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• Supplementary File •

Predefined-Time-Synchronized Control for Euler-Lagrange Systems

Wanyue Jiang[1], Shuzhi Sam Ge[2] & Dongyu Li[3*]

[1]*Institute for Future & Shandong Key Laboratory of Industrial Control Technology, School of Automation, Qingdao University, Qingdao 266071, China*
[2]*Department of Electrical and Computer Engineering, National University of Singapore, Singapore 117576*
[3]*School of Cyber Science and Technology, Beihang University, Beijing 100191, China*

Appendix A Stability Analysis of the PTSS Controller

We introduce the following theorem and lemma for PTSS control.

Theorem A1. System $\dot{x} = f(x,t)$ achieves PTSS in T_c if:

1. there is a Lyapunov function $V(x)$,

$$\dot{V}(x) \leqslant -\lambda(pV^{\alpha}(x) + gV^{\beta}(x))^k, \tag{A1}$$

where α, β, p, g, k are positive parameters with $k\alpha < 1$ and $k\beta > 1$, and $\lambda = \lambda(p, g, k, \alpha, \beta, T_c)$ is a control parameter related with T_c,

$$\lambda(p, g, k, \alpha, \beta, T_c) = \frac{p^k(1-k\alpha) + g^k(k\beta - 1)}{p^k g^k T_c (1-k\alpha)(k\beta-1)}, \tag{A2}$$

2. the state x is ratio persistent with $x/\|x\| = \zeta f(x,t)/\|f(x,t)\|$ for $x \neq 0$ and $\zeta \in \{1, -1\}$.

Proof. Since the derivative of the Lyapunov function satisfies $\dot{V}(x) \leqslant -\lambda(pV^{\alpha}(x) + gV^{\beta}(x))^k$, the following result can be obtained:

$$\dot{V}(x) \leqslant -(p\lambda^{\frac{1}{k}} V^{\alpha}(x) + g\lambda^{\frac{1}{k}} V^{\beta}(x))^k. \tag{A3}$$

From [1], the system is fixed-time stable within time T,

$$T \leqslant \frac{1}{\lambda}\left(\frac{1}{p^k(1-k\alpha)} + \frac{1}{g^k(k\beta-1)}\right) = T_c, \tag{A4}$$

which indicates that the system converges within $T \leqslant T_c$. Moreover, x is ratio persistent, leading to the time-synchronized property according to [2]. Thus the system is PTSS as in the Definition.

Lemma A1. System $\dot{x} = f(x,t)$ is PTSS within T_c if the state x varies in the following manner:

$$\dot{x} = -\lambda\left(p, g, 1, \frac{\alpha+1}{2}, \frac{\beta+1}{2}, 2T_c\right)\left(p\,\mathrm{sig}_{\mathrm{n}}^{\alpha}(x) + g\,\mathrm{sig}_{\mathrm{n}}^{\beta}(x)\right), \tag{A5}$$

where the parameters are detailed in Theorem A1.

Proof. Consider $V(x) = x^{\mathrm{T}}x$, which has the following time-derivative,

$$\dot{V}(x) = x^{\mathrm{T}}\dot{x} = -2\lambda\left(p, g, 1, \frac{\alpha+1}{2}, \frac{\beta+1}{2}, 2T_c\right)\left(p\|x\|^{\alpha+1} + g\|x\|^{\beta+1}\right). \tag{A6}$$

Let $\bar{\alpha} = (\alpha+1)/2$ and $\bar{\beta} = (\beta+1)/2$, we have

$$\dot{V}(x) = -\lambda\left(p, g, 1, \bar{\alpha}, \bar{\beta}, 2T_c\right)\left(pV^{\bar{\alpha}}(x) + gV^{\bar{\beta}}(x)\right), \tag{A7}$$

which formulates the same as (A3) and the system achieves fixed-time stability within T_c according to [1].

The equation (A5) leads to

$$\frac{\dot{x}}{\|\dot{x}\|} = -\frac{p\|x\|^{\alpha}\,\mathrm{sign}_{\mathrm{n}}(x) + g\|x\|^{\beta}\mathrm{sign}_{\mathrm{n}}(x)}{\left\|p\|x\|^{\alpha}\mathrm{sign}_{\mathrm{n}}(x) + g\|x\|^{\beta}\mathrm{sign}_{\mathrm{n}}(x)\right\|} = -\frac{\mathrm{sign}_{\mathrm{n}}(x)}{\|\mathrm{sign}_{\mathrm{n}}(x)\|} = -\frac{x}{\|x\|}, \tag{A8}$$

therefore the system is ratio persistent.

The two conditions in Theorem A1 are all satisfied, as a result, PTSS is achieved.

