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• Supplementary File •

Witnessing nonlocality in quantum network of continuous-variable systems by generalized quasiprobability functions

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Appendix A Proof of Theorem 1

Let $\Xi(y,z)$ be a general network with k independent parties in system of any dimension and assume that it is network local. We prove that, for any dichotomic local measurements $\{M_{x_i} : x_i \in \{0,1\}\}_{i=1}^{y}$ of countable outcomes, the nonlinear Bell-type inequality (2) in the main text holds.

For any $i = 1, 2, \cdots, y$, denote $\Upsilon_{x_i} = \{a_h : a_h \text{ is the eigenvalue of } M_{x_i}\}$. Then $\langle M_{x_1} M_{x_2} \cdots M_{x_y}\rangle = \sum\limits_{a_1\in\Upsilon_{x_1}} \sum\limits_{a_2\in\Upsilon_{x_2}} \cdots \sum\limits_{a_y\in\Upsilon_{x_y}} a_1 a_2 \cdots a_y P(a_1, a_2, \cdots, a_y | x_1, x_2, \cdots, x_y)$ and the expectation of the outcomes of M_{x_i} is

$$\langle M_{x_i}\rangle = \sum_{a_i\in\Upsilon_{x_i}} a_i P(a_i|x_i,\Lambda_i), \tag{A1}$$

where $i = 1, 2, \cdots, y$ and $\Lambda_i = \{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_{e_i}}\}$ denotes the set of local variables associated with the sources that connect to party A_i. Since $|a_i| \leqslant 1$ for any i, we have $|\langle M_{x_i}\rangle| \leqslant 1$.

Denote the index set of k independent parties as $\mathcal{K} = \{i_1, i_2, \cdots, i_k\}$ and let $\bar{\mathcal{K}} = \{1, 2, \cdots, y\}\backslash\mathcal{K}$. By Eq.(3) in the main text and Eq.(A1), we have

$$\begin{aligned}
|\mathcal{I}| =& \frac{1}{2^k}\Big|\int\int\cdots\int d\Lambda_{i_1} d\Lambda_{i_2}\cdots d\Lambda_{i_k} \prod_{m=1}^{k}\mu_{i_m}(\Lambda_{i_m}) \prod_{i_s\in\mathcal{K}}(\langle M_{x_{i_s}=0} + M_{x_{i_s}=1}\rangle) \prod_{j\in\bar{\mathcal{K}}}\langle M_{x_j=0}\rangle\Big| \\
\leqslant& \frac{1}{2^k}\int\int\cdots\int d\Lambda_{i_1} d\Lambda_{i_2}\cdots d\Lambda_{i_k} \prod_{m=1}^{k}\mu_{i_m}(\Lambda_{i_m}) \prod_{i_s\in\mathcal{K}}|\langle M_{x_{i_s}=0} + M_{x_{i_s}=1}\rangle| \prod_{j\in\bar{\mathcal{K}}}|\langle M_{x_j=0}\rangle| \\
\leqslant& \frac{1}{2^k}\int\int\cdots\int d\Lambda_{i_1} d\Lambda_{i_2}\cdots d\Lambda_{i_k} \prod_{m=1}^{k}\mu_{i_m}(\Lambda_{i_m}) \prod_{i_s\in\mathcal{K}}|\langle M_{x_{i_s}=0} + M_{x_{i_s}=1}\rangle|,
\end{aligned}$$

where $\Lambda_{i_m} = \{\lambda_{(i_m)_1}, \lambda_{(i_m)_2}, \cdots, \lambda_{(i_m)_{e_{i_m}}}\}$ denotes the set of local variables associated with the sources that connect to party A_{i_m}, $\mu_{i_m}(\Lambda_{i_m}) = \prod\limits_{l=1}^{e_{i_m}} \mu_{(i_m)_l}(\lambda_{(i_m)_l})$ and $d\Lambda_{i_m} = d\lambda_{(i_m)_1} d\lambda_{(i_m)_2}\cdots d\lambda_{(i_m)_{e_{i_m}}}$. Let $\langle\Delta^{\pm} M_{x_{i_s}}\rangle = \frac{1}{2}(\langle M_{x_{i_s}=0}\rangle \pm \langle M_{x_{i_s}=1}\rangle)$, then

$$|\mathcal{I}| \leqslant \int\int\cdots\int d\Lambda_{i_1} d\Lambda_{i_2}\cdots d\Lambda_{i_k} \prod_{m=1}^{k}\mu_{i_m}(\Lambda_{i_m}) \prod_{i_s\in\mathcal{K}}|\langle\Delta^{+} M_{x_{i_s}}\rangle| \leqslant \prod_{i_s\in\mathcal{K}}\int d\Lambda_{i_s}\mu_{i_s}(\Lambda_{i_s})|\langle\Delta^{+} M_{x_{i_s}}\rangle|. \tag{A2}$$

Similarly, we have

$$|\mathcal{J}| = \Big|\int\int\cdots\int d\Lambda_{i_1} d\Lambda_{i_2}\cdots d\Lambda_{i_k} \prod_{m=1}^{k}\mu_{i_m}(\Lambda_{i_m}) \prod_{i_s\in\mathcal{K}}\langle\Delta^{-} M_{x_{i_s}}\rangle \prod_{j\in\bar{\mathcal{K}}}\langle M_{x_j=1}\rangle\Big| \leqslant \prod_{i_s\in\mathcal{K}}\int d\Lambda_{i_s}\mu_{i_s}(\Lambda_{i_s})|\langle\Delta^{-} M_{x_{i_s}}\rangle|. \tag{A3}$$

Using the Mahler inequality [1], inequalities (A2) and (A3) together lead to

$$\begin{aligned}
|\mathcal{I}|^{\frac{1}{k}} + |\mathcal{J}|^{\frac{1}{k}} \leqslant& [\prod_{i_s\in\mathcal{K}}\int d\Lambda_{i_s}\mu_{i_s}(\Lambda_{i_s})(|\langle\Delta^{+} M_{x_{i_s}}\rangle| + |\langle\Delta^{-} M_{x_{i_s}}\rangle|)]^{\frac{1}{k}} \\
\leqslant& [\prod_{i_s\in\mathcal{K}}\int d\Lambda_{i_s}\mu_{i_s}(\Lambda_{i_s})]^{\frac{1}{k}} \\
=& 1.
\end{aligned} \tag{A4}$$

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The second inequality in inequality (A4) is from the inequality $|\langle\Delta^{+}M_{x_{i_s}}\rangle| + |\langle\Delta^{-}M_{x_{i_s}}\rangle| = \max\{|\langle M_{x_{i_s}=0}\rangle|, |\langle M_{x_{i_s}=1}\rangle|\} \leqslant 1$ for $s = 1, 2, \cdots, k$; and the final equality in inequality (A4) is from the normalization condition of the probability distribution of hidden states. □

