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Structure of finite games with symmetric potential functions

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Abstract The structure of finite games with symmetric potential functions is investigated in this paper. First, by constructing a basis of symmetric functions, a necessary and sufficient condition is presented to verify whether a finite game has symmetric potential functions. Then, a basis of the subspace of finite games with symmetric potential functions is provided. Next, the symmetric potential game is studied. By proving the symmetry of the potential function, a linear system is also presented for the verification of symmetric potential games, as well as a basis. Finally, as an application of the obtained results, the optimization of quasi-symmetric spatial games is considered. A sufficient condition for the utility design is given to turn the spatial game into a weighted potential game with the preassigned objective function as the potential function.

Keywords potential game, symmetric potential function, symmetric potential game, quasi-symmetric spatial game, utility design

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1 Introduction

A key feature of system optimization is the existence of a system objective function defined in the strategy profile space, which the system designer seeks to optimize, such as the sensor coverage problem [1] and graph coloring [2]. A wealth of optimization algorithms with the help of various theories has been proposed, such as the joint design framework of fronthaul and access links in cloud radio access networks [3], distributed optimization [4,5], distributed control [6,7], and game theory control [8,9]. Due to the existence and convergence of Nash equilibria, the potential game theory shows great advantages in solving the system optimization since a system equilibrium behavior is generally substituted by a Nash equilibrium [10–13]. Ref. [14] put forward a solution framework for the system optimization with the finite potential game as an interface of two separate steps: (1) designing a utility function for each individual such that the system becomes a potential game; (2) designing a learning rule such that when individuals optimize their own utility functions, the system objective function reaches its optimal value. In this paper, we only focus on the utility design. Please refer to [15–18] for the learning rules.

At present, the main idea of utility design is to construct utility functions using local information of the system objective function [19]. However, only the existence of a system equilibrium behavior can be guaranteed, not its optimality. Considering that an optimal value point of the potential function is also a Nash equilibrium which is called the optimal Nash equilibrium, it is essential to design the utility functions that can make the system objective function be the potential function to guarantee the correspondence between the optimal Nash equilibrium and the optimal system equilibrium behavior.

In the analysis of dynamics of spatial games, the underlying base game of pairwise interaction is always assumed to be a symmetric two-player game with two strategies to get stronger results because such a game has a symmetric potential function, and then, the associated spatial game is a finite potential game [20, 21]. Then, the models of the base game are extended to symmetric two-player base games with more than two strategies [22, 23]. Later on, Baron et al. went beyond the confines of symmetry and allowed for weighted graphs with asymmetric weights and asymmetric base games with symmetric potential functions [24]. Further analysis shows that if the base game is a finite game with symmetric potential functions, the spatial game is a weighted potential game and converges to the strategy profiles that maximize the potential function. Therefore, considering finite games with symmetric potential

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functions enlarges the application area of characteristics of symmetric potential games (being both a symmetric game and a potential game). However, there is no relevant research on the structure of finite games with symmetric potential functions.

The algebraic and geometric structures of finite potential games are useful for optimizing strategy design and designing for other control problems of games [25, 26]. Studying the game structure provides a new idea for the utility design, which can help us avoid the strong dependence of utility functions on the system objective function in constructive methods. Embedding the game space into a high-dimensional vector space is a common and effective method to study the game structure [27, 28]. Further, with the help of the semi-tensor product of matrices, the analysis of the game structure can be put into the framework of linear algebra [29–31]. Now, there have been many theoretical results regarding the game structure, especially, finite potential games and the relevant applications in system optimization [32,33]. For example, Ref. [34] modeled a network system as a finite potential game and gave a local information-based utility design by studying the structure of finite potential games. Ref. [35] firstly proposed and studied the structure of budget-balanced potential games and gave a utility design for a special network system. Ref. [36] provided a relatively comprehensive introduction to the potential game theory and presented its application in the utility design for radio resource allocation. However, none of the existing results can reveal the game structure of finite games with symmetric potential functions since the symmetry of the potential function makes the structure of utility functions more special, and none of the existing utility design methods can be adopted since the linear structure between the potential function and utility functions in spatial games is different from that in the existing potential game models.

In this paper, we explore the structure of finite games with symmetric potential functions, and apply the obtained results to the optimization of spatial games according to the solution framework presented in [14]. Main contributions of this paper include: (1) an algebraic verification condition of finite games with symmetric potential functions and a basis of the corresponding subspace; (2) a sufficient condition about the utility design in the optimization of spatial games. The algebraic verification condition is fundamental to using techniques of symmetric potential functions to real games because it answers when techniques are applicable. Moreover, it is more convenient and applicable in practice, which is reflected in two aspects: (1) the game verification problem is equivalently transformed to the existence problem of solutions of a linear system, and the linear system has a clear and concise mathematical structure; (2) a concise formula to calculate the symmetric potential function is given for the first time. The basis simplifies the utility design, which is reflected in two ways: (1) the designability of utility functions is transformed to the existence of solutions of a linear system; (2) a concise mathematical formula for calculating utility functions is given. This utility design method can guarantee the correspondence between the optimal system equilibrium behavior and the optimal Nash equilibrium, and can overcome the difficulty that the existing utility design methods can only ensure the existence of the system equilibrium behavior, but cannot guarantee its optimality. The results obtained in this paper are the further exploration of the application of game theory in system optimization, and broaden the application scope of finite potential games in optimization theory.

So far, we have talked about four kinds of finite games, namely, potential games, symmetric games, symmetric potential games and finite games with symmetric potential functions. The relationship between them is shown in Figure 1. A symmetric potential game has a symmetric potential, but a game with symmetric potential functions might not be a symmetric game. Consider the following case. Let p be a symmetric function defined in the space of strategy profiles. Set the utility function of player 1 to be $p + 1$, and the other $n - 1$ players' to be p. Then, the resulting game is a potential game with p as its symmetric potential function, but it is not a symmetric game. Hence, the set of finite games with symmetric potential functions is larger than the set of symmetric potential games.

The outline of this paper is as follows. In Section 2, we introduce the vector space structure of finite games. Section 3 gives the algebraic verification condition of finite games with symmetric potential functions as well as a basis of the corresponding subspace. Section 4 discusses the symmetric potential games. Section 5 considers the optimization of quasi-symmetric spatial games. Section 6 concludes the paper.

2 Preliminaries

For statement ease, we first give some notations. (i) The set of $m \times n$ real matrices is denoted by $\mathcal{M}_{m\times n}$. (ii) The set of columns (rows) of M is denoted by $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$), and its i-th column (row) is denoted by $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$). (iii) $\mathbf{1}_\ell$ is a column vector with all entries equal to 1, and $\mathbf{0}_{p\times q}$ is a $p \times q$ matrix with zero entries. (iv) $\mathbf{S}_n$ is the n-th order symmetric group.

The symmetry of finite games and the symmetry of functions are both described by means of $\mathbf{S}_n$, and the main

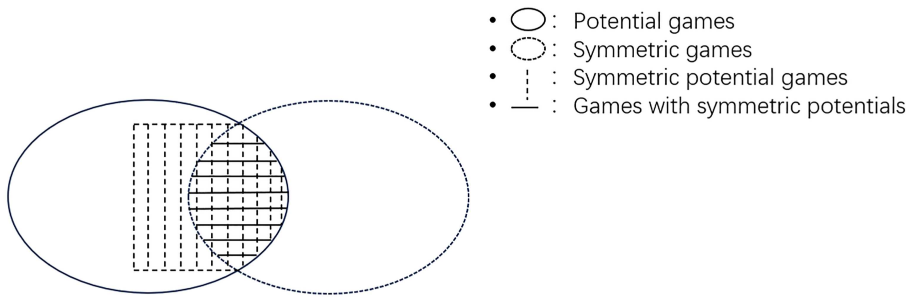

Figure 1 Inclusion relationship.

Table 1 Utility single matrix of G.

	111	112	121	122	211	212	221	222
u_1	4	13	18	20	7	12	17	23
u_2	5	15	8	14	17	21	16	24
u_3	7	10	16	15	19	18	19	22
P	3	6	6	5	6	5	5	8

mathematical tool used is the semi-tensor product of matrices. Please see Appendixes A and B for more details.

A (normal form non-cooperative) finite game is a triple $G = (N, S, U)$, where

(i) $N = \{1, 2, \cdots, n\}$ is the set of players;

(ii) $S_i := \{1, 2, \cdots, k_i\}$ is the finite set of strategies of player i, $i = 1, 2, \cdots, n$. A strategy combination $s = (s_1, s_2, \cdots, s_n)$ is called a strategy profile, where $s_i \in S_i$ is the strategy that player i takes in the strategy profile s. Let $S := \prod_{i=1}^n S_i$ denote the set of strategy profiles, where "$\prod$" is the Cartesian product;

(iii) $U = \{u_1, ..., u_n\}$ is the set of utility functions, where $u_i : S \to \mathbb{R}$ is the utility function of player i.