* Corresponding author (email: dongyuli@buaa.edu.cn)

The Euler-Lagrange system (1) is a second-order system with state vector q. Under the proposed controller (6), the state vector q is PTSS within the predefined time T_c. The analysis can be conducted in two steps.

First, we will force s to the equilibrium before $\bar{T}_c < T_c$.

Consider the Lyapunov candidate $V = s^{\mathrm{T}}s$, which has a derivative of $\dot{V} = 2s^{\mathrm{T}}\dot{s}$. Substitute the system dynamics (1) into (2) yields

$$\dot{s} = \ddot{q} + \zeta_1 \frac{\mathrm{d}}{\mathrm{d}t}\left(p_1 \mathrm{sig}_{\mathrm{n}}^{\alpha_1}(q) + g_1 \mathrm{sig}_{\mathrm{n}}^{\beta_1}(q)\right) = M^{-1}(q)\left(\tau(t) - C(q,\dot{q})\dot{q} - g(q)\right) + \zeta_1 \theta(q)\dot{q}. \tag{A9}$$

Substitute the above equation to the $\dot{V}$ yields

$$\dot{V} = -2s^{\mathrm{T}}\zeta_2\left(p_2 \mathrm{sig}_{\mathrm{n}}^{\alpha_2}(s) + g_2 \mathrm{sig}_{\mathrm{n}}^{\beta_2}(s)\right) = -\zeta_2\left(p_2 V^{\frac{\alpha_2+1}{2}} + g_2 V^{\frac{\beta_2+1}{2}}\right), \tag{A10}$$

therefore the sliding manifold s is PTSS within time $\bar{T}_c$.

Then, after s reaches zero, $\dot{q}$ follows the manner of (A5) in Lemma. A1 and drives q to PTSS in $T_c - \bar{T}_c$.

Besides,

$$\dot{q} = -\zeta_1\left(p_1 \mathrm{sig}_{\mathrm{n}}^{\alpha_1}(q) + g_1 \mathrm{sig}_{\mathrm{n}}^{\beta_1}(q)\right), \tag{A11}$$

or equally,

$$\dot{q}_i = -\zeta_1\left(p_1\|q\|^{\alpha_1} + g_1\|q\|^{\beta_1}\right)\frac{q_i}{\|q\|}. \tag{A12}$$

Let $T(q)$ be the actual settling time of q. For $t \geqslant T(q)$, $q = 0$, thus $\dot{q} = 0$. For $t < T(q)$, $q_i \neq 0$, thus $\dot{q}_i \neq 0$.

Therefore, every element $q_i, i = 1, \cdots, n$, also converges time-synchronously at $t = T(q)$. In other words, the vector $[q, \dot{q}]^{\mathrm{T}}$ achieves time-synchronized convergence. As a result, system (1) is PTSS within T_c.

Appendix B Proof of Theorem 1 and Stability Analysis of the True-PTSS Controller

Proof. According to [3], the system converges in T_c on the first condition of Theorem 2, namely $x(t) = 0$ for $t \in [T_c, +\infty)$. Therefore, Theorem 2 holds if we can prove that every element x_i of the state x satisfies $x_i(t) \neq 0$ for $t \in [0, T_c)$.

The ratio persistent property of x indicates that every state element x_i converges simultaneously. Mathematically, there must be a time instant $T \leqslant T_c$ that $x_i(t) \neq 0$ for $t \in [0, T)$ and $x_i(t) = 0$ for $t \in [T, +\infty)$.

Then, we show that $T = T_c$. Inspired by [3], define $V(t,x) : [0, T_c) \times \mathbb{R}_+ \to \mathbb{R}_+$ as

$$V(t,x) = e^{W(t,\|x\|)} - 1. \tag{B1}$$

The derivative of $V(t,x)$ is

$$\dot{V}(t,x) = e^{W(t,\|x\|)}\mu(t)\frac{e^{W(t,\|x\|)} - 1}{e^{W(t,\|x\|)}} = \mu(t)V(t,x), \tag{B2}$$

which leads to

$$V(t,x) = e^{-\rho(t)}V(0,x_0). \tag{B3}$$

From (10),

$$V(t,x) = e^{-\rho(t)}V(0,x_0) > 0 \quad \text{for} \quad t < T_c. \tag{B4}$$

Recall the relation between $V(t,x)$ and $W(t,\|x\|)$, we have

$$e^{W(t,\|x\|)} > 1 \quad \text{for} \quad t < T_c, \tag{B5}$$

consequently $W(t,\|x\|) \neq 0$ for $t < T_c$.

According to the property of the class-K function that $W(t,\|x\|) = 0$ if and only if $\|x\| = 0$, we can achieve that $\|x\| \neq 0$ for $t < T_c$, which indicates that $\|x\|$ approaches zero at T_c precisely, thus completing the proof.