Appendix B Proof of Theorem 3

For arbitrary chain network scenario $\mathcal{C}ha(y)$ described in Figure 2, it is clear that (1) if y is odd, then $k_{\max} = \frac{y+1}{2}$ with the index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{1, 3, \cdots, y\}$; (2) if y is even, then $k_{\max} = \frac{y}{2}$ with the index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{2, 4, \cdots, y\}$. Denote by $\overline{\mathcal{K}} = \{1, 2, \cdots, y\} \setminus \mathcal{K}$ and $\mathcal{K}'$ the set of indexes from $\mathcal{K}$ with the biggest index excluded. For example, when y is odd, $\mathcal{K}' = \{1, 3, \cdots, y-2\}$. In Figure 2, if y is odd, by Eqs.(3) and (18) in the main text,

$$\begin{aligned}
\mathcal{I}_s =& \frac{1}{2^{\frac{y+1}{2}}} \times \langle \prod_{i\in\mathcal{K}} (M_{x_i=0} + M_{x_i=1}) \prod_{j\in\overline{\mathcal{K}}} M_{x_j=0}\rangle \\
=& \frac{1}{2^{\frac{y+1}{2}}} \times \langle (M_{x_1=0} + M_{x_1=1}) M_{x_2=0} \cdots (M_{x_{y-2}=0} + M_{x_{y-2}=1}) M_{x_{y-1}=0} (M_{x_y=0} + M_{x_y=1})\rangle \\
=& \frac{1}{2^{\frac{y+1}{2}}} \times \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_1}_{x_1=0}; s) + \hat{O}(\alpha^{\lambda_1}_{x_1=1}; s)) \otimes \hat{O}(\alpha^{\lambda_1}_{x_2=0}; s) \otimes \hat{O}(\alpha^{\lambda_2}_{x_2=0}; s) \\
&\otimes ((\hat{O}(\alpha^{\lambda_2}_{x_3=0}; s) \otimes \hat{O}(\alpha^{\lambda_3}_{x_3=0}; s)) + (\hat{O}(\alpha^{\lambda_2}_{x_3=1}; s) \otimes \hat{O}(\alpha^{\lambda_3}_{x_3=1}; s))) \\
&\otimes \hat{O}(\alpha^{\lambda_3}_{x_4=0}; s) \otimes \hat{O}(\alpha^{\lambda_4}_{x_4=0}; s) \otimes \cdots \otimes ((\hat{O}(\alpha^{\lambda_{y-3}}_{x_{y-2}=0}; s) \otimes \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=0}; s)) \\
&+ (\hat{O}(\alpha^{\lambda_{y-3}}_{x_{y-2}=1}; s) \otimes \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=1}; s))) \otimes \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0}; s) \otimes \hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0}; s) \\
&\otimes (\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0}; s) + \hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1}; s))]\rho^{A_1A_2} \otimes \rho^{A_2A_3} \otimes \cdots \otimes \rho^{A_{y-1}A_y}\} \\
=& \frac{1}{2^{\frac{y+1}{2}}} \times \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_1}_{x_1=0}; s) + \hat{O}(\alpha^{\lambda_1}_{x_1=1}; s)) \otimes \hat{O}(\alpha^{\lambda_1}_{x_2=0}; s)]\rho^{A_1A_2}\} \\
&\times \mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_2}_{x_2=0}; s) \otimes (\hat{O}(\alpha^{\lambda_2}_{x_3=0}; s) + \hat{O}(\alpha^{\lambda_2}_{x_3=1}; s))]\rho^{A_2A_3}\} \\
&\times \cdots \times \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=0}; s) + \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=1}; s)) \otimes \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0}; s)]\rho^{A_{y-2}A_{y-1}}\} \\
&\times \mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0}; s) \otimes (\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0}; s) + \hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1}; s))]\rho^{A_{y-1}A_y}\}.
\end{aligned} \tag{B1}$$

Similar to Eq.(B1), when y is even, we have

$$\begin{aligned}
\mathcal{I}_s =& \frac{1}{2^{\frac{y}{2}}} \times \langle \prod_{i\in\mathcal{K}} (M_{x_i=0} + M_{x_i=1}) \prod_{j\in\overline{\mathcal{K}}} M_{x_j=0}\rangle \\
=& \frac{1}{2^{\frac{y}{2}}} \times \langle M_{x_1=0}(M_{x_2=0} + M_{x_2=1}) \cdots M_{x_{y-1}=0}(M_{x_y=0} + M_{x_y=1})\rangle \\
=& \frac{1}{2^{\frac{y}{2}}} \times \mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_1}_{x_1=0}; s) \otimes (\hat{O}(\alpha^{\lambda_1}_{x_2=0}; s) + \hat{O}(\alpha^{\lambda_1}_{x_2=1}; s))]\rho^{A_1A_2}\} \\
&\times \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_2}_{x_2=0}; s) + \hat{O}(\alpha^{\lambda_2}_{x_2=1}; s)) \otimes \hat{O}(\alpha^{\lambda_2}_{x_3=0}; s)]\rho^{A_2A_3}\} \\
&\times \cdots \times \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=0}; s) + \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=1}; s)) \otimes \hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0}; s)]\rho^{A_{y-2}A_{y-1}}\} \\
&\times \mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0}; s) \otimes (\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0}; s) + \hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1}; s))]\rho^{A_{y-1}A_y}\}.
\end{aligned} \tag{B2}$$

By combining Eqs.(B1) and (B2), for arbitrary chain network $\mathcal{C}ha(y)$,

$$\begin{aligned}
\mathcal{I}_s =& \frac{1}{2^{k_{\max}}} \times \langle \prod_{i\in\mathcal{K}} (M_{x_i=0} + M_{x_i=1}) \prod_{j\in\overline{\mathcal{K}}} M_{x_j=0}\rangle \\
=& \frac{1}{2^{k_{\max}}} \times \prod_{i\in\mathcal{K}'} \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0}; s) + \hat{O}(\alpha^{\lambda_i}_{x_i=1}; s)) \otimes \hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0}; s)]\rho^{A_iA_{i+1}}\} \\
&\times \prod_{j\in\overline{\mathcal{K}}} \mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=0}; s) \otimes (\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0}; s) + \hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1}; s))]\rho^{A_jA_{j+1}}\}.
\end{aligned} \tag{B3}$$

