\mathcal{K}_{\max}=\{2,4,\cdots,y\}$. Let $\overline{\mathcal{K}}=\{1,2,\cdots,y\}\setminus\mathcal{K}$ and $\mathcal{K}'$ denote the set of indexes from $\mathcal{K}$ with the biggest index excluded. For example, when y is odd, $\mathcal{K}'=\{1,3,\cdots,y-2\}$. Then, by Theorem 1, we have the following.

Theorem 3. Let $\mathcal{C}ha(y)$ be a chain network as in Figure 2. Assume that each independent source S_j emits $(1+1)$-mode Gaussian state $\rho^{A_jA_{j+1}}$ and let $Q_{\rho^{A_jA_{j+1}}}(\alpha^{\lambda_j}_{x_j},\alpha^{\lambda_j}_{x_{j+1}};s)$, $x_j,x_y\in\{0,1\}$, $j=1,2,\cdots,y-1$ denote its generalized quasiprobability function. If $\mathcal{C}ha(y)$ is network local, then the following nonlinear Bell-type inequality holds:

$$\mathcal{B}^{ch}_{\rho}(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1;s)=|\mathcal{I}_s|^{\frac{1}{k_{\max}}}+|\mathcal{J}_s|^{\frac{1}{k_{\max}}}\leqslant 1, \tag{30}$$

where $\boldsymbol{\alpha}_t=(\alpha^{\lambda_1}_{x_1=t},\alpha^{\lambda_1}_{x_2=t},\cdots,\alpha^{\lambda_{y-1}}_{x_{y-1}=t},\alpha^{\lambda_{y-1}}_{x_y=t})$ for $t=0,1$ and $\rho=\rho^{A_1A_2}\otimes\cdots\otimes\rho^{A_yA_{y-1}}$. In addition, $k_{\max}=\frac{y+1}{2}$ when y is odd, $k_{\max}=\frac{y}{2}$ when y is even; for $-1<s\leqslant 0$,

$$\mathcal{I}_s=F^{+}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)},\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s),\ \ \mathcal{J}_s=F^{-}(\boldsymbol{\zeta}_1^{0};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)},\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s);$$

while for $s\leqslant -1$,

$$\mathcal{I}_s=F^{+}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)},\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s),\ \ \mathcal{J}_s=F^{-}(\boldsymbol{\zeta}_2^{0};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)},\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s).$$

Here,

$$F^{\pm}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)=\frac{1}{2^{k_{\max}}}\prod_{i\in\mathcal{K}'}E^{\pm}_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^{\pm}_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s),$$

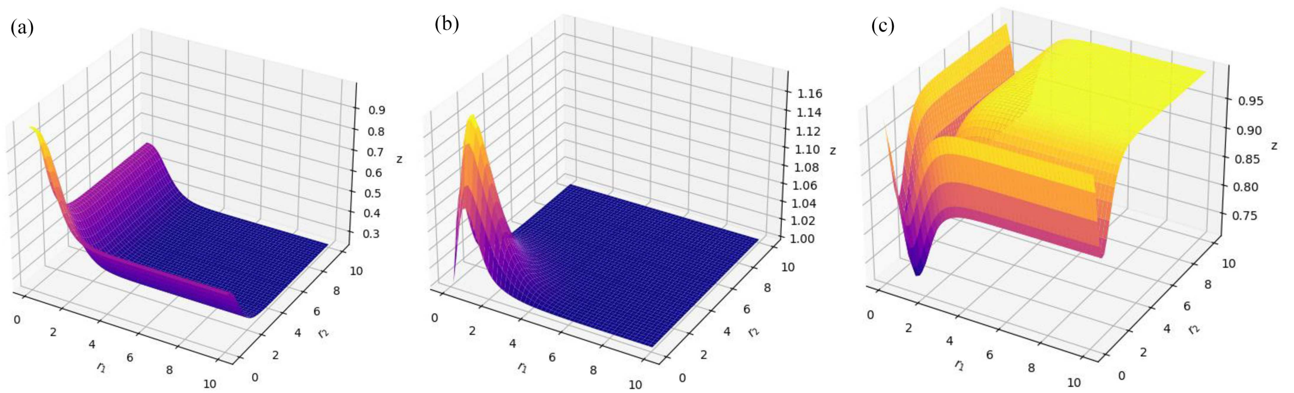

Figure 3 (Color online) Values of $z = B^{en}(s, r_1, r_2)$ in $\mathcal{E}(3)$ for $\rho^{A_1A_2} = \rho^{r_1}$ and $\rho^{A_2A_3} = \rho^{r_2}$, as a function of the parameters r_1 and r_2 for fixed s. (a) $s = -0.5$; (b) $s = -1$; (c) $s = -2$.

and $E^{\pm}$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\boldsymbol{\eta}$ are defined as in (26)–(29).

A proof of Theorem 3 is provided in Appendix B.

By the above result, if the inequality (30) is violated by some choice of parameters s, $\alpha^{\lambda_i}_{x_i=0}$, $\alpha^{\lambda_i}_{x_i=1}$, $\alpha^{\lambda_i}_{x_{i+1}=0}$ and $\alpha^{\lambda_i}_{x_{i+1}=1}$ for $i = 1, 2, \cdots, y-1$, then it demonstrates the nonlocality in the chain network $\mathcal{C}ha(y)$ of CV system (Figure 2). Applying the supremum strategy described by (21) and Theorem 2, one can readily show that if the chain network $\mathcal{C}ha(y)$ is network local, then

$$B^{ch}(s, \rho) = \sup_{\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1} \mathcal{B}^{ch}_{\rho}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s) \leqslant 1. \tag{31}$$

Therefore, $\mathcal{C}ha(y)$ is network nonlocal if $B^{ch}(s, \rho) > 1$ for some $s \leqslant 0$.

As a particular category of $\mathcal{C}ha(y)$, the entanglement swapping network $\mathcal{E}(3) = \mathcal{C}ha(3)$ is a standard tool in quantum information processing. Next, for the scenario of all quantum source states consisting of either pure Gaussian states or STSs, we conduct a detailed analysis of how to use the inequalities (30) and (31) to witness the nonlocality in $\mathcal{E}(3)$ of the CV system.

Example 3. Consider the scenario where source states are arbitrary $(1+1)$-mode pure Gaussian states in $\mathcal{E}(3)$. As every $(1+1)$-mode Gaussian pure state can be transformed into an EPR state via a local Gaussian unitary transformation and these operations do not change the network locality, it suffices to consider the scenario where both source states are (may be different) $(1+1)$-mode EPR states in $\mathcal{E}(3)$.

Assume two sources emit both (1+1)-mode EPR states, namely ρ^{r_1} and ρ^{r_2}. By inequalities (30) and (31), $B^{en}(s, \rho) = B^{ch}(s, \rho)$ is a function of r_1, r_2 and s, denoted as $B^{en}(s, r_1, r_2)$. Then, $B^{en}(s, r_1, r_2) > 1$ witnesses the network nonlocality in $\mathcal{E}(3)$. $B^{en}(s, r_1, r_2)$ can be calculated by (12) and (13).

Figures 3(a)–(c), respectively, exhibit the behaviour of $B^{en}(s, r_1, r_2)$ as a function of r_1 and r_2 for specific, predetermined values of s. To illustrate more intuitively the behavior of $B^{en}(s, r_1, r_2)$ when $s = -1$, we present Figure 4.

By comparing Figures 3(a)–(c), we see that, if $s = -0.5, -2$, the generalized quasiprobability functions cannot witness the network nonlocality. Additionally, by Figures 3(b) and 4, when taking $s = -1$, the inequality $B^{en}(-1, r_1, r_2) > 1$ holds for all $r_1 > 0$ and $r_2 > 0$ and hence, $\mathcal{E}(3)$ demonstrates the network nonlocality nature if the two sources emit $(1+1)$-mode entangled EPR states. Furthermore, since any Gaussian pure state can be transformed into an EPR state through a local Gaussian unitary transformation, the generalized quasiprobability function with $s = -1$ alone can witness all network nonlocality in $\mathcal{E}(3)$ if the two sources emit $(1+1)$-mode entangled Gaussian pure states. This result is consistent with the result in [54]. In addition, it is also analogous to the result for DV systems obtained in [66], where it was shown that if two source states are entangled pure states, then the entanglement swapping network of the DV system is network nonlocal. Figure 4 further reveals that $\mathcal{E}(3)$ may be network nonlocal even if one of the source states is separable as $B^{en}(-1, 0, 0.5) > 1$. Another behavior of $B^{en}(-1, r_1, r_2)$ is that $\lim_{r_1, r_2 \to \infty} B^{en}(-1, r_1, r_2) = 1^+$, which reveals that the generalized quasiprobability functions are not very powerful to detect the network nonlocality for larger r_1, r_2. Anyway, our result suggests that measurements based solely on the generalized quasiprobability functions with $s = -1$ are sufficient to detect the network nonlocality of $\mathcal{E}(3)$ in CV systems if the source states are pure Gaussian states.