Now, we give the vector space structure of finite games. Throughout this paper, we only consider finite games $G = (N, S, U)$ with $|N| = n$, $|S_i| = k$, $i = 1, \cdots, n$, and let $\mathcal{G}_{[n;k]}$ denote the set of such kind of finite games. Set $S_i := \{1, 2, ..., k\}$, $i = 1, 2, ..., n$. Then, for $j \in S_i$, i.e., the j-th strategy of player i, we express this strategy by its vector form δ_k^j. According to Proposition A1, u_i can be expressed into its vector form as $u_i(s_1, \cdots, s_n) = V_i^u \ltimes_{j=1}^n s_j$, $i = 1, \cdots, n$, where $V_i^u \in \mathbb{R}^{k^n}$ is called the structure vector of u_i.

Plugging them together, we define $V_G := [V_1^u, V_2^u, \cdots, V_n^u] \in \mathbb{R}^{nk^n}$, which is called the structure vector of G. Observing that a finite game $G \in \mathcal{G}_{[n;k]}$ is uniquely determined by $V_G \in \mathbb{R}^{nk^n}$; then $\mathcal{G}_{[n;k]}$ has the same topology and vector space structure as $\mathbb{R}^{nk^n}$ if considering a finite game as a point in $\mathbb{R}^{nk^n}$. Hence, the analysis of finite games can be put into the framework of linear algebra.

Definition 1 ([37]). Let $G = (N, S, U) \in \mathcal{G}_{[n;k]}$ be a finite game. If there exists a function $P : S \to \mathbb{R}$, such that for every $i \in N$, every $s_{-i} \in S_{-i}$, and any $s_i, s_i' \in S_i$, we have

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}), \tag{1}$$

where $s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$, then G is called a potential game, and P is called a potential function. Let $\mathcal{G}_{[n;k]}^{\mathcal{P}}$ denote the set of finite potential games.

Further, if there also exists a set of positive numbers $\{\omega_i \mid i = 1, 2, \cdots, n\}$ such that

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \omega_i[P(s_i, s_{-i}) - P(s_i', s_{-i})],$$

then G is called a weighted potential game, and P is called a $\omega-$potential function.

According to Proposition A1, the potential function $P : S \to \mathbb{R}$ can be expressed as

$$P(s_1, \cdots, s_n) = V^P \ltimes_{i=1}^n s_i.$$

Example 1. Consider a finite game $G = (N, S, U) \in \mathcal{G}_{[3,2]}$. The utility information is given in Table 1 which is called a utility single matrix.

Here, we first explain the above representation of utility functions: the first row represents the strategy profiles which are arranged in alphabetic order; the second row gives player 1's utility in each profile; similarly, the third and fourth rows give player 2's and player 3's utilities, respectively. Using this representation, the structure vector of each utility function can be immediately obtained by putting all utilities of the corresponding row in the utility single matrix together. For example, the structure vector of u_1 is $V_1^u := [4, 13, 18, 20, 7, 12, 17, 23]$. According to Definition 1, it can be verified that G is a potential game with a potential function shown in the fifth line of Table 1. Then, we have the structure vector of the potential function as $V^P := [3, 6, 6, 5, 6, 5, 5, 8]$.

3 Finite games with symmetric potential functions

A basis of symmetric functions is constructed. Based on the basis, the algebraic verification condition of finite games with symmetric potential functions is given as well as a basis of the corresponding subspace. The algebraic verification equation turns the verification problem into checking whether the solution of a linear system exists, which is more convenient and applicable in practice.

3.1 Symmetric potential functions

In this subsection, we construct a basis of symmetric functions.

Definition 2. Let $G = (N, S, U) \in \mathcal{G}^{\mathcal{P}}_{[n;k]}$ be a finite potential game with a potential function $P : S \to \mathbb{R}$. If for any $\sigma \in \mathbf{S}_n$, we have $P(s_1, \cdots, s_n) = P(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(2)} \cdots, s_{\sigma^{-1}(n)})$, then P is called a symmetric potential function. Let $\mathcal{G}^{\mathcal{PS}}_{[n;k]}$ denote the set of finite games with symmetric potential functions.

One main feature of a potential function caused by its symmetry is that the function values are the same for any two strategy profiles $s = (s_1, \cdots, s_n)$ and $\sigma(s) = (s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(n)})$, regardless of which permutation is applied to the strategy profile s. In the following, we construct a basis of symmetric functions to describe their structural characteristics. First, we introduce the concept of the strategy multiplicity vector.

Definition 3 ([38])**.** Let $G = (N, S, U) \in \mathcal{G}_{[n;k]}$ be a finite game. For any strategy profile $s = (s_1, \cdots, \mathbf{S}_n) \in S$, the strategy multiplicity vector of s is defined as $\#(s) = (\#(s, 1), \#(s, 2), \cdots, \#(s, k))$, where $\#(s, i) := |\{s_j \mid s_j = i\}|$, $i = 1, \cdots, k$.

Using above notations and definitions, the following result can be obtained.

Proposition 1. Let $P : S \to \mathbb{R}$ be the potential function of a finite potential game $G = (N, S, U) \in \mathcal{G}^{\mathcal{P}}_{[n;k]}$. P is symmetric, if and only if, for any $s,\ s' \in S$, if $\#(s) = \#(s')$, then $P(s) = P(s')$.

Proof. From the introduction of the symmetric group in Appendix B it can be seen that when a permutation acts on a finite set, it only changes the positions of the elements in the set, but does not change their mathematical meanings. Then, the following fact holds:

$$\#(s) = \#(\sigma(s)), \quad \forall\ s \in S,\ \sigma \in \mathbf{S}_n. \tag{2}$$

The sufficiency follows from (2) and Definition 2 immediately.

As for the necessity, we adopt the proof by contradiction. Suppose there exist two strategy profiles $s_1,\ s_2 \in S$ satisfying $\#(s_1) = \#(s_2)$ such that

$$P(s_1) \neq P(s_2). \tag{3}$$

For s_1, it can be obtained from (2) that $\{s \mid \#(s) = \#(s_1)\} = \{s \mid s = \sigma(s_1), \forall \sigma \in \mathbf{S}_n\}$, showing that for any $s \in \{s \mid \#(s) = \#(s_1)\}$ there exists a $\sigma \in \mathbf{S}_n$ such that $s = \sigma(s_1)$. Then, for s_2 there exists a $\sigma \in \mathbf{S}_n$ such that $s_2 = \sigma(s_1)$. According to Definition 2 we have $P(s_1) = P(s_2)$, which contradicts (3).

According to Proposition 1, we can divide all strategy profiles into some different classes according to whether their strategy multiplicity vectors are the same. The corresponding strategy profiles in the same class have the same function value. In this way, the basis of symmetric functions can be constructed. To be specific, define $\mathcal{Q} = \{\#(s) \mid s \in S\}$. It can be seen that $|\mathcal{Q}| =: \alpha := \binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$. Define the elements in $\mathcal{Q}$ as $\mathcal{Q} = \{\#_1, \#_2, \cdots, \#_\alpha\}$. Define a set of basic functions as

$$F_s(\delta_k^{i_1}\delta_k^{i_2}\cdots\delta_k^{i_n}) = \begin{cases} 1, & \#(i_1, i_2, \cdots, i_n) = \#_s, \\ 0, & \text{otherwise}, \end{cases}$$

where $s = 1, \cdots, \alpha$.

Let V_s be the structure vector of F_s. Set

$$H_{[n,k]} = [V_1^{\mathrm{T}}, V_2^{\mathrm{T}}, \cdots, V_\alpha^{\mathrm{T}}]^{\mathrm{T}}. \tag{4}$$

Then, the following result can be verified.

Proposition 2. (1) $\{V_1, V_2, \cdots, V_\alpha\}$ is an orthogonal basis of symmetric functions $f(s_1, \cdots, s_n)$, with $s_i \in \mathcal{D}_k$. Particularly, if P is a symmetric potential function, then there is a row vector $v \in \mathbb{R}^\alpha$, such that $V^P = \nu H_{[n,k]}$.

(2) $H_{[n,k]}H_{[n,k]}^{\mathrm{T}} = \mathrm{diag}\,(\ell_1, \ell_2, \cdots, \ell_\alpha)$, where $\ell_s = \binom{k}{(\#_s)_1}\binom{k-(\#_s)_1}{(\#_s)_2}\cdots\binom{k-(\#_s)_1-\cdots-(\#_s)_{k-2}}{(\#_s)_{k-1}}$, and $(\#_s)_i$ is the i-th component of the vector $\#_s$.

We give a simple example to depict this.

Example 2. Recall Example 1. It can be seen that $n = 3$ and $k = 2$. Then, we have $\mathcal{Q} = \{(3,0), (2,1), (1,2), (0,3)\}$. It can be calculated that $F_1 = V_1 s = [1,0,0,0,0,0,0,0]s$, $F_2 = V_2 s = [0,1,1,0,1,0,0,0]s$, $F_3 = V_3 s = [0,0,0,1,0,1,1,0]s$, $F_4 = V_4 s = [0,0,0,0,0,0,0,1]s$, where $s \in \Delta_8$. Then, $H = [V_1^{\mathrm{T}}, V_2^{\mathrm{T}}, V_3^{\mathrm{T}}, V_4^{\mathrm{T}}]^{\mathrm{T}}$ and $HH^{\mathrm{T}} = \mathrm{diag}(1,3,3,1)$. Set $\nu = [3,6,5,8]$. Then, it can be checked that $V^P = \nu H$.