Now, by Eqs.(7), (19), (22), (24), (27) and (29) in the main text, for $-1 < s \leqslant 0$, we get

$$\begin{aligned}
&\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)+\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}\\
=&\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}+\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_i}_{x_i=1};s)\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}\\
=&\mathrm{Tr}\{[((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)+sI)\otimes((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)+sI)]\rho^{A_iA_{i+1}}\}\\
&+\mathrm{Tr}\{[((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)+sI)\otimes((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)+sI)]\rho^{A_iA_{i+1}}\}\\
=&(1-s)^2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}+s(1-s)\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)\otimes I]\rho^{A_iA_{i+1}}\}\\
&+s(1-s)\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}+s^2+(1-s)^2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\\
&\rho^{A_iA_{i+1}}\}+s(1-s)\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)\otimes I]\rho^{A_iA_{i+1}}\}+s(1-s)\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}+s^2\\
=&\frac{\pi^2(1-s)^4}{4}[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0};s)+Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)]\\
&+\frac{\pi s(1-s)^2}{2}[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0};s)+Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1};s)+2Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]+2s^2\\
=&\frac{\pi^2(1-s)^4}{4}C^{+}_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)+\frac{\pi s(1-s)^2}{2}D^{+}_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)+2s^2\\
=&E^{+}_{\rho^{A_iA_{i+1}}}(\frac{\pi^2(1-s)^4}{4},\frac{\pi s(1-s)^2}{2},2s^2;\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)\\
=&E^{+}_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)};s).
\end{aligned}\tag{B4}$$

Similarly,

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)+\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}=E^{+}_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s).\tag{B5}$$

Let

$$F^{+}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)=\frac{1}{2^{k_{\max}}}\prod_{i\in\mathcal{K}'}E^{+}_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^{+}_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s),\tag{B6}$$

then Eq.(B3) implies that

$$\mathcal{I}_s=\frac{1}{2^{k_{\max}}}\times\prod_{i\in\mathcal{K}'}E^{+}_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^{+}_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s)=F^{+}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)},\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s).\tag{B7}$$

While if $s\leqslant -1$, we have

$$\begin{aligned}
&\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)+\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}\\
=&\mathrm{Tr}\{[(2\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)-I)\otimes(2\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)-I)]\rho^{A_iA_{i+1}}\}+\mathrm{Tr}\{[(2\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)-I)\otimes(2\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=0};s)-I)]\rho^{A_iA_{i+1}}\}\\
=&\pi^2(1-s)^2[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0};s)+Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)]\\
&-\pi(1-s)[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0};s)+Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1};s)+2Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]+2\\
=&\pi^2(1-s)^2C^{+}_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)-\pi(1-s)D^{+}_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)+2\\
=&E^{+}_{\rho^{A_iA_{i+1}}}(\pi^2(1-s)^2,-\pi(1-s),2;\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=0},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=0};s)\\
=&E^{+}_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)};s)
\end{aligned}\tag{B8}$$

and

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)+\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}=E^{+}_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s).\tag{B9}$$

Therefore, Eqs.(B3) and (B6) imply that

$$\mathcal{I}_s=\frac{1}{2^{k_{\max}}}\times\prod_{i\in\mathcal{K}'}E^{+}_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^{+}_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s)=F^{+}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)},\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)};s).\tag{B10}$$

So, we obtain the desired expression of $\mathcal{I}_s$ as in Theorem 3, that is, Eqs.(B7) and (B10).

By a similar argument to Eq.(B3), it can be derived from Eqs.(4) and (18) in the main text that

$$\begin{aligned}
\mathcal{J}_s=&\frac{1}{2^{k_{\max}}}\times\langle\prod_{i\in\mathcal{K}}(M_{x_i=0}-M_{x_i=1})\prod_{j\in\overline{\mathcal{K}}}M_{x_j=1}\rangle\\
=&\frac{1}{2^{k_{\max}}}\times\prod_{i\in\mathcal{K}'}\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}\\
&\times\prod_{j\in\overline{\mathcal{K}}}\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=1};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}.
\end{aligned}\tag{B11}$$

Now, using Eqs.(7), (19), (23), (25), (26), (27) and (29) in the main text, we find that for $-1 < s \leqslant 0$,

$$\begin{aligned}
&\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}\\
=&\mathrm{Tr}\{[((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)+sI)\otimes((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)+sI)]\rho^{A_iA_{i+1}}\}\\
&-\mathrm{Tr}\{[((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)+sI)\otimes((1-s)\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)+sI)]\rho^{A_iA_{i+1}}\}\\
=&(1-s)^2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}+s(1-s)\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)\otimes I]\rho^{A_iA_{i+1}}\}\\
&+s(1-s)\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}+s^2-(1-s)^2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\\
&\rho^{A_iA_{i+1}}\}-s(1-s)\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)\otimes I]\rho^{A_iA_{i+1}}\}-s(1-s)\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}-s^2\\
=&\frac{\pi^2(1-s)^4}{4}[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1};s)-Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)]\\
&+\frac{\pi s(1-s)^2}{2}[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0};s)-Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1};s)]\\
=&\frac{\pi^2(1-s)^4}{4}C^-_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)+\frac{\pi s(1-s)^2}{2}D^-_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)\\
=&E^-_{\rho^{A_iA_{i+1}}}(\frac{\pi^2(1-s)^4}{4},\frac{\pi s(1-s)^2}{2},0;\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)\\
=&E^-_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^0_1;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)};s),
\end{aligned}\tag{B12}$$

and similarly,

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=1};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}=\quad E^-_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^0_1;\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s).\tag{B13}$$

Define

$$F^-(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)=\frac{1}{2^{k_{\max}}}\prod_{i\in\mathcal{K}'}E^-_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^-_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s),\tag{B14}$$

then Eq.(B11) immediately yields

$$\mathcal{J}_s=\frac{1}{2^{k_{\max}}}\times\prod_{i\in\mathcal{K}'}E^-_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^0_1;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^-_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^0_1;\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s)=F^-(\boldsymbol{\zeta}^0_1;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)},\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s).\tag{B15}$$

While when $s\leqslant -1$, we have

$$\begin{aligned}
&\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}\\
=&\mathrm{Tr}\{[(2\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)-I)\otimes(2\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)-I)]\rho^{A_iA_{i+1}}\}-\mathrm{Tr}\{[(2\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)-I)\otimes(2\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)-I)]\rho^{A_iA_{i+1}}\}\\
=&4\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}-2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=0};s)\otimes I]\rho^{A_iA_{i+1}}\}\\
&-2\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}+1-4\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}\\
&+2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_i}_{x_i=1};s)\otimes I]\rho^{A_iA_{i+1}}\}+2\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}-1\\
=&\pi^2(1-s)^2[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1};s)-Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)]\\
&-\pi(1-s)[Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0};s)-Q_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=1};s)]\\
=&\pi^2(1-s)^2C^-_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)-\pi(1-s)D^-_{\rho^{A_iA_{i+1}}}(\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)\\
=&E^-_{\rho^{A_iA_{i+1}}}(\pi^2(1-s)^2,-\pi(1-s),0;\alpha^{\lambda_i}_{x_i=0},\alpha^{\lambda_i}_{x_{i+1}=1},\alpha^{\lambda_i}_{x_i=1},\alpha^{\lambda_i}_{x_{i+1}=1};s)\\
=&E^-_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)};s)
\end{aligned}\tag{B16}$$

and

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=1};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}=E^-_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s).\tag{B17}$$