Example 4. Consider the scenario where source states are $(1+1)$-mode mixed Gaussian states in $\mathcal{E}(3)$. For simplicity's sake, we consider the scenario that the two quantum source states constitute $(1+1)$-mode symmetric

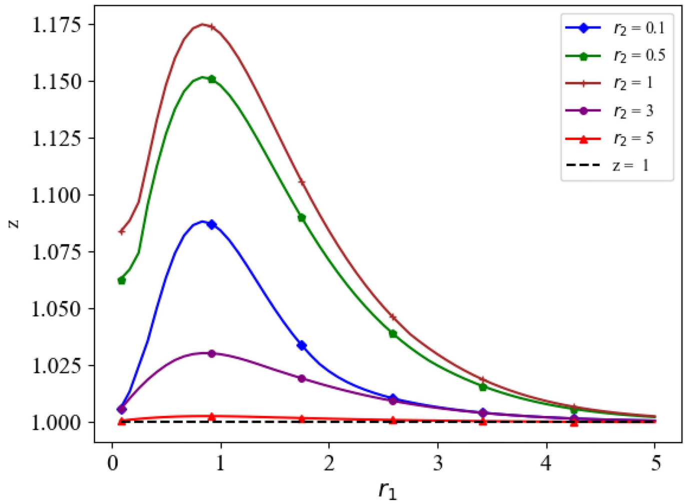

Figure 4 (Color online) Values of $z = B^{en}(-1, r_1, r_2)$ in $\mathcal{E}(3)$ for $\rho^{A_1A_2} = \rho^{r_1}$ and $\rho^{A_2A_3} = \rho^{r_2}$, as a function of the parameter r_1 for fixed $r_2 = 0.1, 0.5, 1, 3, 5$.

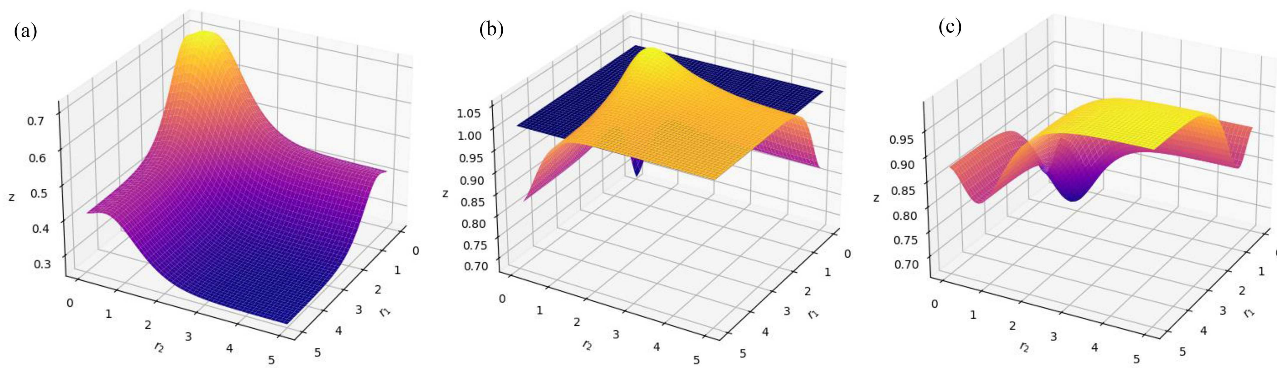

Figure 5 (Color online) Values of $z = B^{en}(s, r_1, r_2)$ in $\mathcal{E}(3)$ for $\rho^{A_1A_2} = \rho(1.2, 1.2, r_1)$ and $\rho^{A_2A_3} = \rho(1.2, 1.2, r_2)$, as a function of the parameters r_1 and r_2 for fixed s. (a) $s = -0.5$; (b) $s = -1$; (c) $s = -2$. The blue part represents the plane $z = 1$ in (b), from which one can easily see the region where $z > 1$.

STSs, specifically, $\rho(1.2, 1.2, r_1)$ and $\rho(1.2, 1.2, r_2)$ as defined in (14). Then $B^{en}(s, r_1, r_2) = B^{ch}(s, \rho)$ in the inequality (31) is expressed as a function that depends on s, r_1 and r_2. Thus, $B^{en}(s, r_1, r_2) > 1$ will witness the network nonlocality in $\mathcal{E}(3)$. Considering $B^{en}(s, r_1, r_2)$ as a function of the parameters r_1 and r_2, with s as constants, the behavior of $B^{en}(s, r_1, r_2)$ is shown in Figure 5. To more intuitively illustrate the behavior of $B^{en}(s, r_1, r_2)$ when $s = -1$, we present Figure 6. As $\rho(1.2, 1.2, r_j)$ is separable if and only if $\cosh(r_j) \leqslant \sqrt{\frac{121}{120}}$, i.e., $0 \leqslant r_j \leqslant r_s = \operatorname{arcosh}(\sqrt{\frac{121}{120}}) \in (0.09116, 0.09117)$, $j = 1, 2$, we see that $\mathcal{E}(3)$ is network local if both $r_1, r_2 \in [0, r_s]$, and thus $B^{en}(s, r_1, r_2) \leqslant 1$ for $(r_1, r_2) \in [0, r_s] \times [0, r_s]$.

By comparing Figures 5(a)–(c), we see that the inequality $B^{en}(s, r_1, r_2) > 1$ may hold only when $s = -1$. And from Figure 5(b), it is known that when both r_1 and r_2 are relatively large, the condition $B^{en}(-1, r_1, r_2) > 1$ is satisfied, demonstrating the network nonlocality in $\mathcal{E}(3)$. Specifically, by Figure 6(a), with $0.8 \leqslant r_2 \leqslant 1.2$, $B^{en}(-1, r_1, r_2) > 1$ holds true for $0.6 \leqslant r_1 \leqslant 2.2$, while $B^{en}(-1, r_1, r_2) > 1$ if $r_2 = 1.2$ and $0.5 \leqslant r_1 \leqslant 3.2$, demonstrating the network nonlocality. For the situation $s = -1$ and $r_2 > 1.2$, Figure 6(b) reveals that $\mathcal{E}(3)$ demonstrates the network nonlocality if $r_2 = 1.4$ and $0.6 \leqslant r_1 \leqslant 3.3$. Additionally, if $2 \leqslant r_2 \leqslant 5$, $B^{en}(-1, r_1, r_2) > 1$ holds for $r_1 \geqslant 2$, demonstrating the network nonlocality. Similar conclusion may be obtained when considering $B^{en}(s, r_1, r_2)$ as a function of r_2.

In summary, if the two sources in $\mathcal{E}(3)$ emit $(1+1)$-mode mixed symmetric STSs, for example, $\rho(1.2, 1.2, r_1)$ and $\rho(1.2, 1.2, r_2)$, using the generalized quasiprobability functions with $s = -1$ enables the detection of network nonlocality for many values of r_1, r_2.

For a chain network $\mathcal{C}ha(y)$ with $y \geqslant 3$, due to its complexity, we only consider cases where all sources emit the same $(1+1)$-mode EPR states (see Example 5) and the same $(1+1)$-mode symmetric STSs (see Example 6).

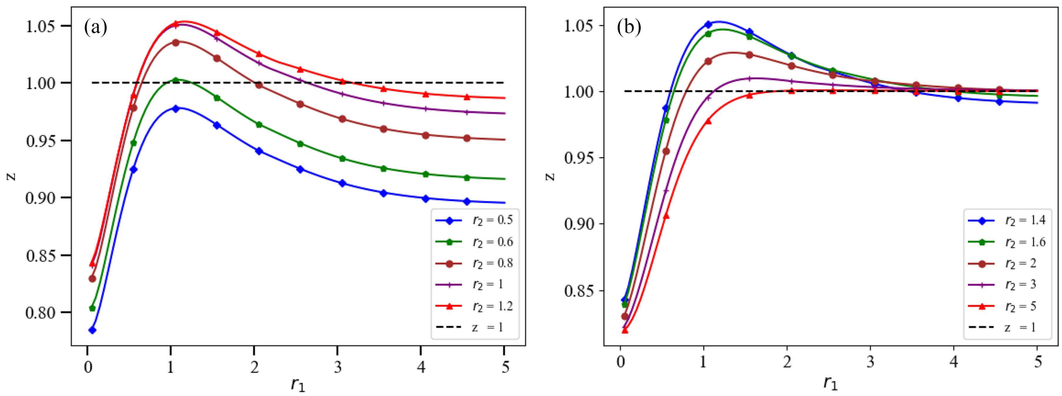

Figure 6 (Color online) Values of $z = B^{en}(-1, r_1, r_2)$ in $\mathcal{E}(3)$ for $\rho^{A_1A_2} = \rho(1.2, 1.2, r_1)$ and $\rho^{A_2A_3} = \rho(1.2, 1.2, r_2)$, as a function of parameters r_1 and r_2 for fixed $s = -1$. (a) The case $r_2 \in [0.5, 1.2]$; (b) the case $r_2 \in [1.4, 5]$. The dashed line represents $z = 1$, from which one can easily see when we have $z > 1$.