3.2 A linear system for finite games with symmetric potential functions

In this subsection, a set of linear equations is given to verify whether a finite game has symmetric potential functions.

Consider a finite game $G = (N, S, U) \in \mathcal{G}_{[n;k]}$. Define functions as $d_i : S_0^{n-1} \to \mathbb{R}, \quad i = 1, 2, \cdots, n$, where $S_0 := \{1, 2, \cdots, k\}$. To make the following statement clearer, we express d_i as $d_i(x_1, \cdots, \widehat{x}_i, \cdots, x_n)$, where a caret denotes missing terms, i.e., d_i is independent of x_i.

For the verification of finite potential games, the following result has been obtained.

Lemma 1 ([39])**.** Let $G = (N, S, U) \in \mathcal{G}_{[n;k]}$ be a finite game. Then, it is a potential game, if and only if, for every $i \in N$, there exists a function $d_i : S_0^{n-1} \to \mathbb{R}$, such that

$$u_i(s_1, \cdots, s_n) = P(s_1, \cdots, s_n) + d_i(s_1, \cdots, \widehat{s}_i, \cdots, s_n),$$

where P is the potential function.

According to Definition 2 and Lemma 1, we can get the following result immediately.

Lemma 2. Let $G = (N, S, U) \in \mathcal{G}_{[n;k]}$ be a finite game. Then, it is with symmetric potential functions, if and only if, for every $i \in N$, there exists a function $d_i : S_0^{n-1} \to \mathbb{R}$, such that for any $\sigma \in \mathbf{S}_n$, we have

$$u_i(s_1, \cdots, s_n) = P(s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(n)}) + d_i(s_1, \cdots, \widehat{s}_i, \cdots, s_n), \tag{5}$$

where P is the potential function.

Now, it is ready to present a linear system for the verification of finite games with symmetric potential functions. According to Proposition 2 and Proposition A1, Eq. (5) can be expressed in its vector form as below:

$$V_i^u \ltimes_{j=1}^n s_j = \nu H_{[n,k]} \ltimes_{j=1}^n s_j + V_i^d \ltimes_{j\neq i} s_j, \tag{6}$$

where $i = 1, 2, \cdots, n$, $V_i^u \in \mathbb{R}^{k^n}$ and $V_i^d \in \mathbb{R}^{k^{n-1}}$ are row vectors.

It can be seen that G is a potential game with symmetric potential functions, if and only if, the solution of (6) exists for unknowns ν and V_i^d.

To prove the existence of solutions, we firstly define two operators as below:

- front deleting operator $D_f^{[p,q]} := \mathbf{1}_p^{\mathrm{T}} \otimes I_q \in \mathcal{L}_{q\times pq}$,
- rear deleting operator $D_r^{[p,q]} := I_p \otimes \mathbf{1}_q^{\mathrm{T}} \in \mathcal{L}_{p\times pq}$.

They have the following properties.

Lemma 3 ([29])**.** Let $X \in \Delta_p$ and $Y \in \Delta_q$. Then, $D_f^{[p,q]}XY = Y$, $D_r^{[p,q]}XY = X$.

Using Lemma 3, we have $\ltimes_{j\neq i}^n s_j = \begin{cases} D_f^{[k,k]} \ltimes_{j=1}^n s_j, & i = 1; \\ D_r^{[k^{i-1},k]} \ltimes_{j=1}^n s_j, & 2 \leqslant i \leqslant n. \end{cases}$ Then, Eq. (6) can be rewritten as

$$V_i^u = \nu H_{[n,k]} + V_i^d M_i, \quad i = 1, 2, \cdots, n, \tag{7}$$

where $M_i = \begin{cases} D_f^{[k,k]}, & i = 1; \\ D_r^{[k^{i-1},k]}, & 2 \leqslant i \leqslant n. \end{cases}$

From Proposition 2, we know that $H_{[n,k]}H_{[n,k]}^{\mathrm{T}}$ is invertible. Solving ν from (7) yields

$$\nu = (V_1^u - V_1^d M_1)H_{[n,k]}^{\mathrm{T}}(H_{[n,k]}H_{[n,k]}^{\mathrm{T}})^{-1}.$$

Plugging it into the rest of (7), we have

$$V_i^u - V_1^u\overline{H} = V_i^d M_i - V_1^d M_1\overline{H}, \quad i = 2, 3, \cdots, n, \tag{8}$$

where $\overline{H} := H_{[n,k]}^{\mathrm{T}}(H_{[n,k]}H_{[n,k]}^{\mathrm{T}})^{-1}H_{[n,k]}$.

Taking transpose on both sides of (8), we have $(V_i^u - V_1^u\overline{H})^{\mathrm{T}} = (V_i^d M_i)^{\mathrm{T}} - \overline{H}^{\mathrm{T}}(V_1^d M_1)^{\mathrm{T}}$. Since $M_1 \in \mathcal{L}_{k\times k^2}$ and $V_1^d \in \mathbb{R}^{k^{n-1}}$, Definition A1 leads to the following result:

$$(V_1^d M_1)^{\mathrm{T}} = M_1^{\mathrm{T}}(V_1^d)^{\mathrm{T}} = (M_1^{\mathrm{T}} \otimes I_{k^{n-2}})(V_1^d)^{\mathrm{T}}.$$

Define $\Phi_1 := M_1^{\mathrm{T}} \otimes I_{k^{n-2}} = \mathbf{1}_k \otimes I_{k^{n-1}}$. Similarly, define $\Phi_2 := M_2^{\mathrm{T}} \otimes I_{k^{n-2}} = I_k \otimes \mathbf{1}_k \otimes I_{k^{n-2}}, \cdots, \Phi_n := M_n^{\mathrm{T}} = I_{k^{n-1}} \otimes \mathbf{1}_k$. Then, Φ_i, $i = 1, \cdots, n$ can be expressed uniformly as $\Phi_i := I_{k^{i-1}} \otimes \mathbf{1}_k \otimes I_{k^{n-i}} \in \mathcal{M}_{k^n\times k^{n-1}}$. Define some vectors as $\xi_i := (V_i^d)^{\mathrm{T}} \in \mathbb{R}^{k^{n-1}}$, $b_i := (V_i^u - V_1^u\overline{H})^{\mathrm{T}} \in \mathbb{R}^{k^n}$, $i = 2, 3, \cdots, n$.

Using above notations, Eq. (8) can be rewritten as a linear system:

$$\Phi\zeta = b, \tag{9}$$

where $\Phi = \begin{bmatrix} -\overline{H}^{\mathrm{T}}\Phi_1 & \Phi_2 & \mathbf{0} & \cdots & \mathbf{0} \\ -\overline{H}^{\mathrm{T}}\Phi_1 & \mathbf{0} & \Phi_3 & \cdots & \mathbf{0} \\ \vdots & & & & \\ -\overline{H}^{\mathrm{T}}\Phi_1 & \mathbf{0} & \mathbf{0} & \cdots & \Phi_n \end{bmatrix}, \zeta = [\xi_1^{\mathrm{T}}, \xi_2^{\mathrm{T}}, \cdots, \xi_n^{\mathrm{T}}]^{\mathrm{T}}, b = [b_2^{\mathrm{T}}, \cdots, b_n^{\mathrm{T}}]^{\mathrm{T}}$.

The above argument leads to the following result.

Theorem 1. Let $G = (N, S, U) \in \mathcal{G}_{[n;k]}$ be a finite game. Then, it is with symmetric potential functions, if and only if, Eq. (9) has a solution. In addition, the symmetric potential function can be calculated by the following formula:

$$V^P = (V_1^u - \xi_1^{\mathrm{T}} D_f^{[k,k]})\overline{H}. \tag{10}$$

3.3 A basis for finite games with symmetric potential functions

Theorem 1 shows that G is a finite game with symmetric potential functions, if and only if,

$$\begin{bmatrix} (V_2^u - V_1^u\overline{H})^{\mathrm{T}} \\ (V_3^u - V_1^u\overline{H})^{\mathrm{T}} \\ \vdots \\ (V_n^u - V_1^u\overline{H})^{\mathrm{T}} \end{bmatrix} \in \mathrm{Span}(\Phi). \tag{11}$$

Adding $(V_1^u)^{\mathrm{T}}$ in (11), Eq. (11) can be rewritten as

$$\begin{bmatrix} (V_1^u)^{\mathrm{T}} \\ (V_2^u - V_1^u\overline{H})^{\mathrm{T}} \\ (V_3^u - V_1^u\overline{H})^{\mathrm{T}} \\ \vdots \\ (V_n^u - V_1^u\overline{H})^{\mathrm{T}} \end{bmatrix} \in \mathrm{Span}(\Phi^p), \tag{12}$$

where $\Phi^p = \begin{bmatrix} H^{\mathrm{T}} & \Phi_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\overline{H}^{\mathrm{T}}\Phi_1 & \Phi_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\overline{H}^{\mathrm{T}}\Phi_1 & \mathbf{0} & \Phi_3 & \cdots & \mathbf{0} \\ \vdots & & & & & \\ \mathbf{0} & -\overline{H}^{\mathrm{T}}\Phi_1 & \mathbf{0} & \mathbf{0} & \cdots & \Phi_n \end{bmatrix}$.