By Eqs.(B11) and (B14), we obtain that

$$\mathcal{J}_s=\frac{1}{2^{k_{\max}}}\times\prod_{i\in\mathcal{K}'}E^-_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)};s)\times\prod_{j\in\overline{\mathcal{K}}}E^-_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s)=F^-(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)},\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)};s).\tag{B18}$$

Therefore, we arrive at the desired expression for $\mathcal{J}_s$ stated in Theorem 3, namely, Eqs. (B15) and (B18), which completes the proof. □

Appendix C Proof of Theorem 4

For a star network $\mathcal{S}(y)$ as in Figure 9, it is clear that the number of maximal independent parties $k_{\max} = y-1$, and the corresponding index set of maximal independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{1, 2, \cdots, y-1\}$. By Eqs.(3) and (18) in the main text, we have

$$\begin{aligned}\mathcal{I}_s =&\frac{1}{2^{y-1}}\langle (M_{x_1=0}+M_{x_1=1})(M_{x_2=0}+M_{x_2=1})\cdots(M_{x_{y-1}=0}+M_{x_{y-1}=1})M_{x_y=0}\rangle\\ =&\frac{1}{2^{y-1}}\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_1}_{x_1=0};s)+\hat{O}(\alpha^{\lambda_1}_{x_1=1};s))\otimes\cdots\otimes(\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)+\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1};s))\\ &\otimes\hat{O}(\alpha^{\lambda_1}_{x_y=0};s)\otimes\hat{O}(\alpha^{\lambda_2}_{x_y=0};s)\otimes\cdots\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0};s)](\rho^{A_1A_y}\otimes\cdots\otimes\rho^{A_{y-1}A_y})\}\\ =&\frac{1}{2^{y-1}}\times\prod_{j\in\mathcal{K}}\{\mathrm{Tr}[((\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)+\hat{O}(\alpha^{\lambda_j}_{x_j=1};s))\otimes\hat{O}(\alpha^{\lambda_j}_{x_y=0};s))\rho^{A_jA_y}]\}.\end{aligned}\tag{C1}$$

Analogous to Eqs.(B4) and (B8), when $-1 < s \leqslant 0$,

$$\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)+\hat{O}(\alpha^{\lambda_j}_{x_j=1};s))\otimes\hat{O}(\alpha^{\lambda_j}_{x_y=0};s)]\rho^{A_jA_y}\} = E^{+}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(j,y,j)}_{(0,0,1,0)};s);\tag{C2}$$

and when $s \leqslant -1$,

$$\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)+\hat{O}(\alpha^{\lambda_j}_{x_j=1};s))\otimes\hat{O}(\alpha^{\lambda_j}_{x_y=0};s)]\rho^{A_jA_y}\} = E^{+}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(j,y,j)}_{(0,0,1,0)};s).\tag{C3}$$

Substituting Eqs.(C2) and (C3) into Eq.(C1), we find that for $-1 < s \leqslant 0$,

$$\mathcal{I}_s = \frac{1}{2^{y-1}}\times\prod_{j\in\mathcal{K}}E^{+}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(j,y,j)}_{(0,0,1,0)};s);\tag{C4}$$

while for $s \leqslant -1$,

$$\mathcal{I}_s = \frac{1}{2^{y-1}}\times\prod_{j\in\mathcal{K}}E^{+}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(j,y,j)}_{(0,0,1,0)};s).\tag{C5}$$

Additionally, by Eqs.(2) and (18) in the main text,

$$\begin{aligned}\mathcal{J}_s =&\frac{1}{2^{y-1}}\langle (M_{x_1=0}-M_{x_1=1})(M_{x_2=0}-M_{x_2=1})\cdots(M_{x_{y-1}=0}-M_{x_{y-1}=1})M_{x_y=1}\rangle\\ =&\frac{1}{2^{y-1}}\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_1}_{x_1=0};s)-\hat{O}(\alpha^{\lambda_1}_{x_1=1};s))\otimes\cdots\otimes(\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)-\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1};s))\\ &\otimes\hat{O}(\alpha^{\lambda_1}_{x_y=1};s)\otimes\hat{O}(\alpha^{\lambda_2}_{x_y=1};s)\otimes\cdots\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1};s)](\rho^{A_1A_y}\otimes\cdots\otimes\rho^{A_{y-1}A_y})\}\\ =&\frac{1}{2^{y-1}}\times\prod_{j\in\mathcal{K}}\{\mathrm{Tr}[((\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_j=1};s))\otimes\hat{O}(\alpha^{\lambda_j}_{x_y=1};s))\rho^{A_jA_y}]\}.\end{aligned}\tag{C6}$$

Analogous to the cases in Eqs.(B12) and (B16), if $-1 < s \leqslant 0$, we have

$$\mathrm{Tr}[((\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_j=1};s))\otimes\hat{O}(\alpha^{\lambda_j}_{x_y=1};s))\rho^{A_jA_y}] = E^{-}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_1^{0};\boldsymbol{\eta}^{(j,y,j)}_{(0,1,1,1)};s);\tag{C7}$$

and if $s \leqslant -1$, we have

$$\mathrm{Tr}[((\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_j=1};s))\otimes\hat{O}(\alpha^{\lambda_j}_{x_y=1};s))\rho^{A_jA_y}] = E^{-}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_2^{0};\boldsymbol{\eta}^{(j,y,j)}_{(0,1,1,1)};s).\tag{C8}$$

Substituting Eqs.(C7) and (C8) into Eq.(C6), we get, for $-1 < s \leqslant 0$,

$$\mathcal{J}_s = \frac{1}{2^{y-1}}\times\prod_{j\in\mathcal{K}}E^{-}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_1^{0};\boldsymbol{\eta}^{(j,y,j)}_{(0,1,1,1)};s);\tag{C9}$$

while for $s \leqslant -1$,

$$J_s = \frac{1}{2^{y-1}}\times\prod_{j\in\mathcal{K}}E^{-}_{\rho^{A_jA_y}}(\boldsymbol{\zeta}_2^{0};\boldsymbol{\eta}^{(j,y,j)}_{(0,1,1,1)};s).\tag{C10}$$

Here, $E^{\pm}$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ and $\boldsymbol{\eta}$ are defined as in Eqs.(26)-(29) of the main text. The proof is completed.