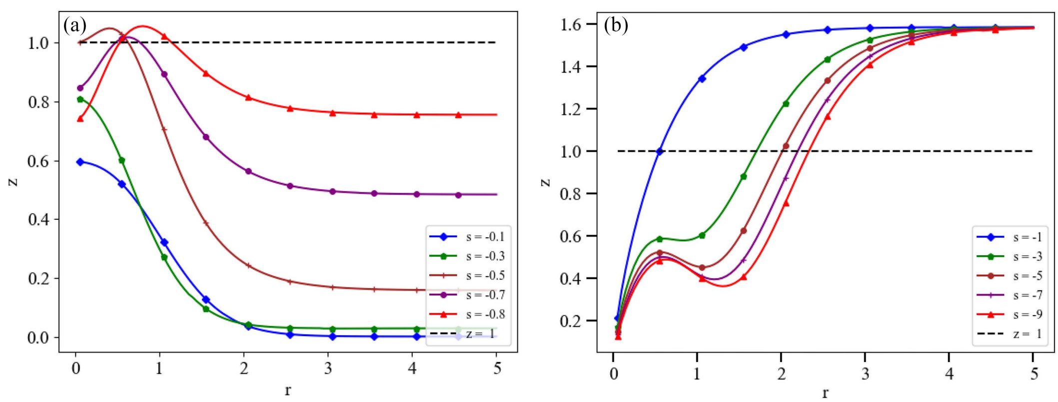

Figure 7 (Color online) Values of $z = B^{ch}(s, r)$ in $\mathcal{C}ha(6)$ for $(1 + 1)$-mode Gaussian state $\rho^{A_iA_{i+1}} = \rho^r (i = 1, 2, 3, 4, 5)$, as a function of the parameter r for fixed s. (a) The behavior of $z = B^{ch}(s, r)$ when $-1 < s \leqslant 0$; (b) the behavior of $z = B^{ch}(s, r)$ when $s \leqslant -1$.

Example 5. Consider the chain network $\mathcal{C}ha(6)$. Assume that all source states are the same EPR state ρ^r as defined in (11), that is, $\rho^{A_iA_{i+1}} = \rho^r$ for each $i = 1, 2, 3, 4, 5$. Then $B^{ch}(s, \rho)$ in the inequality (31) is a function $B^{ch}(s, r)$ of s and r, and $B^{ch}(s, r) > 1$ witnesses the network nonlocality in $\mathcal{C}ha(6)$.

For certain fixed values of s, Figure 7 is derived by considering $B^{ch}(s, r)$ as a function of the parameter r. It reveals that, as a function of r, $B^{ch}(s, r)$ exhibits entirely different behaviors for $s \in (-1, 0]$ and $s \in (-\infty, -1]$. If $s \in (-1, 0]$, $B^{ch}(s, r)$ has advantage to witness the network nonlocality for those r near 0, while, if $s \in (-\infty, -1]$, $B^{ch}(s, r)$ has advantage to witness the network nonlocality for those r far from 0. Observe also that, for $s \in (-\infty, -1]$, as s increases, more values of r make the inequality $B^{ch}(s, r) > 1$ holding true. When $s = -1$, a larger set of r values results in $B^{ch}(s, r) > 1$, that is, using the generalized quasiprobability function with $s = -1$ can witness the network nonlocality for more EPR states.

Furthermore, as depicted in Figure 7(a), when $0 < r \leqslant 0.63$, we observe that $B^{ch}(-0.5, r) > 1$. Similarly, in Figure 7(b), when $r \geqslant 0.55$, we have $B^{ch}(-1, r) > 1$. Consequently, for any $r > 0$, the inequality $B^{ch}(s, r) > 1$ holds for some choice of s, demonstrating the network nonlocality nature. Therefore, we conclude that as long as the five sources in the chain network $\mathcal{C}ha(6)$ emit any identical entangled Gaussian pure states, $\mathcal{C}ha(6)$ will generate network nonlocality, which can be witnessed by the generalized quasiprobability functions. This result is true for any chain network $\mathcal{C}ha(y)$ of CV systems.

Example 6. Assume that all source states of the chain network $\mathcal{C}ha(6)$ are the same symmetric STSs $\rho(v, v, r)$ as given in (14) with $v = 1.2$, i.e., $\rho^{A_iA_{i+1}} = \rho(1.2, 1.2, r)$ for each $i = 1, 2, 3, 4, 5$. Note that $\rho(1.2, 1.2, r)$ is a mixed Gaussian state and it is entangled if and only if $r > r_s$, where $0.09116 < r_s < 0.09117$. It follows that $\mathcal{C}ha(6)$ is network local if $0 \leqslant r \leqslant r_s$. To demonstrate the network nonlocality, we focus on the $B^{ch}(s, \rho)$ in the inequality (31), which is a function $B^{ch}(s, r)$ of s and r. By Theorem 3, $B^{ch}(s, r) > 1$ witnesses the network nonlocality in $\mathcal{C}ha(6)$. Figure 8 is derived by considering $B^{ch}(s, r)$ as a function of the parameter r, with s held constant.

From Figure 8(a), within the range $-1 < s \leqslant 0$, it is observed that only when $s \leqslant -0.7$ there is a small subset of symmetric STSs that satisfy $B^{ch}(s, r) > 1$. However, as depicted in Figure 8(b), when $s \leqslant -1$, $B^{ch}(s, r)$ can detect

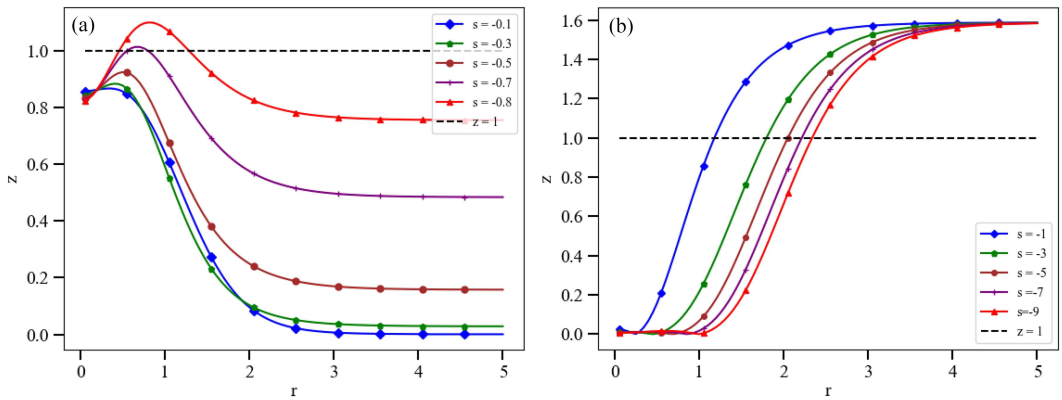

Figure 8 (Color online) Values of $z = B^{ch}(s, r)$ in $\mathcal{C}ha(6)$ for $(1+1)$-mode Gaussian state $\rho^{A_iA_{i+1}} = \rho(1.2, 1.2, r)$ $(i = 1, 2, 3, 4, 5)$, as a function of the parameter r for fixed s. (a) The behavior of $z = B^{ch}(s, r)$ when $-1 < s \leqslant 0$; (b) the behavior of $z = B^{ch}(s, r)$ when $s \leqslant -1$.

network nonlocality in a larger number of STSs. Moreover, within the interval $s \leqslant -1$, as s increases, an increasing number of STSs contribute to the condition $B^{ch}(s, r) > 1$. Therefore, when all sources emit symmetric STSs in $\mathcal{C}ha(6)$, employing the generalized quasiprobability functions with $s = -1$ allows for the detection of network nonlocality in a greater number of instances.

In addition, from Figure 8(a), the inequality $B^{ch}(-0.8, r) > 1$ holds when $0.5 \leqslant r \leqslant 1.25$. And in Figure 8(b), the inequality $B^{ch}(-1, r) > 1$ is satisfied for $r \geqslant 1.18$. Thus we conclude that if all sources emit symmetric STSs $\rho^{A_iA_{i+1}} = \rho(1.2, 1.2, r)$ $(i = 1, 2, 3, 4, 5)$ in $\mathcal{C}ha(6)$ and $r \geqslant 0.5$, then $\mathcal{C}ha(6)$ exhibits network nonlocality, which can be witnessed by the generalized quasiprobability functions. However, for this scenario, we still do not know whether $\mathcal{C}ha(6)$ is network local or not for $r_s < r < 0.5$. We believe that, there exists $r_{1.2}$ with $0 < r_s \leqslant r_{1.2} < 0.5$ such that $\mathcal{C}ha(6)$ is network local if and only if $0 \leqslant r \leqslant r_{1.2}$. Generally, if all source states are the same symmetric STSs $\rho(v, v, r)$, there exists $r_v > 0$ such that $\mathcal{C}ha(6)$ is network local if and only if $0 \leqslant r \leqslant r_v$.