The stability analysis of the true-PTSS controller is presented based on the following Lemma.

Lemma B1. System $\dot{x} = f(x,t)$ is true-PTSS within T_c if the state x varies in the following manner:

$$\dot{x} = \begin{cases} -\frac{p\left(1-e^{-\|x\|^{\alpha}}\right)}{T_c - t}\mathrm{sig}_{\mathrm{n}}^{1-\alpha}(x), & \text{if } 0 \leqslant t < T_c, \\ 0, & \text{otherwise}, \end{cases} \tag{B6}$$

where $0 < \alpha < 1$ and $p > 0$.

Proof. Consider $V(x) = \|x\|^\alpha$, which is a class-K function in terms of $\|x\|$. For $t \in [0, T_c)$,

$$\dot{V}(x) = \frac{d}{dt}\left(x^{\mathrm{T}}x\right)^{\frac{\alpha}{2}} = \alpha\left(x^{\mathrm{T}}x\right)^{\frac{\alpha}{2}-1}x^{\mathrm{T}}\dot{x} = \alpha\|x\|^{\alpha-2}x^{\mathrm{T}}\dot{x}. \tag{B7}$$

It follows from (B6) that at $t < T_c$,

$$\dot{V}(x) = -\frac{\alpha p\left(1-e^{-\|x\|^\alpha}\right)}{T_c - t} = -\frac{\alpha p}{T_c - t}\frac{e^{\|x\|^\alpha}-1}{e^{\|x\|^\alpha}}. \tag{B8}$$

Let $\mu(t) = -\frac{\alpha p}{T_c - t}$, we have

$$\rho(t) = -\int_0^t \mu(s)\mathrm{d}s = \int_0^t \frac{\alpha p}{T_c - s}\mathrm{d}s = -\alpha p\ln(T_c - t), \tag{B9}$$

which satisfies (10) in Theorem 1. Moreover, from (B8),

$$\dot{V}(x) = \mu(t)\frac{e^{V(x)}-1}{e^{V(x)}}, \tag{B10}$$

which satisfies (9) in Theorem 1.

Then, the ratio persistence condition is checked. It is straightforward that for $t < T_c$,

$$\frac{\dot{x}}{\|\dot{x}\|} = -\frac{\mathrm{sig}_{\mathrm{n}}^{1-\alpha}(x)}{\|\mathrm{sig}_{\mathrm{n}}^{1-\alpha}(x)\|} = -\frac{x}{\|x\|}. \tag{B11}$$

Therefore, both conditions of Theorem 1 are satisfied, and the system is true-PTSS.

The Euler-Lagrange system (1) is a second-order system with state vector q. Under the proposed controller (12), the state vector q is true-PTSS at the predefined time T_c. The analysis can be conducted in two steps.

First, we will show that s can be stabilized within $\bar{T}_c < T_c$. The time derivative of s for $t < \bar{T}_c$ in (11) writes

$$\begin{aligned}\dot{s} =& \ddot{q} + \frac{1}{\alpha_1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_1(t)\left(1-e^{-\|q\|^{\alpha_1}}\right)\mathrm{sig}_{\mathrm{n}}^{1-\alpha_1}(q)\right]\\ =& \ddot{q} - \frac{\mu_1(t)}{\alpha_1}\mathrm{sig}_{\mathrm{n}}^{1-\alpha_1}(q)\frac{\mathrm{d}}{\mathrm{d}t}e^{-\|q\|^{\alpha_1}} + \frac{\eta(q)}{\alpha_1}\left[\dot{\mu}_1(t)\mathrm{sig}_{\mathrm{n}}^{1-\alpha_1}(q) + \mu_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{sig}_{\mathrm{n}}^{1-\alpha_1}(q)\right]\\ =& M^{-1}(q)\left[\tau(t) - C(q,\dot{q})\dot{q} - g(q)\right] + \mu_1(t)\left[\theta_1(q) + \theta_2(q)\dot{q}\right].\end{aligned} \tag{B12}$$

From (12) and (B12),

$$\dot{s} = -\frac{p_2\left(1-e^{-\|s\|^{\alpha_2}}\right)}{\bar{T}_c - t}\mathrm{sig}_{\mathrm{n}}^{1-\alpha_2}(s). \tag{B13}$$

Consider a class-K $V = \|s\|^{\alpha_2}$, which has the following derivative

$$\dot{V}(x) = \frac{d}{dt}\left(s^{\mathrm{T}}s\right)^{\frac{\alpha_2}{2}} = \alpha_2\|s\|^{\alpha_2-2}s^{\mathrm{T}}\dot{s}. \tag{B14}$$