Equivalently, Eq. (12) can rewritten as $\begin{bmatrix} I_{k^n} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ -\overline{H}^{\mathrm{T}} & I_{k^n} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & & \\ -\overline{H}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \cdots & I_{k^n} \end{bmatrix} \begin{bmatrix} (V_1^u)^{\mathrm{T}} \\ (V_2^u)^{\mathrm{T}} \\ \vdots \\ (V_n^u)^{\mathrm{T}} \end{bmatrix} \in \mathrm{Span}(\Phi^p)$. That is,

$$(V_G)^{\mathrm{T}} \in \mathrm{Span}(\Phi^e), \tag{13}$$

where

$$\Phi^e := \begin{bmatrix} I_{k^n} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ -\overline{H}^{\mathrm{T}} & I_{k^n} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & & \\ -\overline{H}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \cdots & I_{k^n} \end{bmatrix}^{-1} \begin{bmatrix} H_{[n;k]}^{\mathrm{T}} & \Phi_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\overline{H}^{\mathrm{T}}\Phi_1 & \Phi_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & -\overline{H}^{\mathrm{T}}\Phi_1 & \mathbf{0} & \cdots & \Phi_n \end{bmatrix} = \begin{bmatrix} H_{[n;k]}^{\mathrm{T}} & \Phi_1 & \mathbf{0} & \mathbf{0} & \cdots & 0 \\ H_{[n;k]}^{\mathrm{T}} & \mathbf{0} & \Phi_2 & \mathbf{0} & \cdots & 0 \\ \vdots & & & & & \\ H_{[n;k]}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \Phi_n \end{bmatrix}. \tag{14}$$

Then, the following result can be obtained.

Theorem 2. (i) The dimension of subspace of $\mathcal{G}^{\mathcal{PS}}_{[n;k]}$ is

$$\dim(\mathcal{G}^{\mathcal{PS}}_{[n;k]}) = \binom{n+k-1}{n} + nk^{n-1} - 1. \tag{15}$$

(ii) The subspace of $\mathcal{G}^{\mathcal{PS}}_{[n;k]}$ is

$$\mathcal{G}^{\mathcal{PS}}_{[n;k]} = \mathrm{Span}(\Phi^e), \tag{16}$$

which has $\mathrm{Col}(\Phi^e_0)$ as its basis, and Φ^e_0 is obtained by deleting any one column of Φ^e.

Proof. (i) Let

$$\Upsilon = \begin{bmatrix} -\Phi_1 & \Phi_2 & 0 & \cdots & 0 \\ -\Phi_1 & 0 & \Phi_3 & \cdots & 0 \\ \vdots & & & & \\ -\Phi_1 & 0 & 0 & \cdots & \Phi_n \end{bmatrix}, \widetilde{H} = \begin{bmatrix} H^{\mathrm{T}} \\ H^{\mathrm{T}} \\ \vdots \\ H^{\mathrm{T}} \end{bmatrix}, \Gamma = \begin{bmatrix} \Phi_1 & \mathbf{0} & \cdots & 0 \\ \mathbf{0} & \Phi_2 & \cdots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & \Phi_n \end{bmatrix}.$$

As discussed in [29], deleting any one column of Υ the remaining columns form a basis of $\mathrm{Span}(\Upsilon)$, and $\dim(\Upsilon) = nk^{n-1} - 1$.

On the one hand, using the second equality of (14), it can be verified that

$$\mathrm{Rank}(\Phi^e) = \mathrm{Rank}\left(\begin{bmatrix} H^{\mathrm{T}} & \Phi_1 & \mathbf{0} & \mathbf{0} & \cdots & 0 \\ \mathbf{0} & -\Phi_1 & \Phi_2 & \mathbf{0} & \cdots & 0 \\ \vdots & & & & & \\ \mathbf{0} & -\Phi_1 & \mathbf{0} & \mathbf{0} & \cdots & \Phi_n \end{bmatrix}\right) \geqslant \mathrm{Rank}(H^{\mathrm{T}}) + \mathrm{Rank}(\Upsilon) = \alpha + nk^{n-1} - 1. \tag{17}$$

On the other hand, let $a_i(b_j)$ denote the $i(j)$-th column of $\widetilde{H}(\Gamma)$, $i = 1, 2, \cdots, \alpha$ $(j = 1, 2, \cdots, nk^{n-1})$. It can be calculated that $\sum_{i=1}^{\alpha} a_i = \mathbf{1}_{nk^n}$, $\sum_{j=1}^{nk^{n-1}} b_j = \mathbf{1}_{nk^n}$. Then, we have

$$\sum_{i=1}^{\alpha} a_i - \sum_{j=1}^{nk^{n-1}} b_j = \mathbf{0}. \tag{18}$$

From the second equality of (14) and (18), we get

$$\mathrm{Rank}(\Phi^e) \leqslant \alpha + nk^{n-1} - 1. \tag{19}$$

Eq. (15) follows from (17) and (19).

(ii) Eq. (16) comes from (13). Combining (15) with (18) yields a basis of Φ^e as expected.

4 Symmetric potential games

A finite game is a symmetric potential game if it is both a finite potential game and a symmetric game. Let $\mathcal{G}^{\mathcal{SP}}_{[n;k]}$ denote the set of symmetric potential games. The algebraic expression of symmetric potential games is obtained, showing that $\mathcal{G}^{\mathcal{SP}}_{[n;k]} \subseteq \mathcal{G}^{\mathcal{PS}}_{[n;k]}$. A necessary and sufficient condition for the verification of symmetric potential games is given as well as a basis of the corresponding subspace.

4.1 A linear system for symmetric potential games

In this subsection, a set of linear equations is given to verify whether a finite game is a symmetric potential game.

Definition 4. Let $G = (N, S, U) \in \mathcal{G}_{[n,k]}$ be a finite game. If for any $\sigma \in \mathbf{S}_n$, we have

$$u_i(s_1, \cdots, s_n) = u_{\sigma(i)}(s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(n)}), \tag{20}$$

where $i = 1, 2, \cdots, n$, then G is called a symmetric game.

According to Lemma 1, we can see that each utility function is decomposed into the sum of two functions for a finite potential game. The following result shows that for a symmetric potential game, the symmetry attribute makes the structure of these two functions more special.

Proposition 3. Let $G=(N,S,U)\in\mathcal{G}_{[n,k]}$ be a finite game. Then, it is a symmetric potential game, if and only if, there exist symmetric functions $P:S\to\mathbb{R}$ and $d:S_0^{n-1}\to\mathbb{R}$, such that for every $i\in N$, and any $\sigma\in\mathbf{S}_n$, we have

$$u_i(s_1,\cdots,s_n)=P(s_{\sigma^{-1}(1)},\cdots,s_{\sigma^{-1}(n)})+d(s_{\sigma^{-1}(1)},\cdots,\widehat{s}_i,\cdots,s_{\sigma^{-1}(n)}). \tag{21}$$

Proof. (Necessity) First, we prove the existence of the common function d. Consider a profile $s=(s_1,\cdots,s_j,\cdots,s_{n-1})\in S_0^{n-1}$. Set a profile $y^i=(y_1,\cdots,y_i,y_{i+1},\cdots,y_n)$, where $y_i=y_{i+1}$, and $y_j=\begin{cases}s_j, & j\leqslant i;\\ s_{j-1}, & j>i.\end{cases}$ Then, as G is a symmetric potential game, by using (20) and (21) it can be obtained that

$$\begin{cases}u_i(y^i)=P(y^i)+d_i(y_1,\cdots,\widehat{y}_i,y_{i+1},\cdots,y_n),\\ u_{i+1}(y^i)=P(y^i)+d_{i+1}(y_1,\cdots,y_i,\widehat{y}_{i+1},\cdots,y_n),\\ u_i(y^i)=u_{i+1}(y^i),\end{cases}$$

which implies that $d_i(s)=d_{i+1}(s)$.

Let i run from 1 to $n-1$. Then, we have $d_i(s)=d_j(s)$, $\forall\ i,j=1,2,\cdots,n$. Since $s\in S_0^{n-1}$ is arbitrary, the existence of the common function d can be proven.

Next, we prove the symmetry of d. Let $\sigma=(i,i+1)$, $i=1,2,\cdots,n-1$. Consider two profiles as $s^1=(s_1,\cdots,s_i,s_{i+1},\cdots,s_{n-1})\in S_0^{n-1}$, $s^2:=(s_{\sigma^{-1}(1)},\cdots,s_{\sigma^{-1}(i)},s_{\sigma^{-1}(i+1)},\cdots,s_{\sigma^{-1}(n-1)})=(s_1,\cdots,s_{i+1},s_i,\cdots,s_{n-1})\in S_0^{n-1}$. Set a profile $y^i=(y_1,\cdots,y_i,y_{i+1},y_{i+2},\cdots,y_n)$, where $y_i=y_{i+2}$, and

$$y_j=\begin{cases}s_j, & j\leqslant i+1;\\ s_i, & j=i+2;\\ s_{j-1}, & j>i+2.\end{cases}$$

Then, as G is a symmetric potential game, by using (20) and (21) we have

$$\begin{cases}u_i(y^i)=P(y^i)+d_i(y_1,\cdots,\widehat{y}_i,y_{i+1},\cdots,y_n),\\ u_{i+2}(y^i)=P(y^i)+d_{i+1}(y_1,\cdots,y_i,y_{i+1},\widehat{y}_{i+2},\cdots,y_n),\\ u_i(y^i)=u_{i+2}(y^i).\end{cases}$$

It can be checked that $d(s^1)=d(s^2)$.