5 Witnessing nonlocality in star networks of CV systems

In this section, we consider the star network $\mathcal{S}(y)$ depicted in Figure 9, which consists of y parties $A_1, A_2, \cdots, A_y$, and $y-1$ independent sources $S_1, S_2, \cdots, S_{y-1}$. It is clear that in star network $\mathcal{S}(y)$, the number of maximal independent parties $k_{\max} = y-1$, and the corresponding index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{1, 2, \cdots, y-1\}$. We assume that each S_j emits the $(1+1)$-mode Gaussian state for $j = 1, 2, \cdots, y-1$, and each A_i performs the measurements of the form in (18) for $i = 1, 2, \cdots, y$. Then, by applying Theorem 1, we obtain the following nonlinear Bell-type inequality for the star network $\mathcal{S}(y)$ in Figure 9.

Theorem 4. Let $\mathcal{S}(y)$ be a star network as in Figure 9. Assume that each independent source S_j emits (1+1)-mode Gaussian state $\rho^{A_jA_y}$, and let $Q_{\rho^{A_jA_y}}(\alpha_{x_j}^{\lambda_j}, \alpha_{x_y}^{\lambda_j}; s)$ with $x_j, x_y \in \{0, 1\}$, $j = 1, 2, \cdots, y-1$ denote its generalized quasiprobability function. If $\mathcal{S}(y)$ is network local, then the following nonlinear Bell-type inequality holds:

$$\mathcal{B}_\rho^{st}(\boldsymbol{\alpha}; s) = |\mathcal{I}_s|^{\frac{1}{y-1}} + |\mathcal{J}_s|^{\frac{1}{y-1}} \leqslant 1, \tag{32}$$

where $\boldsymbol{\alpha} = (\alpha_{x_1=0}^{\lambda_1}, \alpha_{x_y=0}^{\lambda_1}, \cdots, \alpha_{x_{y-1}=0}^{\lambda_{y-1}}, \alpha_{x_y=0}^{\lambda_{y-1}}, \alpha_{x_1=1}^{\lambda_1}, \alpha_{x_y=1}^{\lambda_1}, \cdots, \alpha_{x_{y-1}=1}^{\lambda_{y-1}}, \alpha_{x_y=1}^{\lambda_{y-1}})$, $\rho = \rho^{A_1A_y} \otimes \rho^{A_2A_y} \otimes \cdots \otimes \rho^{A_{y-1}A_y}$. Furthermore, the meanings of $\mathcal{I}_s$ and $\mathcal{J}_s$ in (32) are similar to those in Theorem 3, and their specific expressions can be found in (C4), (C5), (C9), and (C10) of Appendix C.

A proof of Theorem 4 can be found in Appendix C.

According to Theorem 4, if we select appropriate parameters s, $\alpha_{x_j=0}^{\lambda_j}$, $\alpha_{x_j=1}^{\lambda_j}$, $\alpha_{x_y=0}^{\lambda_j}$ and $\alpha_{x_y=1}^{\lambda_j}$ for $j = 1, 2, \cdots, y-1$ such that the inequality (32) is violated, then it demonstrates the network nonlocality in the star network $\mathcal{S}(y)$. Similar to (21), we use the supremum strategy described in Section 3 such that

$$B^{st}(s, \rho) = \sup_{\boldsymbol{\alpha}} \mathcal{B}_\rho^{st}(\boldsymbol{\alpha}; s) \leqslant 1, \tag{33}$$

if the star network $\mathcal{S}(y)$ is network local.

Example 7. Consider the star network $\mathcal{S}(6)$ with the same pure Gaussian source states. Without loss of generality, we assume that $\rho^{A_jA_6} = \rho^r$ is an EPR state for each $j = 1, 2, 3, 4, 5$. Then $B^{st}(s, \rho)$ in the inequality (33) is a

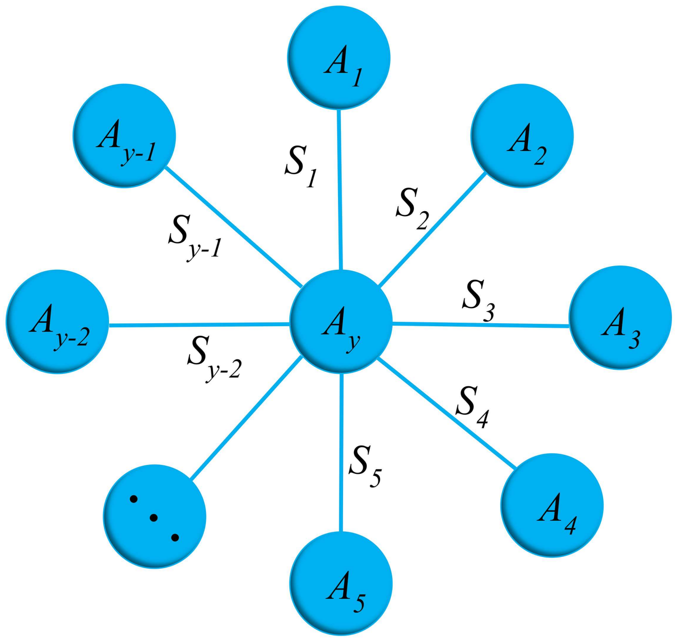

Figure 9 (Color online) A star network $\mathcal{S}(y)$ consists of y parties $A_1, A_2, \cdots, A_y$, and $y-1$ independent sources $S_1, S_2, \cdots, S_{y-1}$. And each independent source S_j emits $(1+1)$-mode Gaussian state $\rho^{A_jA_y}$ for each $j = 1, 2, \cdots, y-1$.

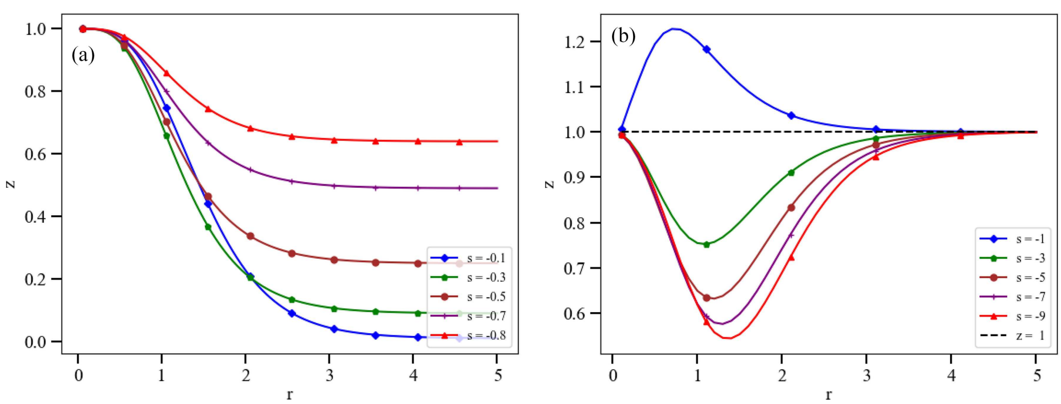

Figure 10 (Color online) Values of $z = B^{st}(s, r)$ in $\mathcal{S}(6)$ for $\rho^{A_jA_6} = \rho^r (j = 1, 2, 3, 4, 5)$, as a function of the parameter r for fixed s. (a) The case of $-1 < s \leqslant 0$; (b) the case of $s \leqslant -1$.

function $B^{st}(s, r)$ of s and r. From Theorem 4, $B^{st}(s, r) > 1$ demonstrates the network nonlocality in $\mathcal{S}(6)$. Figure 10 illustrates the value of $B^{st}(s, r)$ when regarding $B^{st}(s, r)$ as a function of r for some fixed s.

Figure 10 reveals that the inequality $B^{st}(s, r) > 1$ holds true only when $s = -1$. In this case, as r increases, the value of $B^{st}(-1, r)$ initially rises and then declines. The largest violation $B^{st}(-1, r) \approx 1.227$ occurs at $r = 0.75$. Furthermore, for any $r > 0$, it holds that $B^{st}(-1, r) > 1$, indicating that if the five quantum source states are identical entangled EPR states, they will exhibit network nonlocality in $\mathcal{S}(6)$. However, $B^{st}(-1, r) \to 1^+$ quickly as $r \to \infty$. Since every Gaussian pure state can be transformed into an EPR state through a local Gaussian unitary transformation, we conclude that if all five sources in $\mathcal{S}(6)$ emit any identical entangled Gaussian pure states, it will generate network nonlocality.