From (B13) and (12), we have

$$\dot{V} = -\alpha_2\|s\|^{\alpha_2-2}s^{\mathrm{T}}\frac{p_2\left(1-e^{-\|s\|^{\alpha_2}}\right)}{\bar{T}_c - t}\mathrm{sig}_{\mathrm{n}}^{1-\alpha_2}(s) = \mu_2(t)\frac{e^{\|s\|^{\alpha_2}-1}}{e^{\|s\|^{\alpha_2}}} = \mu_2(t)\frac{e^V - 1}{e^V}. \tag{B15}$$

From $\mu_2(t) = -\alpha_2 p_2/(\bar{T}_c - t)$, we have

$$\rho_2(t) = -\int_0^t \mu_2(s)\mathrm{d}s = \alpha_1\alpha_2 p_1\ln(\bar{T}_c - t), \tag{B16}$$

which has the property of (10), leading to the convergence of s within $\bar{T}_c$ according to Theorem 1.

Then, after s reaches zero, $\dot{q}$ follows the manner of (B6) in Lemma. B1 and drives q to true-PTSS within $\bar{T}_c$.

Besides,

$$\dot{q} = -\frac{p_1\left(1-e^{-\|q\|^{\alpha_1}}\right)}{T_c - t}\mathrm{sig}_{\mathrm{n}}^{1-\alpha_1}(q), \text{ for } \bar{T}_c \leqslant t < T_c, \tag{B17}$$

or equally, the i-th state element

$$\dot{q}_i = -\frac{p_1\left(1-e^{-\|q\|^{\alpha_1}}\right)\|q\|^{1-\alpha_1}}{T_c - t}\mathrm{sign}_{\mathrm{n}}(q_i), \text{ for } \bar{T}_c \leqslant t < T_c, \tag{B18}$$

which is non-zero.

Therefore, every element q_i also converges time-synchronously at T_c. In other words, the vector $[q, \dot{q}]^{\mathrm{T}}$ achieves time-synchronized convergence. Thus System (1) is true-PTSS at T_c.

Appendix C Proof of Theorem 2, Convergence Analysis of the Disturbance Observer and the ε-PTSS Controller

Proof. Consider $V(x) = \|x\|^\alpha$, which is a class-K function in terms of $\|x\|$. When $t < T_c$,

$$\dot{V} = -\frac{\alpha p}{T_c + \Delta T - t}\frac{e^{V(x)} - 1}{e^{V(x)}}, \tag{C1}$$

which is derived similar to (B10).

Let $U(t) = e^{V(x(t))} - 1$, we have $\dot{U}(t) = -\frac{\alpha p}{T_c+\Delta T-t}U(t)$, thus

$$U(T_c) = \exp\left(-\int_0^{T_c}\frac{\alpha p}{T_c + \Delta T - t}\mathrm{d}t\right)U(0) = e^{-\alpha p \ln \Delta \mathrm{T}}U(0) = \Delta T^{-\alpha p}U(0). \tag{C2}$$

Invoking the construction of ΔT and V to the above equation yields

$$V(x(T_c)) = \ln\left(U(T_c) + 1\right) = \varepsilon^\alpha. \tag{C3}$$

Recall that $V(x) = \|x\|^\alpha$, we have $\|x(T_c)\| = \varepsilon$.

The structure of $\dot{x}$ in (13) is similar to (B6), we can also achieve the ratio persistent property of x.

From to the analysis above, $\dot{V} \leqslant 0$ for $t < T_c$, which indicates that $\|x\|$ is monotonously decreasing for $t < T_c$. Since $\|x(T_c)\| = \varepsilon$, we have $\|x\| \geqslant \varepsilon$ for $t < T_c$. According to the ratio persistent property of x, x_i/x_j = constant for any i, j and $x_j(0) \neq 0$. Therefore, every initially non-zero state element x_i satisfies $x_i \neq 0$ for $t < T_c$, which is consistent with the first requirement in the ε-PTSS Definition.

When $t \geqslant T_c$,

$$\dot{V}(t) = -\frac{\alpha p}{\Delta T}\frac{e^{V(x)} - 1}{e^{V(x)}} \leqslant 0, \tag{C4}$$

which indicates that $\|x\|$ will not increase and stays in the region $\|x\| \leqslant \varepsilon$. This is consistent with the second requirement of the ε-PTSS Definition, thus completing the proof.

Then, the convergence analysis of the disturbance observer is conducted. Define the observation error of the first/second-order state as $\tilde{z}_0 = z_0 - \dot{q}$ and $\tilde{z}_1 = z_1 - M^{-1}(q)d(t)$, respectively. We introduce the following lemma.