Note that $\mathbf{S}_n=\langle(i,i+1)\mid 1\leqslant i\leqslant n-1\rangle$, and $s^1\in S_0^{n-1}$ is arbitrary. The symmetry of the function d can be verified.

Using (21), the symmetry of the potential function P can be concluded from the symmetry of the game G and the function d.

(Sufficiency) Eq. (20) follows immediately from the symmetry of these two functions.

Let $G=(N,S,U)\in\mathcal{G}^{\mathcal{SP}}_{[n;k]}$ be a symmetric potential game with the common symmetric function $d:S_0^{n-1}\to\mathbb{R}$, and V^d be the structure vector of d. Then, according to Proposition 2, we know that there exists a row vector $\eta\in\mathbb{R}^\beta$, such that

$$V^d=\eta H_{[n-1,k]}, \tag{22}$$

where $\beta:=\binom{n+k-2}{n-1}=\frac{(n+k-2)!}{(n-1)!(k-1)!}$.

Substitute (22) into (7). Define $\xi:=\eta^{\mathrm{T}}\in\mathbb{R}^\beta$, $b_i:=(V_i^u-V_1^u\overline{H})^{\mathrm{T}}\in\mathbb{R}^{k^n}$, $i=2,3,\cdots,n$. Then, similar to the derivation of Theorem 2, we obtain the following liner system:

$$\Psi\xi=b, \tag{23}$$

where $\Psi=[(\Phi_2-\overline{H}^{\mathrm{T}}\Phi_1)^{\mathrm{T}},\cdots,(\Phi_n-\overline{H}^{\mathrm{T}}\Phi_1)^{\mathrm{T}}]^{\mathrm{T}}H^{\mathrm{T}}_{[n-1,\kappa]}$, $b=[b_2^{\mathrm{T}},b_3^{\mathrm{T}},\cdots,b_n^{\mathrm{T}}]^{\mathrm{T}}$.

The above argument leads to the following result.

Theorem 3. Let $G=(N,S,U)\in\mathcal{G}_{[n;k]}$ be a finite game. Then, it is a symmetric potential game, if and only if, Eq. (23) has a solution. Moreover, the symmetric potential function can be calculated by the following formula:

$$V^P=(V_1^u-\xi^{\mathrm{T}}H_{[n-1,k]}D_f^{[k,k]})\overline{H}. \tag{24}$$

Remark 1. Theorems 1 and 3 are more convenient and applicable in practice. On the one hand, the verification process is simplified since it is easy to check whether a linear system has a solution, and the linear system has a clear and concise mathematical structure. The coefficient matrix is only related to n and k. Then, we can use n and k to classify the type of games, and each type has a uniform coefficient matrix. On the other hand, a concise mathematical formula for calculating the symmetric potential function is given for the first time.

4.2 A basis for symmetric potential games

According to Theorem 3, it can be shown that G is a symmetric potential game, if and only if,

$$\begin{bmatrix} (V_2^u - V_1^u \overline{H})^{\mathrm{T}} \\ (V_3^u - V_1^u \overline{H})^{\mathrm{T}} \\ \vdots \\ (V_n^u - V_1^u \overline{H})^{\mathrm{T}} \end{bmatrix} \in \mathrm{Span}(\Psi). \tag{25}$$

Adding $(V_1^u)^{\mathrm{T}}$ into (25) yields

$$\begin{bmatrix} (V_1^u)^{\mathrm{T}} \\ (V_2^u - V_1^u \overline{H})^{\mathrm{T}} \\ (V_3^u - V_1^u \overline{H})^{\mathrm{T}} \\ \vdots \\ (V_n^u - V_1^u \overline{H})^{\mathrm{T}} \end{bmatrix} \in \mathrm{Span}(\Psi^p), \tag{26}$$

where $\Psi^p := \begin{bmatrix} H^{\mathrm{T}} & \Phi_1 H_{[n-1,k]}^{\mathrm{T}} \\ \mathbf{0} & \Psi \end{bmatrix}$.

Equivalently, Eq. (26) can be rewritten as $\begin{bmatrix} I_{k^n} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ -\overline{H}^{\mathrm{T}} & I_{k^n} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots \\ -\overline{H}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \cdots & I_{k^n} \end{bmatrix} \begin{bmatrix} (V_1^u)^{\mathrm{T}} \\ (V_2^u)^{\mathrm{T}} \\ \vdots \\ (V_n^u)^{\mathrm{T}} \end{bmatrix} \in \mathrm{Span}(\Psi^p)$. That is,

$$(V_G)^{\mathrm{T}} \in \mathrm{Span}(\Psi^e), \tag{27}$$

where

$$\Psi^e := \begin{bmatrix} I_{k^n} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ -\overline{H}^{\mathrm{T}} & I_{k^n} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots \\ -\overline{H}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \cdots & I_{k^n} \end{bmatrix}^{-1} \begin{bmatrix} H^{\mathrm{T}} & \Phi_1 \\ \mathbf{0} & \Psi \end{bmatrix} = \begin{bmatrix} H^{\mathrm{T}} & \Phi_1 H_{[n-1,k]}^{\mathrm{T}} \\ H^{\mathrm{T}} & \Phi_2 H_{[n-1,k]}^{\mathrm{T}} \\ \vdots \\ H^{\mathrm{T}} & \Phi_n H_{[n-1,k]}^{\mathrm{T}} \end{bmatrix}. \tag{28}$$

Then, the following result can be proven.

Theorem 4. (1) The dimension of subspace of $\mathcal{G}_{[n;k]}^{\mathcal{SP}}$ is

$$\dim(\mathcal{G}_{[n;k]}^{\mathcal{SP}}) = \binom{n+k-1}{n} + \binom{n+k-2}{n-1} - 1. \tag{29}$$

(2) The subspace of $\mathcal{G}_{[n;k]}^{\mathcal{SP}}$ is

$$\mathcal{G}_{[n;k]}^{\mathcal{SP}} = \mathrm{Span}(\Psi^e), \tag{30}$$

which has $\mathrm{Col}(\Psi_0^e)$ as its basis, and Ψ_0^e is obtained by deleting any one column of Ψ^e.

Proof. (1) Let $\widetilde{H} := [H_{[n,k]}, \cdots, H_{[n,k]}]^{\mathrm{T}}$, $\Xi := [H_{[n-1,k]}\Phi_1^{\mathrm{T}}, H_{[n-1,k]}\Phi_2^{\mathrm{T}}, \cdots, H_{[n-1,k]}\Phi_n^{\mathrm{T}}]^{\mathrm{T}}$. On the one hand, we proof that

$$\mathrm{Rank}(\Psi^e) \leqslant \alpha + \beta - 1. \tag{31}$$

Since Φ_i has full column rank, we have $\mathrm{Rank}(\Phi_i H_{[n-1,k]}^{\mathrm{T}}) = \mathrm{Rank}(H_{[n-1,k]}^{\mathrm{T}}) = \beta$, which implies that

$$\mathrm{Rank}(\Xi) = \beta. \tag{32}$$

Let a_i (b_j) be the i (j)-th column of $\widetilde{H}$ (Ξ), where $i = 1, 2, \cdots, \alpha$ ($j = 1, 2, \cdots, \beta$). Then, it can be calculated that $\sum_{i=1}^{\alpha} a_i = \mathbf{1}_{nk^n}$, $\sum_{j=1}^{k^{n-1}} b_j = \mathbf{1}_{nk^n}$. Hence, we have

$$\sum_{i=1}^{\alpha} a_i - \sum_{j=1}^{\beta} b_j = \mathbf{0}. \tag{33}$$

Eq. (31) follows from (32) and (33).