□

Appendix D Proof of Theorem 5

In tree-shaped network $\mathcal{T}(m,f)$, it is clear that (1) if m is odd, then $k_{\max}=\frac{f^{m+1}-1}{f^2-1}$ with the index set of maximal independent parties $\mathcal{K}=\mathcal{K}_{\max}=\{1,\frac{1-f^{t-1}}{1-f}+1,\frac{1-f^{t-1}}{1-f}+2,\cdots,\frac{1-f^{t}}{1-f}\}_{t=3,5,\cdots,m}$; (2) if m is even, then $k_{\max}=\frac{f-f^{m+1}}{1-f^2}$ with the index set of maximal independent parties $\mathcal{K}=\mathcal{K}_{\max}=\{\frac{1-f^{t-1}}{1-f}+1,\frac{1-f^{t-1}}{1-f}+2,\cdots,\frac{1-f^{t}}{1-f}\}_{t=2,4,\cdots,m}$. Denote by $\overline{\mathcal{K}}=\{1,2,\ldots\frac{1-f^m}{1-f}\}\setminus\mathcal{K}$ and $\mathcal{K}'$ the set of indexes from $\mathcal{K}$ with the last layer of indexes excluded. For example, when m is odd, $\mathcal{K}'=\{1,\frac{1-f^{t-1}}{1-f}+1,\frac{1-f^{t-1}}{1-f}+2,\cdots,\frac{1-f^{t}}{1-f}\}_{t=3,5,\cdots,m-2}$. By Eqs.(3) and (18) in the main text, we have

$$\begin{aligned}\mathcal{I}_s=&\frac{1}{2^{k_{\max}}}\langle\prod_{i\in\mathcal{K}}(M_{x_i=0}+M_{x_i=1})\prod_{j\in\overline{\mathcal{K}}}M_{x_j=0}\rangle\\=&\frac{1}{2^{k_{\max}}}\times\prod_{j=(i-1)f+2}^{if+1}\prod_{i\in\mathcal{K}'}\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=0};s)+\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_{j-1}}_{x_j=0};s)]\rho^{A_iA_j}\}\\&\times\prod_{q=(p-1)f+2}^{pf+1}\prod_{p\in\overline{\mathcal{K}}}\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{q-1}}_{x_p=0};s)\otimes(\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=0};s)+\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=1};s))]\rho^{A_pA_q}\}.\end{aligned}\tag{D1}$$

Analogous to Eqs.(B4), (B5), (B8) and (B9), when $-1<s\leqslant 0$,

$$\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=0};s)+\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_{j-1}}_{x_j=0};s)]\rho^{A_iA_j}\}=E^+_{\rho^{A_iA_j}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,j,j-1)}_{(0,0,1,0)};s)\tag{D2}$$

and

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{q-1}}_{x_p=0};s)\otimes(\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=0};s)+\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=1};s))]\rho^{A_pA_q}\}E^+_{\rho^{A_pA_q}}(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(p,q,q-1)}_{(0,0,0,1)};s);\tag{D3}$$

while for $s\leqslant -1$,

$$\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=0};s)+\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_{j-1}}_{x_j=0};s)]\rho^{A_iA_j}\}=E^+_{\rho^{A_iA_j}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(i,j,j-1)}_{(0,0,1,0)};s)\tag{D4}$$

and

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{q-1}}_{x_p=0};s)\otimes(\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=0};s)+\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=1};s))]\rho^{A_pA_q}\}=E^+_{\rho^{A_pA_q}}(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(p,q,q-1)}_{(0,0,0,1)};s).\tag{D5}$$

Substituting Eqs.(D2)-(D5) into Eq.(D1) and defining

$$\begin{aligned}&F^+(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(i,j,j-1)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(p,q,q-1)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)\\=&\frac{1}{2^{k_{\max}}}\times\prod_{j=(i-1)f+2}^{if+1}\prod_{i\in\mathcal{K}'}E^+_{\rho^{A_iA_j}}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(i,j,j-1)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{q=(p-1)f+2}^{pf+1}\prod_{p\in\overline{\mathcal{K}}}E^+_{\rho^{A_pA_q}}(\boldsymbol{\zeta}_{v_2}^{v_1};\boldsymbol{\eta}^{(p,q,q-1)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s),\end{aligned}$$

we obtain for $-1<s\leqslant 0$,

$$\mathcal{I}_s=F^+(\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,j,j-1)}_{(0,0,1,0)},\boldsymbol{\eta}^{(p,q,q-1)}_{(0,0,0,1)};s);\tag{D6}$$

while if $s\leqslant -1$,

$$\mathcal{I}_s=F^+(\boldsymbol{\zeta}_2^{2};\boldsymbol{\eta}^{(i,j,j-1)}_{(0,0,1,0)},\boldsymbol{\eta}^{(p,q,q-1)}_{(0,0,0,1)};s).\tag{D7}$$

Similarly,

$$\begin{aligned}\mathcal{J}_s=&\frac{1}{2^{k_{\max}}}\langle\prod_{i\in\mathcal{K}}(M_{x_i=0}-M_{x_i=1})\prod_{j\in\overline{\mathcal{K}}}M_{x_j=1}\rangle\\=&\frac{1}{2^{k_{\max}}}\times\prod_{j=(i-1)f+2}^{if+1}\prod_{i\in\mathcal{K}'}\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_{j-1}}_{x_j=1};s)]\rho^{A_iA_j}\}\\&\times\prod_{q=(p-1)f+2}^{pf+1}\prod_{p\in\overline{\mathcal{K}}}\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{q-1}}_{x_p=1};s)\otimes(\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=0};s)-\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=1};s))]\rho^{A_pA_q}\}.\end{aligned}\tag{D8}$$

Analogous to the cases in Eqs.(B12), (B13), (B16) and (B17), when $-1<s\leqslant 0$,

$$\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_{j-1}}_{x_j=1};s)]\rho^{A_iA_j}\}=E^-_{\rho^{A_iA_j}}(\boldsymbol{\zeta}_1^{0};\boldsymbol{\eta}^{(i,j,j-1)}_{(0,1,1,1)};s)\tag{D9}$$

and

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{q-1}}_{x_p=1};s)\otimes(\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=0};s)-\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=1};s))]\rho^{A_pA_q}\}=E^-_{\rho^{A_pA_q}}(\boldsymbol{\zeta}_1^{0};\boldsymbol{\eta}^{(p,q,q-1)}_{(1,0,1,1)};s);\tag{D10}$$

while if $s\leqslant -1$,

$$\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_{j-1}}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_{j-1}}_{x_j=1};s)]\rho^{A_iA_j}\}=E^-_{\rho^{A_iA_j}}(\boldsymbol{\zeta}_2^{0};\boldsymbol{\eta}^{(i,j,j-1)}_{(0,1,1,1)};s)\tag{D11}$$

and

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{q-1}}_{x_p=1};s)\otimes(\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=0};s)-\hat{O}(\alpha^{\lambda_{q-1}}_{x_q=1};s))]\rho^{A_pA_q}\}=E^{-}_{\rho^{A_pA_q}}(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(p,q,q-1)}_{(1,0,1,1)};s). \tag{D12}$$