Lemma C1. Considering the system (1) and the disturbance observer (15)-(16), under Assumption 1, $\tilde{z}_0$ and $\tilde{z}_1$ converge to zero within the following time-bound

$$T_{ob} \leqslant \gamma_{ob} + \frac{\sigma + M\gamma_{ob}}{(1 - Mh/k_{z1})\,m} + \frac{\sigma}{m}, \tag{C5}$$

where h, M, m and γ_{ob} have the forms

$$h = \frac{1}{k_{z1}} + \left(\frac{2e}{mk_{z1}}\right)^{\frac{1}{3}}, \qquad \gamma_{ob} = \gamma\left(2k_{z1}, 2k_{z2}, \frac{3}{4}, \frac{1+\nu}{2}, 1\right),$$
$$M = k_{z3} + \delta, \qquad m = k_{z3} - \delta, \tag{C6}$$

and the following inequalities hold:

$$k_{z3} > \delta, k_{z1} > Mh. \tag{C7}$$

Proof. From (15)-(16), the error dynamics is further derived

$$\begin{aligned}\dot{\tilde{z}}_0 &= -k_{z1}\mathrm{sig}_c^{\frac{1}{2}}\left(\tilde{z}_0\right) - k_{z2}\mathrm{sig}_c^{\nu}\left(\tilde{z}_0\right) + \tilde{z}_1,\\ \dot{\tilde{z}}_1 &= -k_{z3}\mathrm{sign_c}\left(\tilde{z}_0\right) - \dot{d}(t).\end{aligned} \tag{C8}$$

The bound in (C5) is calculated by considering the following three situations:

Case 1: For every element i, $i = 1, \cdots, n$, $\mathrm{sign_c}(\tilde{z}_{1,i}(0)) = 0$ or $\mathrm{sign_c}(\tilde{z}_{1,i}(0)) = -\mathrm{sign_c}(\tilde{z}_{0,i}(0))$.

Let's construct the Lyapunov function $V_{ob} = \tilde{z}_0^{\mathrm{T}}\tilde{z}_0$ with the following derivative

$$\dot{V}_{ob} = -2k_{z1}\tilde{z}_0^{\mathrm{T}}\mathrm{sig}_c^{\frac{1}{2}}\left(\tilde{z}_0\right) - 2k_{z2}\tilde{z}_0^{\mathrm{T}}\mathrm{sig}_c^{\nu}\left(\tilde{z}_0\right) + 2\tilde{z}_0^{\mathrm{T}}\tilde{z}_1.$$

In this case, each element $\mathrm{sign_c}(\tilde{z}_{1,i})$ has either the opposite sign with respect to $\mathrm{sign_c}(\tilde{z}_{0,i})$ or the zero value, which leads to $\tilde{z}_0^{\mathrm{T}}\tilde{z}_1 \leqslant 0$. Define $V_{ob,i} = \tilde{z}_{0,i}^2$ for $i = 1, \cdots, i$, where $\tilde{z}_{0,i}$ is the i-th element of $\tilde{z}_0$. Therefore we have

$$\dot{V}_{ob,i} \leqslant -2k_{z1}\tilde{z}_{0,i}\mathrm{sig}_c^{\frac{1}{2}}\left(\tilde{z}_{0,i}\right) - 2k_{z2}\tilde{z}_{0,i}\mathrm{sig}_c^{\nu}\left(\tilde{z}_{0,i}\right) = -2k_{z1}V_{ob,i}^{\frac{3}{4}} - 2k_2V_{ob,i}^{\frac{1+\nu}{2}}.$$

From the above equation, $\tilde{z}_{0,i}$ is fixed-time stable. The settling time of $\tilde{z}_0$ can be calculated as

$$T_{ob1} \leqslant \gamma_{ob} = \gamma\left(2k_{z1}, 2k_{z2}, \frac{3}{4}, \frac{1+\nu}{2}, 1\right), \tag{C9}$$

where the formulation of γ can be found in the following equation, $\gamma(p,g,\alpha,\beta,k)$ is defined as

$$\gamma(p,g,\alpha,\beta,k)=\frac{B\left(\omega_\alpha,\omega_\beta\right)}{p^{k-\omega_\alpha}g^{\omega_\beta}(\beta-\alpha)}, \tag{C10}$$

where α,β,p,g,k are positive parameters with $k\alpha<1$ and $k\beta>1$, $\omega_\alpha=\frac{1-k\alpha}{\beta-\alpha}$, $\omega_\beta=\frac{k\beta-1}{\beta-\alpha}$. The detailed construction of the Beta function B can be found in [4].

In the case of $\|\tilde{z}_0(0)\|\to+\infty$, we have $T_{ob1}=\gamma_{ob}$.