Example 8. Consider the star network $\mathcal{S}(6)$ with the same mixed Gaussian states as the source states. Assume $\rho^{A_jA_6} = \rho(1.2, 1.2, r)$ is a STS as defined in (14) for each $j = 1, 2, 3, 4, 5$. In this context, $B^{st}(s, \rho)$ in the inequality (33) is expressed as a function $B^{st}(s, r)$ that depends on both s and r. According to Theorem 4, when $B^{st}(s, r) > 1$, it witnesses the network nonlocality in $\mathcal{S}(6)$. For certain specific, fixed values of s, we can obtain Figure 11 by plotting $B^{st}(s, r)$ as a function of the parameter r.

From Figure 11, it is evident that the inequality $B^{st}(s, r) > 1$ holds exclusively when $s = -1$ and $r \geqslant 0.55$, indicating the presence of the nonlocality in $\mathcal{S}(6)$ when r meets or exceeds this threshold. Additionally, as r increases,

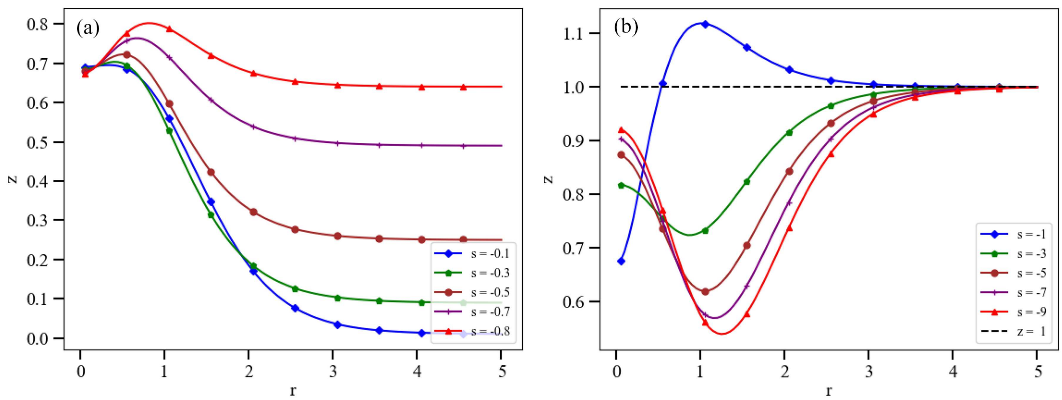

Figure 11 (Color online) Values of $z = B^{st}(s,r)$ in $\mathcal{S}(6)$ for $\rho^{A_jA_6} = \rho(1.2, 1.2, r)$ $(j = 1, 2, 3, 4, 5)$, as a function of the parameter r for fixed s. (a) The case of $-1 < s \leqslant 0$; (b) the case of $s \leqslant -1$.

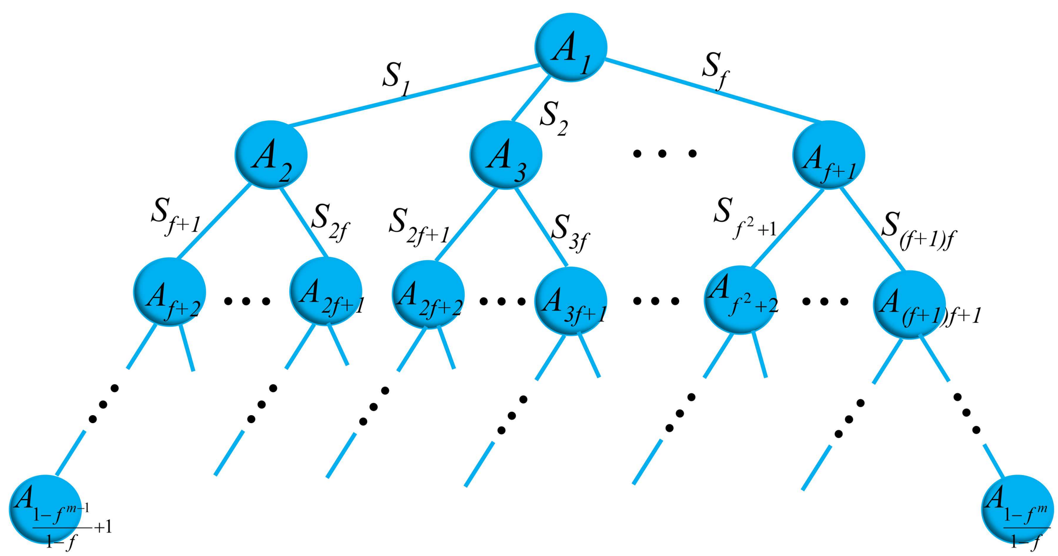

Figure 12 (Color online) A tree-shaped network $\mathcal{T}(m, f)$ consists of $\frac{1-f^m}{1-f}$ parties consisting of $A_1, A_2, \cdots, A_{\frac{1-f^m}{1-f}}$ and $\frac{f-f^m}{1-f}$ independent sources $S_1, S_2, \cdots, S_{\frac{f-f^m}{1-f}}$. And each independent source S_i emits $(1+1)$-mode Gaussian state $\rho^{A_{\lceil \frac{i}{f} \rceil}A_{i+1}}$ for each $i = 1, 2, \cdots, \frac{f-f^m}{1-f}$.

the value of $B^{st}(-1, r)$ exhibits a trend of initial increase followed by a decrease, and ultimately approaches 1. The maximum value $B^{st}(-1, r) \approx 1.118$ is achieved at $r = 1$.

6 Witnessing nonlocality in tree-shaped networks of CV systems

In this section, we discuss the network nonlocality in the deterministic m-layer f-forked tree-shaped network $\mathcal{T}(m, f)$, as depicted in Figure 12, which is a network consisting of $\frac{1-f^m}{1-f}$ parties $A_1, A_2, \cdots, A_{\frac{1-f^m}{1-f}}$ and $\frac{f-f^m}{1-f}$ sources $S_1, S_2, \cdots, S_{\frac{f-f^m}{1-f}}$. Employing the same methodology as in Sections 4 and 5, we establish corresponding Bell-type inequality for the tree-shaped network $\mathcal{T}(m, f)$ of CV systems.

In tree-shaped network $\mathcal{T}(m, f)$, it is clear that (1) if m is odd, then $k_{\max} = \frac{1-f^{m+1}}{1-f^2}$ with the index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{1, \frac{1-f^{t-1}}{1-f}+1, \frac{1-f^{t-1}}{1-f}+2, \cdots, \frac{1-f^t}{1-f}\}_{t=3,5,\cdots,m}$; (2) if m is even, then $k_{\max} = \frac{f-f^{m+1}}{1-f^2}$ with the index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{\frac{1-f^{t-1}}{1-f} + 1, \frac{1-f^{t-1}}{1-f} + 2, \cdots, \frac{1-f^t}{1-f}\}_{t=2,4,\cdots,m}$. Let $\overline{\mathcal{K}} = \{1, 2, \cdots, \frac{1-f^m}{1-f}\} \setminus \mathcal{K}$ and $\mathcal{K}'$ denote the set of indexes from $\mathcal{K}$ with the last layer of indexes excluded. For example, when m is odd, $\mathcal{K}' = \{1, \frac{1-f^{t-1}}{1-f} + 1, \frac{1-f^{t-1}}{1-f} + 2, \cdots, \frac{1-f^t}{1-f}\}_{t=3,5,\cdots,m-2}$. Then, by applying Theorem 1, we have the following.

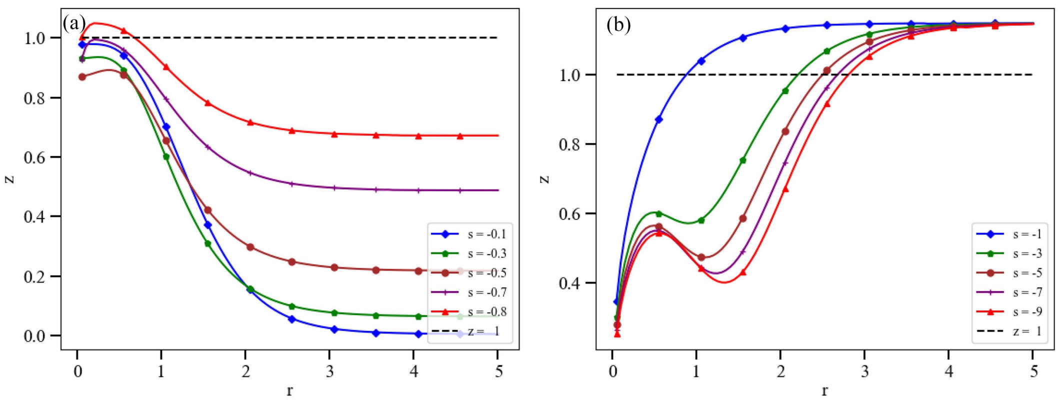

Figure 13 (Color online) Values of $z = B^{tr}(s,r)$ in $\mathcal{T}(3,2)$ for $(1+1)$-mode Gaussian state $\rho^{A_1A_2} = \rho^{A_1A_3} = \rho^{A_2A_4} = \rho^{A_2A_5} = \rho^{A_3A_6} = \rho^{A_3A_7} = \rho^r$, as a function of the parameter r for fixed s. (a) The case of $-1 < s \leqslant 0$; (b) the case of $s \leqslant -1$.