Substituting Eqs.(D9)-(D12) into Eq.(D8) and defining

$$\begin{aligned}&F^{-}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(i,j,j-1)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(p,q,q-1)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)\\ =&\frac{1}{2^{k_{\max}}}\times\prod_{j=(i-1)f+2}^{if+1}\prod_{i\in\mathcal{K}'}E^{-}_{\rho^{A_iA_j}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(i,j,j-1)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{q=(p-1)f+2}^{pf+1}\prod_{p\in\overline{\mathcal{K}}}E^{-}_{\rho^{A_pA_q}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(p,q,q-1)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s),\end{aligned}$$

we obtain for $-1<s\leqslant 0$,

$$\mathcal{J}_s=F^{-}(\boldsymbol{\zeta}^0_1;\boldsymbol{\eta}^{(i,j,j-1)}_{(0,1,1,1)},\boldsymbol{\eta}^{(p,q,q-1)}_{(1,0,1,1)};s); \tag{D13}$$

while for $s\leqslant -1$,

$$\mathcal{J}_s=F^{-}(\boldsymbol{\zeta}^0_2;\boldsymbol{\eta}^{(i,j,j-1)}_{(0,1,1,1)},\boldsymbol{\eta}^{(p,q,q-1)}_{(1,0,1,1)};s). \tag{D14}$$

Here, $E^{\pm}$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\boldsymbol{\eta}$ are defined as in Eqs.(26)-(29) of the main text. The proof is completed. □

Appendix E Proof of Theorem 6

In a cyclic network $\mathcal{C}yc(y)$ (see Figure 15), there are y parties A_1, A_2, $\cdots$, A_y with A_i and A_{i+1} sharing a source S_i for $i=1,2,\cdots,y-1$, A_y and A_1 sharing a source S_y. Here, we have to address the scenarios where y is odd or even as two distinct cases in $\mathcal{C}yc(y)$.

(a) y is odd.

When y is odd, it is straightforward that $k_{\max}=\frac{y-1}{2}$ with the index set of maximal independent parties $\mathcal{K}=\mathcal{K}_{\max}=\{1,3,5,\cdots,y-2\}$. Let $\overline{\mathcal{K}}=\{1,2,\cdots,y\}\setminus\mathcal{K}$ and $\overline{\mathcal{K}}'=\overline{\mathcal{K}}\setminus\{y-1\}$, then by Eqs.(3) and (18) in the main text,

$$\begin{aligned}\mathcal{I}_s=&\frac{1}{2^{\frac{y-1}{2}}}\langle(M_{x_1=0}+M_{x_1=1})M_{x_2=0}\cdots(M_{x_{y-2}=0}+M_{x_{y-2}=1})M_{x_{y-1}=0}M_{x_y=0}\rangle\\ =&\frac{1}{2^{\frac{y-1}{2}}}\mathrm{Tr}\{[((\hat{O}(\alpha^{\lambda_y}_{x_1=0};s)\otimes\hat{O}(\alpha^{\lambda_1}_{x_1=0};s))+(\hat{O}(\alpha^{\lambda_y}_{x_1=1};s)\otimes\hat{O}(\alpha^{\lambda_1}_{x_1=1};s)))\\ &\otimes(\hat{O}(\alpha^{\lambda_1}_{x_2=0};s)\otimes\hat{O}(\alpha^{\lambda_2}_{x_2=0};s))\otimes\cdots\otimes((\hat{O}(\alpha^{\lambda_{y-3}}_{x_{y-2}=0};s)\otimes\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=0};s))\\ &+(\hat{O}(\alpha^{\lambda_{y-3}}_{x_{y-2}=1};s)\otimes\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=1};s)))\otimes(\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s))\\ &\otimes(\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0};s)\otimes\hat{O}(\alpha^{\lambda_y}_{x_y=0};s))](\rho^{A_1A_2}\otimes\cdots\otimes\rho^{A_{y-1}A_y}\otimes\rho^{A_yA_1})\}\\ =&\frac{1}{2^{\frac{y-1}{2}}}\times\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_1}_{x_1=0};s)+\hat{O}(\alpha^{\lambda_1}_{x_1=1};s))\otimes\hat{O}(\alpha^{\lambda_1}_{x_2=0};s)]\rho^{A_1A_2}\}\\ &\times\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_2}_{x_2=0};s)\otimes(\hat{O}(\alpha^{\lambda_2}_{x_3=0};s)+\hat{O}(\alpha^{\lambda_2}_{x_3=1};s))]\rho^{A_2A_3}\}\\ &\times\cdots\times\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=0};s)+\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-2}=1};s))\otimes\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0};s)]\rho^{A_{y-2}A_{y-1}}\}\\ &\times\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}\times\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_y}_{x_y=0};s)\otimes(\hat{O}(\alpha^{\lambda_y}_{x_1=0};s)+\hat{O}(\alpha^{\lambda_y}_{x_1=1};s))]\rho^{A_yA_1}\}\\ =&\frac{1}{2^{\frac{y-1}{2}}}\times\prod_{i\in\mathcal{K}}\mathrm{Tr}\{[((\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)+\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0};s)]\rho^{A_iA_{i+1}}\}\\ &\times\prod_{j\in\overline{\mathcal{K}}'}\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=0};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)+\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}\times\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}.\end{aligned} \tag{E1}$$

Additionally, in Eq.(E1), $y+1$ actually represents 1 due to the cyclic nature of the network.

By Eqs.(7), (19), (22), (24), (26)-(29) in the main text, if $-1<s\leqslant 0$, then

$$\begin{aligned}&\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}\\ =&\mathrm{Tr}\{[((1-s)\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)+sI)\otimes((1-s)\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_y=0};s)+sI)]\rho^{A_{y-1}A_y}\}\\ =&(1-s)^2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}+s(1-s)\\ &\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes I]\rho^{A_{y-1}A_y}\}+s(1-s)\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}+s^2\\ =&\frac{\pi^2(1-s)^4}{4}Q_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)+\frac{\pi s(1-s)^2}{2}[Q_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)+Q_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]+s^2\\ =&\frac{\pi^2(1-s)^4}{8}C^{+}_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0},\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)+\frac{\pi s(1-s)^2}{4}D^{+}_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0},\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)+s^2\\ =&E^{+}_{\rho^{A_{y-1}A_y}}(\frac{\pi^2(1-s)^4}{8},\frac{\pi s(1-s)^2}{4},s^2;\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0},\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)\\ =&E^{+}_{\rho^{A_{y-1}A_y}}(\frac{1}{2}\times\boldsymbol{\zeta}^{2s^2}_1;\boldsymbol{\eta}^{(y-1,y,y-1)}_{(0,0,0,0)};s);\end{aligned} \tag{E2}$$

while if $s \leqslant -1$, then

$$\begin{aligned}
&\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}\\
=&\mathrm{Tr}\{[(2\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)-I)\otimes(2\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_y=0};s)-I)]\rho^{A_{y-1}A_y}\}\\
=&4\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}-2\mathrm{Tr}\{[\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)\otimes I]\rho^{A_{y-1}A_y}\}-2\mathrm{Tr}\{[I\otimes\hat{\Pi}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]\rho^{A_{y-1}A_y}\}+1\\
=&\pi^2(1-s)^2Q_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)-\pi(1-s)[Q_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0};s)+Q_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_y=0};s)]+1\\
=&\frac{\pi^2(1-s)^2}{2}C^+_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0},\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)-\frac{\pi(1-s)}{2}D^+_{\rho^{A_{y-1}A_y}}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0},\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)+1\\
=&E^+_{\rho^{A_{y-1}A_y}}(\frac{\pi^2(1-s)^2}{2},-\frac{\pi(1-s)}{2},1;\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0},\alpha^{\lambda_{y-1}}_{x_{y-1}=0},\alpha^{\lambda_{y-1}}_{x_y=0};s)\\
=&E^+_{\rho^{A_{y-1}A_y}}(\frac{1}{2}\times\boldsymbol{\zeta}_2^2;\boldsymbol{\eta}^{(y-1,y,y-1)}_{(0,0,0,0)};s).
\end{aligned}\tag{E3}$$