When $t=T_{ob1}$,

$$|\tilde{z}_{1,i}(T_{ob1})|\leqslant\sigma+MT_{ob1}.$$

Considering $t>T_{ob1}$, $\tilde{z}_{1,i}(T_{ob1})$ starts at $(0,\sigma+MT_{ob1})$ for $i=1,\cdots,i$. Following Theorem 4.5 in [5] and the derivations in [6], with the condition of $Mh/k_{z1}<1$, the settling time of $\tilde{z}_{1,i}(T_{ob1})$ satisfies

$$T_{ob2}\leqslant\frac{\sigma+MT_{ob1}}{\left(1-Mh/k_{z1}\right)m}. \tag{C11}$$

Therefore the settling time of the disturbance observer satisfies

$$T_{ob}^{(1)}\leqslant T_{ob1}+T_{ob2}\leqslant\gamma_{ob}+\frac{\sigma+M\gamma_{ob}}{\left(1-Mh/k_{z1}\right)m}.$$

Case 2: There exist an element i, such that $\mathrm{sign_c}(\tilde{z}_{1,i}(0))=\mathrm{sign_c}(\tilde{z}_{0,i}(0))$.

We consider the case that there exists a dimension that $\mathrm{sign_c}(\tilde{z}_1(0))$ and $\mathrm{sign_c}(\tilde{z}_0(0))$ have the same sign, and $\tilde{z}_{0,i}$ cannot reach zero before $\mathrm{sign_c}(\tilde{z}_{1,i})=-\mathrm{sign_c}(\tilde{z}_{0,i})$. Thus $\tilde{z}_{1,i}$ converges before $\tilde{z}_{0,i}$. From (C8), we can derive that $\tilde{z}_{1,i}$ arrives at the origin within

$$T_{ob}^{(2)}\leqslant\frac{\sigma}{m}.$$

When $t\geqslant T_{ob0}$, we have $\mathrm{sign_c}\left(\tilde{z}_1\left(T_{ob}^{(2)}\right)\right)=0$ or for $i=1,\cdots,n$, $\mathrm{sign_c}\left(\tilde{z}_{1,i}\left(T_{ob}^{(2)}\right)\right)=-\mathrm{sign_c}\left(\tilde{z}_{0,i}\left(T_{ob}^{(2)}\right)\right)$, which has been discussed in Case 1.

Case 3: There exist an element i, such that $\mathrm{sign_c}(\tilde{z}_{1,i}(0))\neq0$ and $\tilde{z}_{0,i}(0)=0$.

We have $\mathrm{sig}_c^{\frac{1}{2}}\left(\tilde{z}_{0,i}\right)=\mathrm{sig}_c^{\nu}\left(\tilde{z}_{0,i}\right)=0$, thus $\dot{\tilde{z}}_{0,i}=\tilde{z}_{1,i}\neq0$. In other words, $\tilde{z}_{0,i}(0)=0$ is a short enough transient process and it becomes Case 1 or Case 2 within $T_{ob}^{(3)}\to0$.

From $T_{ob}^{(1)}$, $T_{ob}^{(2)}$ and $T_{ob}^{(3)}$, the settling time

$$T_{ob}\leqslant\gamma_{ob}+\frac{\sigma}{m}+\frac{\sigma+M\gamma_{ob}}{\left(1-Mh/k_{z1}\right)m}, \tag{C12}$$

which completes the proof.

The above lemma estimates the disturbance before T_{ob}. Since T_{ob} is a function of the parameters k_{z1},k_{z2}, and k_{z3}, the disturbance can be estimated accurately by designing the parameters such that $T_{ob}<T_c$.

Finally, the stability analysis of the ε-PTSS controller is conducted.

According to Theorem 2, the state vector q of system (1) is ε-PTSS at T_c if $s=0$. We will show that s can be stabilized within T_c.

Similar to (B12), $\dot{s}$ of (16) writes

$$\dot{s}=M^{-1}(q)\left[\tau(t)-C(q,\dot{q})\dot{q}-g(q)+d(t)\right]+\theta_3(q)+\theta_4(q)\dot{q}. \tag{C13}$$

Note that z_1 is designed to satisfy $\tilde{z}_1=z_1-M^{-1}(q)d(t)=0$ for $t\in[T_{ob},T_c]$, $d(t)$ is accurately estimated before T_c. Substitute $\tau(t)$ of (19) into (C13) yields

$$\dot{s}=-\lambda_\varepsilon M(q)\left(p_2\mathrm{sig}_n^{\alpha_2}(s)+g_2\mathrm{sig}_n^{\beta_2}(s)\right), \tag{C14}$$

where every element of s is monotonously moving towards zero.