On the other hand, we verify that

$$\mathrm{Rank}(\Psi^e) \geqslant \alpha + \beta - 1. \tag{34}$$

Let

$$\Lambda := \begin{bmatrix} H^{\mathrm{T}}_{[n,k]} & \Phi_1 H^{\mathrm{T}}_{[n-1,k]} & \cdots & \mathbf{0} \\ \vdots & & & \\ H^{\mathrm{T}}_{[n,\kappa]} & \mathbf{0} & \cdots & \Phi_n H^{\mathrm{T}}_{[n-1,k]} \end{bmatrix} \in \mathcal{M}_{k^n \times (\alpha + n\beta)}, \quad \widetilde{I} := \begin{bmatrix} I_\alpha & \mathbf{0} \\ \mathbf{0} & I_\beta \\ \vdots & \vdots \\ \mathbf{0} & I_\beta \end{bmatrix} \in \mathcal{M}_{(\alpha+n\beta)\times(\alpha+\beta)}.$$

Then, it can be calculated that $\Psi^e = \Lambda \widetilde{I}$. Hence, we have $\mathrm{Rank}(\Psi^e) = \mathrm{Rank}(\Lambda\widetilde{I}) \geqslant \mathrm{Rank}(\Lambda) + \mathrm{Rank}(\widetilde{I}) - (\alpha + n\beta)$. As $\mathrm{Rank}(\widetilde{I}) = \alpha + \beta$, it can be seen that Eq. (34) holds, if the following inequality holds:

$$\mathrm{Rank}(\Lambda) \geqslant \alpha + n\beta - 1. \tag{35}$$

In the following, we only consider (35). First, we have

$$\mathrm{Rank}(\Lambda) = \mathrm{Rank}(\Delta), \tag{36}$$

where

$$\Delta := \begin{bmatrix} H^{\mathrm{T}}_{[n,k]} & \Phi_1 H^{\mathrm{T}}_{[n-1,k]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\Phi_1 H^{\mathrm{T}}_{[n-1,k]} & \Phi_2 H^{\mathrm{T}}_{[n-1,k]} & \cdots & \mathbf{0} \\ \vdots & & & & \\ \mathbf{0} & -\Phi_1 H^{\mathrm{T}}_{[n-1,k]} & \mathbf{0} & \cdots & \Phi_n H^{\mathrm{T}}_{[n-1,k]} \end{bmatrix}.$$

Then, we have

$$\mathrm{Rank}(\Lambda) \geqslant \mathrm{Rank}(H^{\mathrm{T}}_{[n,k]}) + \mathrm{Rank}(\Pi), \tag{37}$$

where $\Pi = \begin{bmatrix} -\Phi_1 H^{\mathrm{T}}_{[n-1,k]} & \Phi_2 H^{\mathrm{T}}_{[n-1,k]} & \cdots & \mathbf{0} \\ \vdots & & & \\ -\Phi_1 H^{\mathrm{T}}_{[n-1,k]} & \mathbf{0} & \cdots & \Phi_n H^{\mathrm{T}}_{[n-1,k]} \end{bmatrix}$.

Let

$$\widehat{H} := \begin{bmatrix} H^{\mathrm{T}}_{[n,k]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & H^{\mathrm{T}}_{[n,k]} & \cdots & \mathbf{0} \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & H^{\mathrm{T}}_{[n,k]} \end{bmatrix} \in \mathcal{M}_{nk^{n-1} \times n\beta}, \quad \widetilde{\Phi} := \begin{bmatrix} -\Phi_1 & \Phi_2 & \cdots & \mathbf{0} \\ -\Phi_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ -\Phi_1 & \mathbf{0} & \cdots & \Phi_n \end{bmatrix} \in \mathcal{M}_{(n-1)k^n \times nk^{n-1}}.$$

Then, it can be calculated that $\Pi = \widetilde{\Phi}\widehat{H}$.

Hence,

$$\mathrm{Rank}(\Pi) \geqslant \mathrm{Rank}(\widehat{H}) + \mathrm{Rank}(\widetilde{\Phi}) - nk^{n-1} = n\beta - 1. \tag{38}$$

Eq. (35) follows from (37) and (38). Consequently, Eq. (34) holds.

(2) Eq. (30) follows from (27). Applying (29) with (33) yields a basis of Ψ^e.

5 Application to optimization of spatial games

In this section, we consider the optimization of quasi-symmetric spatial games. A sufficient condition about the designability of utility functions is presented as well as a concise formula to calculate utility functions.

• **Spatial game**: The spatial game is a tuple $G = (N, (N, \mathcal{E}), g, S, U)$, where $(N, \mathcal{E})$ is a given graph with each player as a vertex, g is a finite two-player game which is called the base game, and $\mathcal{E}$ is the set of edges with each edge consisting of two vertices (players). If an edge consists of players i and j, then it is denoted by (i, j) or (j, i) for an undirected graph, and (v_1, v_2) with $v_1 \in \{i, j\}$ as the initial vertex and $v_2 \in \{i, j\}$ as the terminal vertex for a directed graph. Define the set of neighbours of player i as $N_i = \{j \in N : (i, j) \in \mathcal{E} \text{ or } (j, i) \in \mathcal{E}\}$. Assume each player is paired with each of his or her neighbours to play the base game, and he sticks to a strategy during the whole period.

Define the incidence matrix of a graph as $A = \{a_{ij}\} \in \mathcal{M}_{n\times n}$ with nonnegative real numbers, where $(i, j) \in \mathcal{E} \Leftrightarrow a_{ij} \neq 0$ and $a_{ji} \neq 0$. For each interaction pair (i, j), a_{ij} indicates the extent to which player j affects player i. If

$a_{ij} > a_{ik}$ for $j, k \in N$, then we say i is affected more by neighbour j than by neighbour k. Similarly, a_{ji} describes how much influence player i has on player j. Let $\alpha_i : S_0 \times S_0 \to \mathbb{R}$ be the utility function of g, $i = 1, 2$. Then, the one-period utility of player i in a strategy profile s is defined as below:

$$u_i(s_i, s_{-i}) = \sum_{j \in N} a_{ij} \alpha_{id(i)}(s_i, s_j), \tag{39}$$

where $id(i)$ is the identity of player i in the base game when he (she) plays the base game with player j. To be specific, if the graph is undirected, then the player's utility depends only on his (her) strategy, not on his (her) identity, meaning that the base game is a symmetric game. Otherwise, $\alpha_{id(i)} = \alpha_1$ if i is the initial vertex of the edge (i, j), and $\alpha_{id(i)} = \alpha_2$ if i is the terminal vertex.

A spatial game is called a quasi-symmetric spatial game, if A is quasi-symmetric, that is, there exist $\lambda_1 > 0, \cdots, \lambda_n > 0$ such that $\lambda_i a_{ij} = \lambda_j a_{ji}$ for all $i, j \in N$.

• **Evolution**: Consider a quasi-symmetric spatial game with the base game being a finite game with symmetric potential functions. It has been proven in [24] that such games are weighted potential games. We adopt the asynchronous updating described as below: at each time t, only one player is selected with probability q_i to update his (her) strategy, and he (she) chooses the strategy $t_i \in S_i$ in a strategy profile s with the probability $p_i^{t_i}(s)$, where $p_i^{t_i}(s) = \frac{exp[P(t_i, s_{-i})/\varepsilon]}{\sum_{s_i} exp[P(s_i, s_{-i})/\varepsilon]}$. Here, $\epsilon > 0$ and P is the potential function of the quasi-symmetric spatial game. Then, the above asynchronous updating determines a Markov process, and the set of stochastically stable states (long-run equilibria) is exactly the set of strategy profiles that maximize P.

• **Utility design**: Let $W : S \to \mathbb{R}$ be a pre-assigned global objective function of the quasi-symmetric spatial game, which needs to be optimized. The problem is how to design the utility functions α_1 and α_2 such that the spatial game can, in the evolution process, converge to strategy profiles that maximize W when each rational player only tries to optimize his (her) utility.

The technique developed in this paper is designing proper utility functions of the base game such that the spatial game becomes a w-potential game with the given global objective criterion W as its w-potential function. Taking W as the potential function can guarantee the correspondence between the maximum or minimum points of the potential function and those of the global objective function.

The following result has been obtained about the constitution of the w-potential function.

Proposition 4 ([24]). Let $G = (N, (V, \mathcal{E}), g, S, U)$ be a quasi-symmetric spatial game and λ_i, $i \in N$, be positive numbers that make the quasi-symmetric conditions $\lambda_i a_{ij} = \lambda_j a_{ji}$ hold. Furthermore, let $\rho : S_0 \times S_0$ be a symmetric function. Suppose the base game is a potential game admitting the potential function ρ. Then, G is a w-potential game with $\omega_i = \frac{1}{2\lambda_i}$ for all $i \in N$, and

$$P(s) = \sum_i \sum_j \lambda_i a_{ij} \rho(s_i, s_j). \tag{40}$$

Proposition 4 gives the linear relationship between the potential function of the quasi-symmetric spatial game and that of the base game. Therefore, the next important work is to find the algebraic relationship between the potential function ρ of the base game and its utility functions α_1 and α_2. In fact, Definition 1 outlines a linear system where the existence of a potential function is equivalent to that of a solution. In the following, we give the matrix form of (1) when $n = 2$.

Let $i = 1$. Then, using vector forms of strategies, Eq. (1) can be expressed into its vector form as

$$V_1^\alpha(s_1 - s_1')s_2 = V^\rho(s_1 - s_1')s_2. \tag{41}$$

Fix $s_2 = \delta_k^1$, $s_1 = \delta_k^1$, $s_1' = \delta_k^j$. Let s_2 run from δ_k^1 to δ_k^k. Then, using (41) we have

$$V_1^\alpha(\delta_k^1 - \delta_k^j)I_k = V^\rho(\delta_k^1 - \delta_k^j)I_k. \tag{42}$$

According to Definition A1, Eq. (42) can be expressed as $V_1^\alpha[(\delta_k^1 - \delta_k^j) \otimes I_k] = V^\rho[(\delta_k^1 - \delta_k^j) \otimes I_k]$.