Now, substituting Eqs.(B4)-(B5), (B8)-(B9), (E2)-(E3) into Eq.(E1) and defining

$$\begin{aligned}
&F^+(\boldsymbol{\zeta}^{v_1}_{v_2},\boldsymbol{\zeta}^{v_3}_{v_2};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)},\boldsymbol{\eta}^{(y-1,y,y-1)}_{(\eta_9,\eta_{10},\eta_{11},\eta_{12})};s)\\
=&\frac{1}{2^{\frac{y-1}{2}}}\times\prod_{i\in\mathcal{K}}E^+_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{j\in\overline{\mathcal{K}}'}E^+_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)\times E^+_{\rho^{A_{y-1}A_y}}(\frac{1}{2}\times\boldsymbol{\zeta}^{v_3}_{v_2};\boldsymbol{\eta}^{(y-1,y,y-1)}_{(\eta_9,\eta_{10},\eta_{11},\eta_{12})};s),
\end{aligned}$$

we obtain for $-1 < s \leqslant 0$,

$$\mathcal{I}_s=F^+(\boldsymbol{\zeta}_1^{2s^2},\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)},\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)},\boldsymbol{\eta}^{(y-1,y,y-1)}_{(0,0,0,0)};s);\tag{E4}$$

while for $s \leqslant -1$,

$$\mathcal{I}_s=F^+(\boldsymbol{\zeta}_2^2,\boldsymbol{\zeta}_2^2;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)},\boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)},\boldsymbol{\eta}^{(y-1,y,y-1)}_{(0,0,0,0)};s).\tag{E5}$$

Similarly, by Eqs.(4) and (18) in the main text,

$$\begin{aligned}
\mathcal{J}_s=&\frac{1}{2^{\frac{y-1}{2}}}\langle(M_{x_1=0}-M_{x_1=1})M_{x_2=1}\cdots(M_{x_{y-2}=0}-M_{x_{y-2}=1})M_{x_{y-1}=1}M_{x_y=1}\rangle\\
=&\frac{1}{2^{\frac{y-1}{2}}}\times\prod_{i\in\mathcal{K}}\mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0};s)-\hat{O}(\alpha^{\lambda_i}_{x_i=1};s))\otimes\hat{O}(\alpha^{\lambda_i}_{x_{i+1}=1};s)]\rho^{A_iA_{i+1}}\}\\
&\times\prod_{j\in\overline{\mathcal{K}}'}\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=1};s)\otimes(\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0};s)-\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1};s))]\rho^{A_jA_{j+1}}\}\times\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1};s)]\rho^{A_{y-1}A_y}\}.
\end{aligned}\tag{E6}$$

Likewise, in Eq.(E6), $y+1$ actually represents 1 due to the cyclic nature of the network.

Analogous to Eqs.(E2) and (E3), when $-1 < s \leqslant 0$,

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1};s)]\rho^{A_{y-1}A_y}\}=E^+_{\rho^{A_{y-1}A_y}}(\frac{1}{2}\times\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(y-1,y,y-1)}_{(1,1,1,1)};s);\tag{E7}$$

while if $s \leqslant -1$,

$$\mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1};s)\otimes\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1};s)]\rho^{A_{y-1}A_y}\}=E^+_{\rho^{A_{y-1}A_y}}(\frac{1}{2}\times\boldsymbol{\zeta}_2^2;\boldsymbol{\eta}^{(y-1,y,y-1)}_{(1,1,1,1)};s).\tag{E8}$$

Substituting Eqs.(B12)-(B13), (B16)-(B17), (E7)-(E8) into Eq.(E6) and defining

$$\begin{aligned}
&F^-(\boldsymbol{\zeta}^{v_1}_{v_2},\boldsymbol{\zeta}^{v_3}_{v_2};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)},\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)},\boldsymbol{\eta}^{(y-1,y,y-1)}_{(\eta_9,\eta_{10},\eta_{11},\eta_{12})};s)\\
=&\frac{1}{2^{\frac{y-1}{2}}}\times\prod_{i\in\mathcal{K}}E^-_{\rho^{A_iA_{i+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)};s)\times\prod_{j\in\overline{\mathcal{K}}'}E^-_{\rho^{A_jA_{j+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2};\boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)};s)\\
&\times E^+_{\rho^{A_{y-1}A_y}}(\frac{1}{2}\times\boldsymbol{\zeta}^{v_3}_{v_2};\boldsymbol{\eta}^{(y-1,y,y-1)}_{(\eta_9,\eta_{10},\eta_{11},\eta_{12})};s),
\end{aligned}$$

we obtain for $-1 < s \leqslant 0$,

$$\mathcal{J}_s=F^-(\boldsymbol{\zeta}_1^0,\boldsymbol{\zeta}_1^{2s^2};\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)},\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)},\boldsymbol{\eta}^{(y-1,y,y-1)}_{(1,1,1,1)};s);\tag{E9}$$

while for $s \leqslant -1$,

$$\mathcal{J}_s=F^-(\boldsymbol{\zeta}_2^0,\boldsymbol{\zeta}_2^2;\boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)},\boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)},\boldsymbol{\eta}^{(y-1,y,y-1)}_{(1,1,1,1)};s).\tag{E10}$$

Here, $E^{\pm}$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\boldsymbol{\eta}$ are defined as in Eqs.(26)-(29) of the main text. □

(b) y is even.