Let's calculate the supremum of $\|q\|$ before $s=0$. For any state element q_i, In the case of $\mathrm{sign}\left(\dot{q}_i\right)=-\mathrm{sign}\left(q_i\right)$ or $\mathrm{sign}\left(\dot{q}_i\right)=0$, $|q_i|$ will not increase. In the other case, either $q_i=0$ or $\dot{q}_i,q_i$, and s_i have the same sign. From (14), $|\dot{q}_i|\leqslant|s_i|$. Since $|s_i|$ is monotonously decreasing, $|\dot{q}_i|<|s_i(0)|$. Thus we have

$$|q_i|<|q_i(0)|+|s_i(0)|T_c \tag{C15}$$

for $t<T_c$. Define $\bar{q}_i=|q_i(0)|+|s_i(0)|T_c$ and $\bar{q}=[\bar{q}_1,\cdots,\bar{q}_n]^{\mathrm{T}}$, we can achieve that $\|q\|<\|\bar{q}\|$ for $t<T_c$. Then, ΔT is calculated as

$$\Delta T=\left(\frac{e^{\varepsilon^{\alpha_1}}-1}{e^{\|\bar{q}\|^{\alpha_1}}-1}\right)^{-\frac{1}{\alpha_1p_1}}. \tag{C16}$$

Consider the Lyapunov function $V=s^{\mathrm{T}}s$. From (C13), $\dot{V}$ has the following derivative

$$\dot{V}=-\lambda_\varepsilon\left(2p_2V^{\frac{\alpha_2+1}{2}}+2g_2V^{\frac{\beta_2+1}{2}}\right), \tag{C17}$$

s converges within T_c. Moreover, $\dot{q}$ is non-zero at $t<T_c$ as in the true-PTSS case. System (1) is ε-PTSS at T_c.

Appendix D Simulation Details

We use the satellite in the local-vertical-local-horizontal rotating frame for controller verification [7]. The dynamics described in (1) have the following formulations,

$$M = mI_3, C = m \begin{bmatrix} 0 & -2\omega_0 & 2 \\ 2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, g = m \begin{bmatrix} -\omega_0^2 q_1 + \frac{\mu_e (R_0 + q_1)}{R^3}) - \frac{\mu_e}{R_0^2} \\ -\omega_0^2 q_2 + \frac{\mu_e q_2}{R^3} \\ \frac{\mu_e q_3}{R^3} \end{bmatrix}, \tag{D1}$$

where $q = [q_1, q_2, q_3]^{\mathrm{T}}$ is the relative coordinate, $\omega_0 = \sqrt{\mu_3 / R_0^3}$ is the orbital angular velocity, R_0 is the orbit radius, and R is the distance between the geocenter and the satellite. The control input τ is a three-dimensional force vector.

The control parameters used for the PTSS controller are demonstrated in Table. D1.

Table D1 Control parameters for PTSS

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
p_1	0.1	p_2	0.1	α_1	0.6	α_2	0.6
g_1	0.1	g_2	0.1	β_1	1.08	β_2	1.08
T_c	200s	ρ	0.5				

The system states under the PTSS controller are illustrated in the Letter. The initial state vector is $q_0 = [5, -15, 25]^{\mathrm{T}}$m in the simulation. The settling time is smaller than T_c, and the state elements under the proposed controllers converge time-synchronously, thus the goal of predefined-time-synchronized convergence is achieved. The time-synchronization property can be observed more clearly in the sub-figure, which is zoomed by 10^{-3}. The actual settling time is $t = 30.8$s for the PTSS controller.

The control parameters for true-PTSS are presented in Table. D2 and the initial state vector is $q_0 = [5, -15, 25]^{\mathrm{T}}$ m.

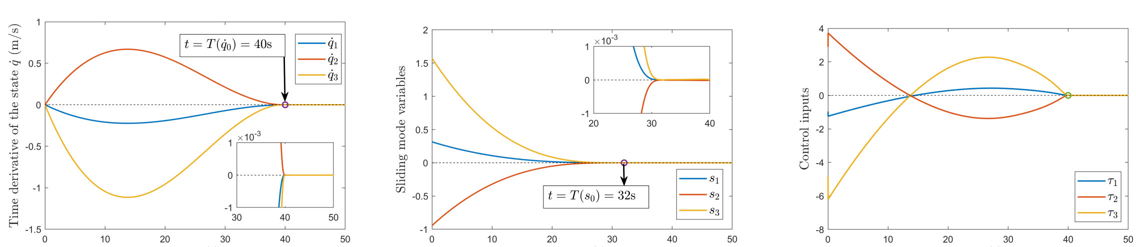
The system states under the true-PTSS controller are plotted in Fig. 1 (top right) of the Letter, where the settling time of the state elements q_1, q_2, and q_3 are all 40s as predefined. We zoom the states near the origin by 10^{-3} in the sub-figure. The results indicate that every state element converges precisely at the predefined time by using the proposed true-PTSS controller. From Fig. D1(a),$\dot{q}_i, i = 1, 2, 3$, also reach the origin at $t = T_c = 40$s. Thus $q(t) = \dot{q}(t) \to 0$ for $t \to T_c$ and remain zero for $t \geqslant T_c$. The true-predefined-time-synchronized convergence is achieved by the proposed controller.