Theorem 5. Let $\mathcal{T}(m,f)$ be a tree-shaped network as in Figure 12. Assume that each independent source S_i emits $(1+1)$-mode Gaussian state $\rho^{A_{\lceil \frac{i}{f}\rceil}A_{i+1}}$ and let $Q_{\rho^{A_{\lceil \frac{i}{f}\rceil}A_{i+1}}}(\alpha^{\lambda_i}_{x_{\lceil \frac{i}{f}\rceil}}, \alpha^{\lambda_i}_{x_{i+1}}; s)$ denote its generalized quasiprobability function, where $x_{\lceil \frac{i}{f}\rceil}, x_{i+1} \in \{0,1\}$, $i = 1,2,\cdots,\frac{f-f^m}{1-f}$ and $\lceil\cdot\rceil$ represents the ceiling function. If $\mathcal{T}(m,f)$ is network local, then the following nonlinear Bell-type inequality holds:

$$\mathcal{B}^{tr}_{\rho}(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1;s) = |\mathcal{I}_s|^{\frac{1}{k_{\max}}} + |\mathcal{J}_s|^{\frac{1}{k_{\max}}} \leqslant 1, \tag{34}$$

where $\rho = \otimes_{i=1}^{\frac{f-f^m}{1-f}}(\rho^{A_{\lceil \frac{i}{f}\rceil}A_{i+1}})$, and $\boldsymbol{\alpha}_t = (\alpha^{\lambda_1}_{x_1=t}, \alpha^{\lambda_1}_{x_2=t}, \cdots, \alpha^{\lambda_i}_{x_{\lceil \frac{i}{f}\rceil}=t}, \alpha^{\lambda_i}_{x_{i+1}=t}, \cdots, \alpha^{\lambda_{\frac{f-f^m}{1-f}}}_{x_{\frac{f-f^{m-1}}{1-f}}=t}, \alpha^{\lambda_{\frac{f-f^m}{1-f}}}_{x_{\frac{f-f^m}{1-f}+1}=t})$ for $t=0,1$, $k_{\max} = \frac{f^{m+1}-1}{f^2-1}$ when m is odd, $k_{\max} = \frac{f-f^{m+1}}{1-f^2}$ when m is even. Furthermore, the meanings of $\mathcal{I}_s$ and $\mathcal{J}_s$ in (34) are analogous to their counterparts in Theorem 3, and their specific expressions can be found in (D6), (D7), (D13), and (D14) of Appendix D.

A proof of Theorem 5 is provided in Appendix D.

By the above theorem, a violation of inequality (34) witnesses the nonlocality in $\mathcal{T}(m,f)$ of the CV system.

The following two examples illustrate how to use Theorem 5 and the supremum strategy to specifically analyze the network nonlocality in $\mathcal{T}(m,f)$.

Example 9. Consider the tree-shaped network $\mathcal{T}(3,2)$ with source states that are the same EPR states. In this scenario, $\rho^{A_1A_2} = \rho^{A_1A_3} = \rho^{A_2A_4} = \rho^{A_2A_5} = \rho^{A_3A_6} = \rho^{A_3A_7} = \rho^r$, an EPR state as defined in (11). Let

$$\boldsymbol{\alpha}_t = (\alpha^{\lambda_1}_{x_1=t}, \alpha^{\lambda_1}_{x_2=t}, \alpha^{\lambda_2}_{x_1=t}, \alpha^{\lambda_2}_{x_3=t}, \alpha^{\lambda_3}_{x_2=t}, \alpha^{\lambda_3}_{x_4=t}, \alpha^{\lambda_4}_{x_2=t}, \alpha^{\lambda_4}_{x_5=t}, \alpha^{\lambda_5}_{x_3=t}, \alpha^{\lambda_5}_{x_6=t}, \alpha^{\lambda_6}_{x_3=t}, \alpha^{\lambda_6}_{x_7=t}) \in \mathbb{C}^{12},$$

where $t \in \{0,1\}$. Then

$$B^{tr}(s,r) = \sup_{\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1} \mathcal{B}^{tr}_{\rho}(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1;s)$$

is a function of s and r. According to Theorem 5, $B^{tr}(s,r) > 1$ will demonstrate the network nonlocality in $\mathcal{T}(3,2)$. Figure 13 shows the image of $B^{tr}(s,r)$ when considering $B^{tr}(s,r)$ as a function of r for the cases $-1 < s \leqslant 0$ and $s \leqslant -1$, respectively.

From Figure 13(a), it is evident that the inequality $B^{tr}(-0.8,r) > 1$ holds when $0 < r \leqslant 0.75$. Additionally, in Figure 13(b), the inequality $B^{tr}(-1,r) > 1$ is satisfied for $r \geqslant 0.7$. Consequently, taking $s = -0.8$ and -1 will detect the network nonlocality in $\mathcal{T}(3,2)$ for any $r > 0$. Furthermore, since any Gaussian pure state can be transformed into an EPR state through a local Gaussian unitary transformation, we conclude that if the six sources in the tree-shaped network $\mathcal{T}(3,2)$ emit the same entangled Gaussian pure states, then the network is network nonlocal.

Example 10. Now let us consider the tree-shaped network $\mathcal{T}(3,2)$ with all source states being the same mixed Gaussian states, say a symmetric STS $\rho(v,v,r)$ as defined in (14). For simplicity, let $\rho^{A_1A_2} = \rho^{A_1A_3} = \rho^{A_2A_4} = \rho^{A_2A_5} = \rho^{A_3A_6} = \rho^{A_3A_7} = \rho(1.2,1.2,r)$. Just like what we have done in Example 9, by taking the supremum of $\mathcal{B}^{tr}_{\rho}(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1;s)$ in the inequality (34) over the parameters $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, one gets a function $B^{tr}(s,r)$ of s and r. Applying Theorem 5, if $B^{tr}(s,r) > 1$, then it demonstrates network nonlocality. For some fixed values of s, Figure 14 is obtained by treating $B^{tr}(s,r)$ as a function of the parameter r.

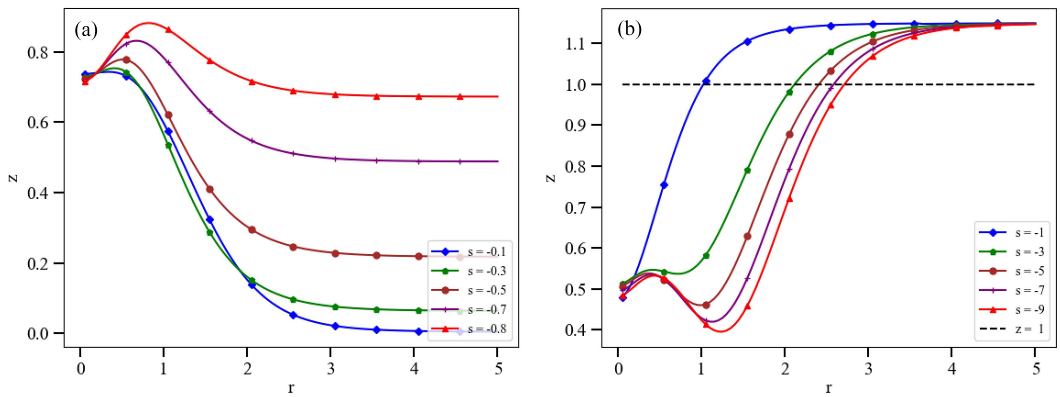

Figure 14 (Color online) Values of $z = B^{tr}(s,r)$ in $\mathcal{T}(3,2)$ for $(1+1)$-mode Gaussian state $\rho^{A_1A_2} = \rho^{A_1A_3} = \rho^{A_2A_4} = \rho^{A_2A_5} = \rho^{A_3A_6} = \rho^{A_3A_7} = \rho(1.2, 1.2, r)$, as a function of the parameter r for fixed s. (a) The case of $-1 < s \leqslant 0$; (b) the case of $s \leqslant -1$.