For cyclic network $\mathcal{C}yc(y)$, if y is even, then $k_{\max} = \frac{y}{2}$ with the index set of maximal set of independent parties $\mathcal{K} = \mathcal{K}_{\max} = \{2, 4, 6, \cdots, y\}$. Denote by $\overline{\mathcal{K}} = \{1, 2, \cdots, y\} \setminus \mathcal{K}$. By Eqs.(3) and (18),

$$\begin{aligned}\mathcal{I}_s =& \frac{1}{2^{\frac{y}{2}}} \langle (M_{x_1=0} + M_{x_1=1}) M_{x_2=0} \cdots (M_{x_{y-1}=0} + M_{x_{y-1}=1}) M_{x_y=0} \rangle \\ =& \frac{1}{2^{\frac{y}{2}}} \mathrm{Tr}\{[((\hat{O}(\alpha^{\lambda_y}_{x_1=0}; s) \otimes \hat{O}(\alpha^{\lambda_1}_{x_1=0}; s)) + (\hat{O}(\alpha^{\lambda_y}_{x_1=1}; s) \otimes \hat{O}(\alpha^{\lambda_1}_{x_1=1}; s))) \\ & \otimes (\hat{O}(\alpha^{\lambda_1}_{x_2=0}; s) \otimes \hat{O}(\alpha^{\lambda_2}_{x_2=0}; s)) \otimes \cdots \otimes ((\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0}; s) \otimes \hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0}; s)) \\ & + (\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=1}; s) \otimes \hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1}; s))) \otimes (\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=0}; s) \otimes \hat{O}(\alpha^{\lambda_y}_{x_y=0}; s))] \\ & (\rho^{A_1 A_2} \otimes \cdots \otimes \rho^{A_{y-1} A_y} \otimes \rho^{A_y A_1})\} \\ =& \frac{1}{2^{\frac{y}{2}}} \prod_{i \in \mathcal{K}} \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0}; s) + \hat{O}(\alpha^{\lambda_i}_{x_i=1}; s)) \otimes \hat{O}(\alpha^{\lambda_i}_{x_{i+1}=0}; s)] \rho^{A_i A_{i+1}}\} \\ & \times \prod_{j \in \overline{\mathcal{K}}} \mathrm{Tr}\{[\hat{O}(\alpha^{\lambda_j}_{x_j=0}; s) \otimes (\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0}; s) + \hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1}; s))] \rho^{A_j A_{j+1}}\},\end{aligned} \tag{E11}$$

with $y + 1$ actually represents 1 (mod y). Substituting Eqs.(B4), (B5), (B8), (B9) into Eq.(E11) and defining

$$F^+(\boldsymbol{\zeta}^{v_1}_{v_2}; \boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)}, \boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)}; s) = \frac{1}{2^{\frac{y}{2}}} \prod_{i \in \mathcal{K}'} E^+_{\rho^{A_i A_{i+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2}; \boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)}; s) \times \prod_{j \in \overline{\mathcal{K}}} E^+_{\rho^{A_j A_{j+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2}; \boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)}; s),$$

we obtain for $-1 < s \leqslant 0$,

$$\mathcal{I}_s = F^+(\boldsymbol{\zeta}^{2s^2}_1; \boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)}, \boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)}; s); \tag{E12}$$

while for $s \leqslant -1$,

$$\mathcal{I}_s = F^+(\boldsymbol{\zeta}^2_2; \boldsymbol{\eta}^{(i,i+1,i)}_{(0,0,1,0)}, \boldsymbol{\eta}^{(j,j+1,j)}_{(0,0,0,1)}; s). \tag{E13}$$

Similarly, by Eqs.(4) and (18),

$$\begin{aligned}\mathcal{J}_s =& \frac{1}{2^{\frac{y}{2}}} \langle (M_{x_1=0} - M_{x_1=1}) M_{x_2=1} \cdots (M_{x_{y-1}=0} - M_{x_{y-1}=1}) M_{x_y=1} \rangle \\ =& \frac{1}{2^{\frac{y}{2}}} \mathrm{Tr}\{[((\hat{O}(\alpha^{\lambda_y}_{x_1=0}; s) \otimes \hat{O}(\alpha^{\lambda_1}_{x_1=0}; s)) - (\hat{O}(\alpha^{\lambda_y}_{x_1=1}; s) \otimes \hat{O}(\alpha^{\lambda_1}_{x_1=1}; s))) \\ & \otimes (\hat{O}(\alpha^{\lambda_1}_{x_2=1}; s) \otimes \hat{O}(\alpha^{\lambda_2}_{x_2=1}; s)) \otimes \cdots \otimes ((\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=0}; s) \otimes \hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=0}; s)) \\ & - (\hat{O}(\alpha^{\lambda_{y-2}}_{x_{y-1}=1}; s) \otimes \hat{O}(\alpha^{\lambda_{y-1}}_{x_{y-1}=1}; s))) \otimes (\hat{O}(\alpha^{\lambda_{y-1}}_{x_y=1}; s) \otimes \hat{O}(\alpha^{\lambda_y}_{x_y=1}; s))] \\ & (\rho^{A_1 A_2} \otimes \cdots \otimes \rho^{A_{y-1} A_y} \otimes \rho^{A_y A_1})\} \\ =& \frac{1}{2^{\frac{y}{2}}} \times \prod_{i \in \mathcal{K}} \mathrm{Tr}\{[(\hat{O}(\alpha^{\lambda_i}_{x_i=0}; s) - \hat{O}(\alpha^{\lambda_i}_{x_i=1}; s)) \otimes \hat{O}(\alpha^{\lambda_i}_{x_{i+1}=1}; s)] \rho^{A_i A_{i+1}}\} \\ & \times \prod_{j \in \overline{\mathcal{K}}} \mathrm{Tr}[(\hat{O}(\alpha^{\lambda_j}_{x_j=1}; s) \otimes (\hat{O}(\alpha^{\lambda_j}_{x_{j+1}=0}; s) - \hat{O}(\alpha^{\lambda_j}_{x_{j+1}=1}; s))) \rho^{A_j A_{j+1}}]\}.\end{aligned} \tag{E14}$$

Substituting Eqs.(B12), (B13), (B16), (B17) into Eq.(E14) and defining

$$F^-(\boldsymbol{\zeta}^{v_1}_{v_2}; \boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)}, \boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)}; s) = \frac{1}{2^{\frac{y}{2}}} \prod_{i \in \mathcal{K}'} E^-_{\rho^{A_i A_{i+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2}; \boldsymbol{\eta}^{(i,i+1,i)}_{(\eta_1,\eta_2,\eta_3,\eta_4)}; s) \times \prod_{j \in \overline{\mathcal{K}}} E^-_{\rho^{A_j A_{j+1}}}(\boldsymbol{\zeta}^{v_1}_{v_2}; \boldsymbol{\eta}^{(j,j+1,j)}_{(\eta_5,\eta_6,\eta_7,\eta_8)}; s),$$

we have for $-1 < s \leqslant 0$,

$$\mathcal{J}_s = F^-(\boldsymbol{\zeta}^0_1; \boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)}, \boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)}; s); \tag{E15}$$

while for $s \leqslant -1$,

$$\mathcal{J}_s = F^-(\boldsymbol{\zeta}^0_2; \boldsymbol{\eta}^{(i,i+1,i)}_{(0,1,1,1)}, \boldsymbol{\eta}^{(j,j+1,j)}_{(1,0,1,1)}; s). \tag{E16}$$

Here, $E^{\pm}$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\boldsymbol{\eta}$ are defined as in Eqs.(26)-(29) of the main text. The proof is completed. □

References

1 https://en.wikipedia.org/wiki/Mahler%27s inequality.