Let $\Gamma_1^j := (\delta_k^1 - \delta_k^j) \otimes I_k$. Let j run from 2 to k. Then, we have

$$V_1^\alpha[\Gamma_1^2, \cdots, \Gamma_1^k] = V^\rho[\Gamma_1^2, \cdots, \Gamma_1^k]. \tag{43}$$

Similarly, for $i = 2$, it can be obtained that

$$V_2^\alpha[\Gamma_2^2, \cdots, \Gamma_2^k] = V^\rho[\Gamma_2^2, \cdots, \Gamma_2^k], \tag{44}$$

where $\Gamma_2^j = I_k \otimes (\delta_k^1 - \delta_k^j)$, $j = 2, \cdots, k$.

Let $\Upsilon_1 = [\Gamma_1^{\mathrm{T}}, \Gamma_2^{\mathrm{T}}]^{\mathrm{T}}$, $\Upsilon_2 = \mathrm{diag}(\Gamma_1, \Gamma_2)$, where $\Gamma_i = [\Gamma_i^2, \cdots, \Gamma_i^k]^{\mathrm{T}}$, $i = 1, 2$. Then, combining (43) and (44) yields a linear system:

$$\Upsilon_1 (V^\rho)^{\mathrm{T}} = \Upsilon_2 (V_g)^{\mathrm{T}}. \tag{45}$$

In the following, we give the matrix form of (40). Since ρ is symmetric, it can be verified that

$$\rho(s_i, s_j) = \rho(s_j, s_i), \quad \forall\ s_i, s_j \in S_0.$$

Hence, Eq. (40) can be expressed into its vector form as $V^P \ltimes_{t=1}^n s_t = V^\rho [\sum_{(i,j)\in\mathcal{E}} (\lambda_i a_{ij} + \lambda_j a_{ji}) s_{\min\{i,j\}} \ltimes s_{\max\{i,j\}}]$. Using deleting operators, we can further obtain

$$V^P \ltimes_{t=1}^n s_t = V^\rho [\sum_{(i,j)\in\mathcal{E}} (\lambda_i a_{ij} + \lambda_j a_{ji}) E_{\{\min\{i,j\},\max\{i,j\}\}}] \ltimes_{t=1}^n s_t, \tag{46}$$

where $E_{\{\min\{i,j\},\max\{i,j\}\}} = D_r^{[k^2, k^{n-\max\{i,j\}}]} D_r^{[k, k^{\max\{i,j\}-\min\{i,j\}-1}]} D_f^{[k^{\min\{i,j\}-1}, k]}$.

Let

$$E = \sum_{(i,j)\in\mathcal{E}} (\lambda_i a_{ij} + \lambda_j a_{ji}) E_{\{\min\{i,j\},\max\{i,j\}\}}. \tag{47}$$

Then, Eq. (46) can be expressed as

$$V^P = V^\rho E. \tag{48}$$

The type of the network graph is closely related to the type of the base game. To be specific, an undirected graph corresponds to a symmetric base game while a directed graph corresponds to an asymmetric base game. Hence, we consider the following two cases, respectively.

• When the graph of a quasi-symmetric game is directed, the base game is an asymmetric game. Applying Proposition 2 and Theorem 2 to (45) and (48), respectively, we obtain a linear system with unknowns x and ν:

$$\begin{cases} V^P = \nu H E, \\ \Upsilon_1 (\nu H)^{\mathrm{T}} = \Upsilon_2 \Phi_0^e x. \end{cases} \tag{49}$$

Equivalently, it can be rewritten as

$$\begin{bmatrix} E^{\mathrm{T}} H^{\mathrm{T}} & \mathbf{0} \\ \Upsilon_1 H^{\mathrm{T}} & -\Upsilon_2 \Phi_0^e \end{bmatrix} \begin{bmatrix} \nu^{\mathrm{T}} \\ x \end{bmatrix} = \begin{bmatrix} (V^P)^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix}. \tag{50}$$

Further, to assure the base game is asymmetric, we propose the following condition: for any given solution of (50) denoted by x_0, we have

$$\mathrm{Rank}(\Psi_0^e) \neq \mathrm{Rank}([\Psi_0^e, \Phi_0^e x_0]). \tag{51}$$

• When the graph of a quasi-symmetric game is undirected, the base game is a symmetric game. In this case, replacing Φ_0^e with Ψ_0^e in (50), we obtain

$$\begin{bmatrix} E^{\mathrm{T}} H^{\mathrm{T}} & \mathbf{0} \\ \Upsilon_1 H^{\mathrm{T}} & -\Upsilon_2 \Psi_0^e \end{bmatrix} \begin{bmatrix} \nu^{\mathrm{T}} \\ x \end{bmatrix} = \begin{bmatrix} (V^P)^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix}. \tag{52}$$

Let V^w be the structure vector of the global objective function $W : S \to \mathbb{R}$, and substitute V^P with V^w. Then, the above arguments lead to the following theorems. Though each result is only a sufficient condition, it is enough in application.

Theorem 5. For a quasi-symmetric spatial game with a directed graph, if Eq. (50) has a solution and Eq. (51) holds, then there exist utility functions α_1 and α_2 to convert a quasi-symmetric spatial game with utility functions defined in (39) into a w-potential game with W as its potential function. Moreover, the utility functions can be calculated by the following formula: $V_g = \Phi_0^e x$.

Theorem 6. For a quasi-symmetric spatial game with an undirected graph, if Eq. (52) has solutions, then there exist utility functions α_1 and α_2 to convert a quasi-symmetric spatial game with utility functions defined in (39) into a w-potential game with W as its potential function. Moreover, the utility functions can be calculated by the following formula: $V_g = \Psi_0^e x$.

Table 2 Global objective function W.

	111	112	113	121	122	123	131	132	133
W	96	88	36	44	12	84	28	100	76
	211	212	213	221	222	223	231	232	233
W	76	36	68	40	48	16	56	0	72
	311	312	313	321	322	323	331	332	333
W	92	52	84	24	32	0	92	36	108

Table 3 Utility matrix of g.

	11	12	13	21	22	23	31	32	33
α_1	5	3	7	0	4	0	4	0	9
α_2	9	4	8	4	5	1	6	-1	8
ρ	8	3	7	3	4	0	7	0	9

Example 3. Consider a quasi-symmetric spatial game G with $N = \{1,2,3\}$, $S_i = S_0 = \{1,2,3\}$, $i = 1,2,3$. Assume the incidence matrix $A = \begin{bmatrix} 0&2&0\\ 1&0&2\\ 0&2&0 \end{bmatrix}$. Set $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 2$. Then, it can be verified that A is quasi-symmetric, and G is a quasi-symmetric game. Assume the pre-assigned global objective function W is given in Table 2.

(1) Assume the graph is directed as shown in Figure 2. Then, using (4), (43) and (47), it can be calculated that

$$H_{[2,3]} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, \Upsilon_1 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}, \Phi_0^e = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&1&0&1&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\ 1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&00\\ 0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1 \end{bmatrix}^{\mathrm{T}},$$

$$E = (\lambda_1 a_{12} + \lambda_2 a_{21})E_{12} + (\lambda_2 a_{23} + \lambda_3 a_{32})E_{23} = \begin{bmatrix} 12&4&4&0&0&0&0&0&0&8&0&0&0&0&0&0&0&0&8&0&0&0&0&0&0&0&0\\ 0&8&0&4&4&4&0&0&0&0&8&0&0&0&0&0&0&0&0&8&0&0&0&0&0&0&0\\ 0&0&8&0&0&0&4&4&4&0&0&8&0&0&0&0&0&0&0&0&8&0&0&0&0&0&0\\ 0&0&0&8&0&0&0&0&0&4&4&4&8&0&0&0&0&0&0&0&0&8&0&0&0&0&0\\ 0&0&0&0&8&0&0&0&0&0&0&0&4&12&4&0&0&0&0&0&0&0&8&0&0&0&0\\ 0&0&0&0&0&8&0&0&0&0&0&0&0&0&8&4&4&4&0&0&0&0&0&8&0&0&0\\ 0&0&4&0&0&0&8&0&0&0&0&0&0&0&0&8&0&0&4&4&4&0&0&0&8&0&0\\ 0&0&0&0&0&0&0&8&0&0&0&0&0&0&0&0&8&0&0&0&0&4&4&4&0&8&0\\ 0&0&0&0&0&0&0&0&8&0&0&0&0&0&0&0&0&8&0&0&0&0&0&0&4&4&12 \end{bmatrix},$$

$$\Upsilon_2 = \begin{bmatrix} 1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&0&0&0&0&1&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&-1 \end{bmatrix}.$$

(a) In [24], the utility functions of the base game g is designed as in Table 3.

According to Theorem 1, it can be checked that g is a finite game with the symmetric potential function ρ which is shown in Table 3.

(b) Inserting the above matrices into (50), it can be verified that Eq. (50) has solutions. For instance, $\nu_0 = [8,3,7,4,0,9]$, $x_0 = [7,2,6,3,-1,8,-2,1,1,2,2]^{\mathrm{T}}$. It can be calculated that the utility functions and the potential function are the same with that shown in Table 3, implying that the results in [24] can also be obtained by using the proposed method in this paper.