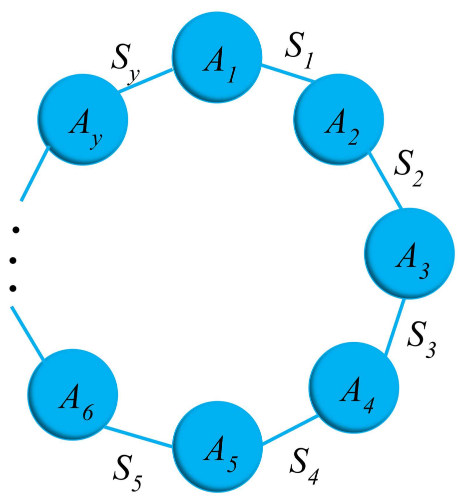

Figure 15 (Color online) A cyclic network $\mathcal{C}yc(y)$ consists of y parties consisting of $A_1, A_2, \cdots, A_y$ and y independent sources $S_1, S_2, \cdots, S_y$. And each independent source S_j emits $(1+1)$-mode Gaussian state $\rho^{A_jA_{j+1}}$ for each $j = 1, 2, \cdots, y-1$ and S_y emits $(1+1)$-mode Gaussian states $\rho^{A_yA_1}$.

From Figures 14(a) and (b), we can see that the inequality $B^{tr}(s,r) > 1$ holds for some r only when $s \leqslant -1$, and as s increases, more symmetric STSs $\rho(1.2, 1.2, r)$ satisfy the inequality $B^{tr}(s,r) > 1$. Therefore, using the generalized quasiprobability function with $s = -1$ allows for witnessing network nonlocality of a larger number of STSs in $\mathcal{T}(3,2)$. Furthermore, Figure 14(b) demonstrates that the inequality $B^{tr}(-1,r) > 1$ holds whenever $r \geqslant 1.025$. Then, we conclude that, with every source state being symmetric STS $\rho(1.2, 1.2, r)$, if $r \geqslant 1.025$, the tree-shaped network $\mathcal{T}(3,2)$ exhibits network nonlocality.

7 Witnessing nonlocality in cyclic networks of CV systems

In a cyclic network $\mathcal{C}yc(y)$ (see Figure 15), there are y parties $A_1, A_2, \cdots, A_y$ with A_i and A_{i+1} sharing a source S_i for $i = 1, 2, \cdots, y-1$, and A_y and A_1 sharing a source S_y. It is clear that (1) if y is odd, then $k_{\max} = \frac{y-1}{2}$ with the index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{1, 3, 5, \cdots, y-2\}$; (2) if y is even, then $k_{\max} = \frac{y}{2}$ with the index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{1, 3, 5, \cdots, y-1\}$ or $\{2, 4, 6, \cdots, y\}$. Let $\overline{\mathcal{K}} = \{1, 2, \cdots, y\} \setminus \mathcal{K}$. Using a measurement scheme based on generalized quasiprobability functions and applying Theorem 1, we can establish the following nonlinear Bell-type inequality for the cyclic network $\mathcal{C}yc(y)$ of CV systems.

Theorem 6. Let $\mathcal{C}yc(y)$ be a cyclic network as in Figure 15. Assume that each independent source S_j emits $(1+1)$-mode Gaussian state $\rho^{A_jA_{j+1}}$ and let $Q_{\rho^{A_jA_{j+1}}}(\alpha_{x_j}^{\lambda_j}, \alpha_{x_{j+1}}^{\lambda_j}; s)$ for $j = 1, 2, \cdots, y-1$ denote its generalized quasiprobability function; S_y emits $(1+1)$-mode Gaussian state $\rho^{A_yA_1}$ and let $Q_{\rho^{A_yA_1}}(\alpha_{x_y}^{\lambda_y}, \alpha_{x_1}^{\lambda_y}; s)$ denote its generalized quasiprobability function, where $x_j, x_y \in \{0, 1\}$, $j = 1, 2, \cdots, y-1$. Then $\mathcal{C}yc(y)$ is network local

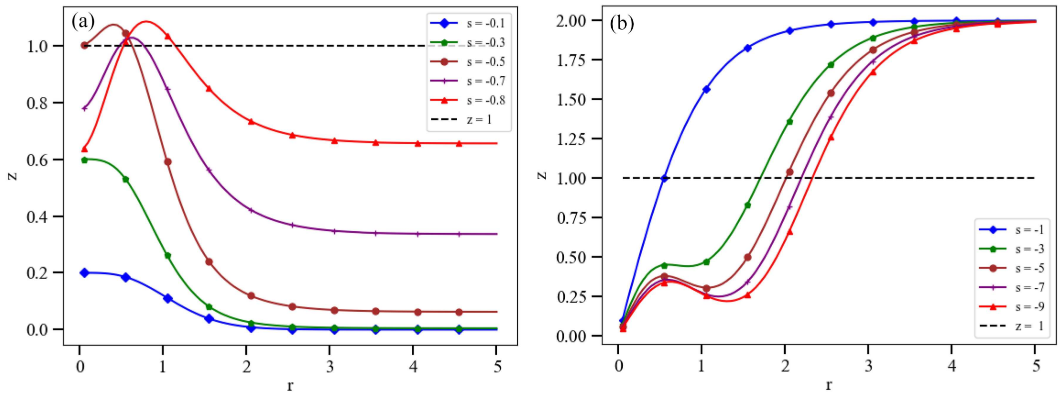

Figure 16 (Color online) Values of $z = B^{cy}(s,r)$ in $\mathcal{C}yc(5)$ for $(1+1)$-mode Gaussian state $\rho^{A_jA_{j+1}} = \rho^{A_5A_1} = \rho^r$ $(j=1,2,3,4)$, as a function of the parameter r for fixed s. (a) The case of $-1 < s \leqslant 0$; (b) the case of $s \leqslant -1$.

implies that the following nonlinear Bell-type inequality holds:

$$\mathcal{B}^{cy}_{\rho}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s) = |\mathcal{I}_s|^{\frac{1}{k_{\max}}} + |\mathcal{J}_s|^{\frac{1}{k_{\max}}} \leqslant 1, \tag{35}$$

where $\rho = \rho^{A_1A_2} \otimes \rho^{A_2A_3} \otimes \cdots \otimes \rho^{A_{y-1}A_y} \otimes \rho^{A_yA_1}$, $\boldsymbol{\alpha}_t = (\alpha^{\lambda_1}_{x_1=t}, \alpha^{\lambda_1}_{x_2=t}, \cdots, \alpha^{\lambda_{y-1}}_{x_{y-1}=t}, \alpha^{\lambda_{y-1}}_{x_y=t}, \alpha^{\lambda_y}_{x_y=t}, \alpha^{\lambda_y}_{x_1=t})$ for $t \in \{0,1\}$. In addition, $k_{\max} = \frac{y-1}{2}$ when y is odd, $k_{\max} = \frac{y}{2}$ when y is even. Furthermore, the quantities $\mathcal{I}_s$ and $\mathcal{J}_s$ in (35) have meanings analogous to those defined in Theorem 3. Specifically, their explicit expressions are provided in Appendix E: for odd y, refer to (E4), (E5), (E9), and (E10); for even y, see (E12), (E13), (E15), and (E16).

Our proof of Theorem 6 is provided in Appendix E.

According to Theorem 6, if the inequality (35) is violated by some specific choice of parameters s, $\alpha^{\lambda_i}_{x_i=0}$, $\alpha^{\lambda_i}_{x_i=1}$, $\alpha^{\lambda_i}_{x_{i+1}=0}$, $\alpha^{\lambda_i}_{x_{i+1}=1}$ for $i = 1,2,\cdots,y-1$ and $\alpha^{\lambda_y}_{x_y=0}$, $\alpha^{\lambda_y}_{x_y=1}$, $\alpha^{\lambda_y}_{x_1=0}$, $\alpha^{\lambda_y}_{x_1=1}$, then it demonstrates the network nonlocality in $\mathcal{C}yc(y)$ of CV system. Moving forward, we will provide two specific examples to illustrate how to use the inequality (35) to witness the nonlocality in $\mathcal{C}yc(5)$ of CV systems.

Example 11. Consider the cyclic network $\mathcal{C}yc(5)$ and assume that $\rho^{A_jA_{j+1}} = \rho^{A_5A_1} = \rho^r$ is an EPR state as in (11) for each $j = 1,2,3,4$. Let

$$\boldsymbol{\alpha}_t = (\alpha^{\lambda_1}_{x_1=t}, \alpha^{\lambda_1}_{x_2=t}, \alpha^{\lambda_2}_{x_2=t}, \alpha^{\lambda_2}_{x_3=t}, \alpha^{\lambda_3}_{x_3=t}, \alpha^{\lambda_3}_{x_4=t}, \alpha^{\lambda_4}_{x_4=t}, \alpha^{\lambda_4}_{x_5=t}, \alpha^{\lambda_5}_{x_5=t}, \alpha^{\lambda_5}_{x_1=t}) \in \mathbb{C}^{10},$$

where $t \in \{0,1\}$. Then

$$B^{cy}(s,r) = \sup_{\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1} \mathcal{B}^{cy}_{\rho}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s)$$

is a function of s and r. By Theorem 6, $B^{cy}(s,r) > 1$ will demonstrate the network nonlocality in $\mathcal{C}yc(5)$. For some specific fixed values of s, Figure 16 plots $B^{cy}(s,r)$ as a function of the parameter r.