Similar results can be found in the plots of s in Fig. D1(b). The predefined settling time of s is $\bar{T}_c(s_0) = 32$s. In the zoomed sub-figure, both s_1 and s_2 reach the origin at $t = 32$s and then stay at zero. At $t < 32$s, $s_2(t)$ and $s_1(t)$ keeps a persistent ratio of $s_2(t)/s_1(t) = 3$ in the whole process, which equals $q_2(t)/q_1(t)$. The persistent ratios indicate that different elements of the state vector converge at the same speed, which results in their simultaneous arrival. Similarly, $s_2(t)/s_3(t) = q_2(t)/q_3(t) = -0.6$ and $s_3(t)/s_1(t) = q_3(t)/q_1(t) = 5$ at $t < 32$s.

The control inputs are plotted in Fig. D1(c), where $\tau = [\tau_1, \tau_2, \tau_3]$ varies slowly according to the predefined time.

Table D2 Control parameters of true-PTSS

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
p_1	3	p_2	3	α_1	0.04	α_2	0.04
T_c	40s	$\bar{T}_c$	32s				



(a) The state derivatives (b) The sliding mode variables (c) The control inputs

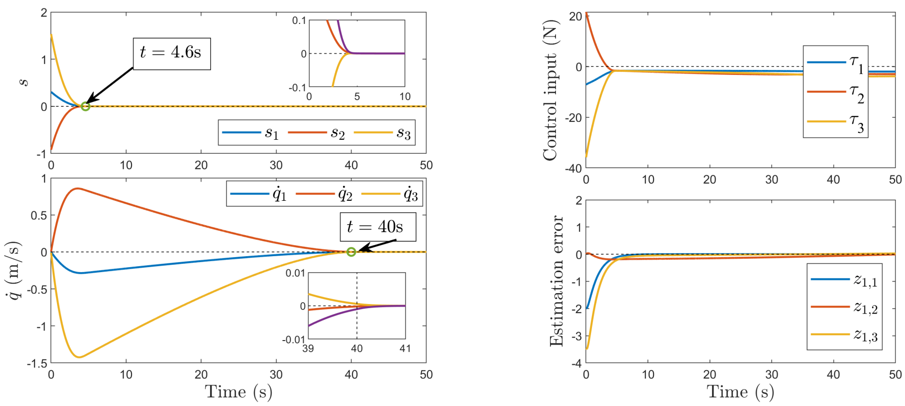
Figure D1 The state derivatives, the sliding mode variables, and the control inputs under the true-PTSS controller.

The disturbance and control parameters for the ε-PTSS controller are listed in Tab. D3.

The system state q, state derivative $\dot{q}$, and sliding-mode surface s under ε-PTSS controller are illustrated in Fig. D2 (a). In the simulation, the desired settling time is $T_c = 40$s and the steady state error is allowed to be $\varepsilon = 0.01$. The extra time is calculated as $\Delta T \approx 1$s. From the figures, the state $\|q\| < \varepsilon$ in T_c, and $\dot{q}$ is also stabilized to a small neighborhood of the equilibrium. The sliding-mode surface s reaches the origin at $t = 4.6$s, which is earlier than the predefined time. Besides, the time-synchronous property is well kept for $q, \dot{q}$ and s.

Table D3 Control parameters for ε-PTSS

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
p	3	α	0.04	T_c	40s	$d_2(t)$	$3\sin(0.01\pi t)$
p_2	0.1	α_2	0.6	$d_1(t)$	2	$d_3(t)$	$4\sin(0.005\pi t + \pi/3)$
k_{z1}	2	k_{z2}	2	k_{z3}	2	ν	1.2
g_2	0.1	β_2	1.08				



(a) The system state q, state derivative $\dot{q}$, and sliding-mode surface s

(b) The control input τ and disturbance estimation error z_1

Figure D2 Control Performance under ε-PTSS controller.

The control input $\tau(t)$ and the estimation error of the disturbance $\tilde{z}_1$ is illustrated in Fig. D2 (b). In this case, the control inputs do not stay at the origin when $t > T_c$, instead, $\tau(t)$ varies to compensate for the disturbance.

From the figures, ε-PTSS is achieved for the system with disturbance.

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