In the following, we show that more design methods can be obtained by using our approach. To simplify the idea, we fix $\nu = \nu_0$ in the first equality of (49). Then, we obtain all solutions that satisfy the second equality of (49) and (51), and can be expressed as $x_1 = x_0 + \eta_1\beta_1 + \cdots + \eta_6\beta_6$, where $\eta_t \in \mathbb{R}$ are arbitrary, $t = 1,2,\cdots,6$, $\beta_1 = [1,1,1,1,1,1,0,0,0,0,0]^{\mathrm{T}}$, $\beta_2 = [0,0,0,0,0,0,1,0,0,0,0]^{\mathrm{T}}$, $\beta_3 = [0,0,0,0,0,0,0,1,0,0,0]^{\mathrm{T}}$, $\beta_4 = [0,0,0,0,0,0,0,0,1,0,0]^{\mathrm{T}}$, $\beta_5 = [0,0,0,0,0,0,0,0,0,1,0]^{\mathrm{T}}$, $\beta_6 = [0,0,0,0,0,0,0,0,0,0,1]^{\mathrm{T}}$, and $\mathrm{Rank}(\Psi_0^e) \neq \mathrm{Rank}([\Psi_0^e, \Phi_0^e(x_0 + B\eta)])$. Here, $B = [\beta_1, \beta_2, \cdots, \beta_6]$, $\eta = [\eta_1, \cdots, \eta_6]^{\mathrm{T}}$. For example, let $\eta_1 = 1, \eta_2 = 3, \eta_3 = 3, \eta_4 = 4, \eta_5 = 5, \eta_6 = 6$. Then, we have $x_1 = [8,3,7,4,0,9,0,4,5,7,8]$. Now, the utility functions of the base game is $V_g = \Phi_0^e x = [8,7,12,3,8,5,7,4,14,15,10,14,11,12,8,7,0,9]$.

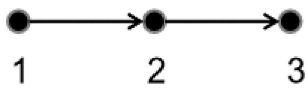

Figure 2 Directed graph.

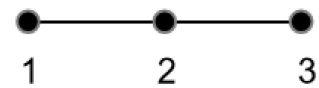

Figure 3 Undirected graph.

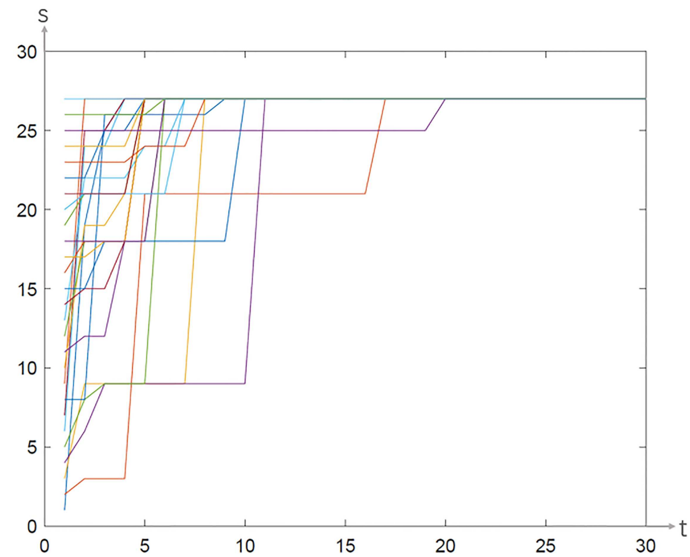

Figure 4 (Color online) Evolution process. By designing suitable utility functions of the base game, the global objective function W becomes the potential function. Then, by adopting the asynchronous updating rule, the spatial game evolves to the stable state (3,3,3) which is exactly the strategy profile that maximizes W.

(2) Assume the graph is undirected as shown in Figure 3. Then, using (4) and (28), it can be calculated that

$$\Psi_0^e = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1 \\ 1&0&0&1&0&0&1&0&0&1&1&1&0&0&0&0&0&0 \\ 0&1&0&0&1&0&0&1&0&0&0&0&1&1&1&0&0&0 \end{bmatrix}^{\mathrm{T}}, H_{[1,3]} = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}.$$

For the given W, fix $\nu = \nu_0$. Then, inserting above matrices into the second part of (52), it can be checked that Eq. (52) has solutions. For instance, $x_2 = [7, 2, 6, 3, -1, 8, 9, 10]^{\mathrm{T}}$. Now, the utility functions of the base game is $V_g = \Psi_0^e x = [16, 12, 6, 11, 13, -1, 15, 9, 8, 16, 11, 15, 12, 13, 9, 6, -1, 8]$.

Finally, set $q_i = \frac{1}{3}$, $i = 1, 2, 3$, and $\epsilon = 0.1$. A simulation result is given in Figure 4 which shows that the spatial game evolves to the unique optimal Nash equilibrium $(3, 3, 3)$ $(= \delta_{27}^{27})$. Here, the horizontal axis represents the time and the vertical axis represents the strategy profiles.

6 Conclusion

Finite games with symmetric potential functions are studied in this paper. First, by constructing a basis of symmetric functions, a linear system is proposed, which turns the verification of finite games with symmetric potential functions into checking the existence of their solutions. A basis of the corresponding subspace is obtained. Then, symmetric potential games are discussed, including the linear system to verify whether a finite game is a symmetric potential one, and a basis of the corresponding subspace. Finally, the theoretical results are applied to the optimization of quasi-symmetric spatial games over directed and undirected graphs, respectively. A sufficient condition for the w-potential game modeling is given as well as the formula to calculate utility functions.

We leave some problems for further study. For instance, (i) a basis for the quotient space generated by finite games with symmetric potential functions and symmetric potential games; (ii) the optimization of quasi-symmetric spatial games over hypergraphs; (iii) the research on practical applications to urban planning, ecosystems, and traffic flow, etc.

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Appendix A Semi-tensor product of matrices

Definition A1 (see the footnote1)). Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$. Then, the semi-tensor product of M and N is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \tag{A1}$$

where $t = lcm(n, p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Throughout this paper, the default matrix product is the semi-tensor product, that is $AB := A \ltimes B$.

Next, we consider the matrix expression of logical relations. Let $\mathcal{D}_k := \{1, 2, \cdots, k\}$ be a finite set with k elements, in which the concrete number j has no quantitative meaning, but is only used to represent the j-th element in the set. In this paper, $\mathcal{D}_k$ represents a finite set of k logical variables. For example, in classical logic relation, logic variables can only take values on a finite set $A = \{T(True), F(False)\}$. Usually, T is set to be 1 and F to be 0. Here, for the sake of uniformity, F corresponds to 2. There is no essential difference. Then, we have $A = \{1, 2\}$.

1) Cheng D, Qi H, Zhao Y. An Introduction to Semi-tensor Product of Matrices and its Applications. Singapore: World Scientific, 2012.

For a logical variable $j \in \mathcal{D}_k$, the vector form expression of j, denoted by $\vec{j}$, is defined as $\vec{j} := \delta_k^j \in \Delta_k$. Corresponding $j \in \mathcal{D}_k$ to $\delta_k^j \in \Delta_k$, that is, identifying $j \sim \delta_k^j$, $j = 1, 2, \cdots, k$, it can be verified that $\mathcal{D}_k \sim \Delta_k$. Here, $\Delta_k := \{\delta_k^i \mid i = 1, 2, \cdots, k\}$ is the set of all columns of identity matrix I_k, and the symbol $\sim$ stands for an equivalence relation between two finite sets. For instance, in classical logic relation, we have $1 \sim \vec{1} := \delta_1^2 = [1, 0]^{\mathrm{T}}$, $2 \sim \vec{2} := \delta_2^2 = [0, 1]^{\mathrm{T}}$.

Proposition A1 (see the footnote[1])**.** For a function $f : \mathcal{D}_k^n \to \mathbb{R}$, there exists a unique row vector $V^f \in \mathbb{R}^{k^n}$, such that (in vector form) $f(x_1, \cdots, x_n) = V^f \ltimes_{i=1}^n x_i$, where $x_i \in \mathcal{D}_k,\ i = 1, 2, \cdots, n$; $\ltimes_{i=1}^n x_i := x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n$. V^f is called the structure vector of f.

Appendix B Symmetric group

Let $\Omega = \{1, 2, \cdots, n\}$. Then, a permutation on Ω is a one-to-one mapping, say, $\sigma : \Omega \to \Omega$. If $\sigma(j) = i_j$, $j = 1, \cdots, n$, then σ can be expressed by $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$. Let $\mathbf{S}_n$ denote the set of permutations on Ω. Define a product of any two permutations μ and σ as their composition, that is $(\sigma \circ \mu)(\alpha) = \sigma(\mu(\alpha))$, $\alpha \in \Omega$. Then, $\mathbf{S}_n$ becomes a group, called an n-th order symmetric group.

The following result gives some standard sets of generators of $\mathbf{S}_n$.

Proposition B1 (see the footnote[2])**.** (1) $\mathbf{S}_n$ is generated by transpositions. That is, $\mathbf{S}_n = \langle (i, j) \mid 1 \leqslant i < j \leqslant n \rangle$. (2) $\mathbf{S}_n$ is generated by transpositions with 1. That is, $\mathbf{S}_n = \langle (1, j) \mid 1 < j \leqslant n \rangle$.

2) Jacobson N. Basic Algebra I. 2nd ed. San Francisco: Freeman, 1985.