In Figure 16(a), when s is fixed within the interval $(-1,0]$, the inequality $B^{cy}(s,r) > 1$ holds for some r when $s \leqslant -0.5$ and this inequality is only satisfied when r takes on relatively small values. Figure 16(b) shows that, when $s \leqslant -1$, the inequality $B^{cy}(s,r) > 1$ holds when r is sufficiently large, and as s increases, there are more values of r that satisfy $B^{cy}(s,r) > 1$. So we can see that using the generalized quasiprobability function with $s = -1$ enables us to witness more network nonlocality.

Moreover, from Figure 16(a), the inequality $B^{cy}(-0.5,r) > 1$ holds for $0 < r \leqslant 0.6$, while from Figure 16(b), the inequality $B^{cy}(-1,r) > 1$ holds for $r \geqslant 0.55$. In conclusion, by taking $s = -0.5$ and -1, the inequality $B^{cy}(s,r) > 1$ holds for any $r > 0$. This reveals that if all five sources in $\mathcal{C}yc(5)$ emit any identical entangled EPR states, they generate a network nonlocality correlation in $\mathcal{C}yc(5)$. Therefore, we in fact obtain that if the five sources in $\mathcal{C}yc(5)$ emit any identical entangled Gaussian pure states, the cyclic network $\mathcal{C}yc(5)$ demonstrates the network nonlocality.

Example 12. Let us consider the scenario that all sources in the cyclic network $\mathcal{C}yc(5)$ emit identical mixed Gaussian states. For simplicity, we assume that $\rho^{A_jA_{j+1}} = \rho^{A_5A_1} = \rho(1.2,1.2,r)$ $(j=1,2,3,4)$, where $\rho(1.2,1.2,r)$ is a symmetric STS as described in (14). Similar to Example 11, $B^{cy}(s,r) = \sup_{\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1} \mathcal{B}^{cy}_{\rho}(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1; s)$ is a function of s and r, and by Theorem 6, $B^{cy}(s,r) > 1$ demonstrates the network nonlocality in $\mathcal{C}yc(5)$. Figure 17 depicts the value of $B^{cy}(s,r)$ as a function of the parameter r for several fixed s.

It is observed from Figures 17(a) and (b) that when $s = -0.8$ and $s \leqslant -1$, $B^{cy}(s,r) > 1$ for some r. Furthermore, Figure 17(a) indicates that the inequality $B^{cy}(-0.8,r) > 1$ holds for $0.5 \leqslant r \leqslant 1.25$, while Figure 17(b) reveals that

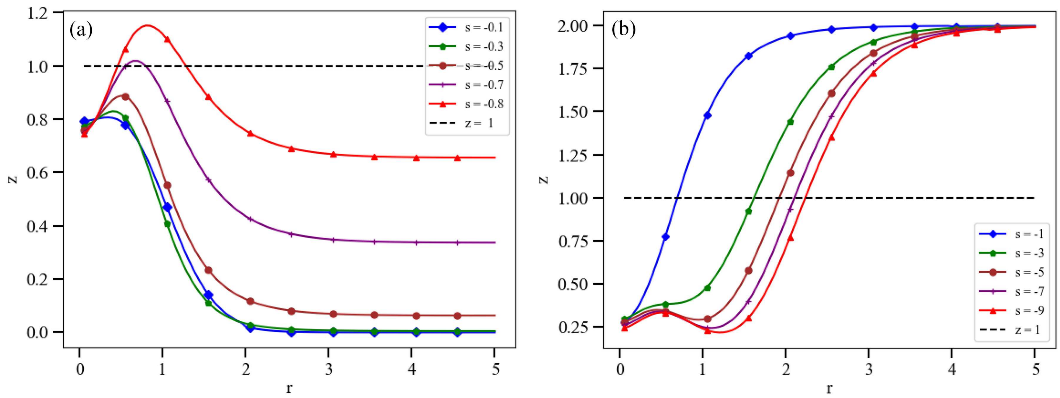

Figure 17 (Color online) Values of $z = B^{cy}(s, r)$ in $\mathcal{C}yc(5)$ for $(1+1)$-mode Gaussian state $\rho^{A_jA_{j+1}} = \rho^{A_5A_1} = \rho(1.2, 1.2, r)$ $(j = 1, 2, 3, 4)$, as a function of the parameter r for fixed s. (a) The case of $-1 < s \leqslant 0$; (b) the case of $s \leqslant -1$.

the inequality $B^{cy}(-1, r) > 1$ holds for $r \geqslant 0.7$. Combining these two facts reveals that, the cyclic network $\mathcal{C}yc(5)$ exhibits network nonlocality if $r \geqslant 0.5$.

8 Conclusion and discussion

Gaussian states serve as a highly valuable resource in the field of quantum information and quantum communication protocols, and the quantum networks that will be operated in the future are likely to be hybrid networks. However, so far, there has been a significant limitation of research focusing on the network nonlocality for CV systems. The challenge comes from the fact that Gaussian states cannot recognize nonlocality when Gaussian measurements are performed. In this work, we aim to address the problem of detecting network nonlocality in CV systems by proposing an effective nonlinear Bell-type inequality for a general network of finite or infinite dimensional system, where each party has two inputs and unrestricted outputs (which may be infinitely many). This inequality provides a general approach for detecting the network nonlocality of CV systems, where the quantum source states are multi-mode Gaussian states and the measurement scheme is based on generalized quasiprobability functions. Notably, assessing the network nonlocality in CV systems using this inequality is straightforward, as this inequality is solely related to the generalized quasiprobability functions of Gaussian states. This allows us to propose a general approach called the supremum strategy. To illustrate how to utilize established nonlinear Bell-type inequality and the supremum strategy to detect the network nonlocality of CV systems, we establish several nonlinear Bell-type inequalities for various network configurations of depth 2, including the chain, star, tree-shaped, and cyclic networks of CV systems with source states being arbitrary $(1+1)$-mode Gaussian states. In addition, the violation of these inequalities was demonstrated for identical entangled EPR states and certain symmetric STSs. Through these examples, we infer that, for a general quantum network, if all source states are identical $(1+1)$-mode entangled Gaussian pure states, then the Bell-type inequalities of the network will be violated, and this violation thereby naturally witnesses the network nonlocality. We also observe that, using only special measurements based on the generalized quasiprobability functions, for instance with some choice of s, is sufficient to recognize the network nonlocality of networks with pure Gaussian source states. When source states are identical mixed Gaussian states such as the symmetric STSs $\rho(v, v, r)$, the situation becomes more complex. However, by utilizing the generalized quasiprobability function, one still can recognize the network nonlocality for all $r > r_v$ for some $r_v > 0$. For the entanglement swapping network, the scenarios where sources emit different pure Gaussian states or symmetric STSs are discussed in detail.

Our scheme demonstrates that for Gaussian networks, measurements based on generalized quasiprobability functions can effectively detect network nonlocality. This approach may provide new insights into protocols including quantum sensing [35] and device-independent quantum key distribution [36].

Moreover, under ideal conditions, our approach is physically feasible. As described in [63], when all sources emit $(1+1)$-mode Gaussian states in the quantum network, the experimental implementation based on this scheme requires only a beam-splitter coupled to the photodetector. When the parameter $s = -1$, the measurement scheme only requires a beam-splitter coupled to an intense coherent state, followed by a photodetector that distinguishes between vacuum and non-vacuum states. Although the proposed test presents implementation challenges due to inevitable optical losses and imperfect detection, recent advances in photon-number-resolving detectors have brought it closer to practical realization. Significant efforts are underway to develop high-efficiency photon-number-resolving devices using superconducting-nanowire detectors or transition-edge sensors [67–72]. Notably, recent experimental progress has yielded single-photon detectors with approximately 98% detection efficiency and photon-number resolution up to 32 [72]. Nevertheless, further in-depth exploration is still required to push single-photon detection toward even higher photon-number resolution and to achieve high-fidelity detection in multi-photon regimes.

It is worth to point out, although our work only focuses on the network nonlocality of CV systems, our scheme is well-suited for detecting network nonlocality within "hybrid" quantum networks that are integrated with both DV and CV systems. For example, consider the entanglement swapping network with one source emitting a single-photon entangled state $\rho = |\Psi\rangle\langle\Psi|$ with $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ (its two-mode generalized quasiprobability function and its marginal single-mode distribution are given in [63]) and the other source distributes a $(1+1)$-mode EPR state. Then, similar to the proof of Theorem 3, one can establish a nonlinear Bell-type inequality by applying Theorem 1 to detect the network nonlocality in this hybrid entanglement swapping network.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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