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Double-observer-based consensus of switched positive multiagent systems with switched topologies

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Abstract This study investigates the consensus of switched positive multiagent systems with switched topologies. Each agent in the system is governed by its own switching law, while the network topology switches according to a separate and independent switching law. Double observers are first designed individually for the state and disturbance. Linear programming and copositive Lyapunov functions are utilized to handle the leader-following consensus of systems with a single leader and multiple leaders. The contributions of this work are as follows. (i) A double-observer framework is established for switched positive multiagents. (ii) The leader-following consensus is solved using a linear approach. (iii) A linear programming-based design approach is proposed for the gain design of observers and control protocols. Finally, an example is provided to demonstrate the validity of the design.

Keywords switched positive multi-agent systems, switched topology, copositive Lyapunov function, linear programming

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1 Introduction

In recent years, multiagent systems (MASs) have attracted considerable attention owing to their theoretical complexity and wide applications in formation control [1], fuzzy control [2], clustering [3], etc. [4–8]. In practical applications, many dynamic processes involve non-negative quantities such as insect population, electric current, and container volumes. It is essential to employ positive systems to model these processes [9, 10]. Positive systems are an important class of systems whose states and outputs are always nonnegative [11–13]. Control problems related to vehicle formations, vessel fleets, and unmanned-aircraft clusters can be described as agents with positive characteristics. Thus, MASs with positive properties hold both theoretical and practical significance. A previous study [14] proposed a control reconstruction technique based on linear programming (LP) and addressed the positive consensus of homogeneous positive MASs (PMASs). Another work [15] proposed certain necessary and sufficient conditions and an effective algorithm to solve the positivity-preserving consensus problem in PMASs. More results on PMASs can be found in [16–19]. With the development of MASs, an important class of hybrid MASs, namely, switched MASs (SMASs), was introduced by Olfati-Saber [20]. SMASs possess the properties of both switched systems and MASs and have powerful modeling ability to describe agents with multiple modes and the status of multiple studies of MASs [21–30]. However, research on switched PMASs (SPMASs) is sparse. To the best of the authors' knowledge, there is no unified consensus on SPMASs. Moreover, the existing consensus design of MASs is unsuitable for SPMASs owing to the unique characteristics of PMASs.

The problem of consensus is an important issue for MASs. Some results of attempts to tackle the consensus problem are presented in [21–30]. For the scenario in which each agent contains switched subsystems, multiple Lyapunov functions (MLFs) were employed to achieve consensus of SMASs in [21]. Owing to the heterogeneity of the switching agents, a switching-dependent controller was constructed in [22] to solve the practical output synchronization problem. Another study [23] focused on devising suitable consensus protocols to reach resilient consensus in continuous-time and discrete-time SMASs. Further, Ref. [24] first investigated unknown asynchronous switching in a nonlinear SMAS with completely switched nonlinearities, and common Lyapunov functions were utilized to establish system consensus. A novel asynchronous impulsive containment control strategy was proposed for time-delay SMASs in [25]. Meanwhile, MLFs and the Razumikhin technique were utilized to establish a containment

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control formulation and investigate SMAS consensus. Previous studies [21–25] hypothesized that the communication topologies of SMASs were fixed. However, in real systems, these topologies potentially change under the influence of the external environment. Although such systems with dynamic communication topologies are also named SMASs, more complicated techniques are required to solve their consensus problem [26–30]. It is necessary to consider the switching of communication topologies between agents in an SMAS. For the consensus problem in MASs with switching topologies, Ref. [26] designed a novel distributed control strategy and a switching topology with heterogeneous Markov chains. A previous study [27] presented a distributed consensus scheme for addressing asynchronous consensus problems in continuous-time SMASs with discontinuous information transmission. By using the common quadratic Lyapunov function, Ref. [28] solved the problem of consensus for homogeneous MASs under arbitrary switching communication topologies. The distributed consensus problem and Zeno phenomenon in MASs were studied using fixed and switching directed graphs, respectively, in [29]. Meanwhile, Ref. [30] addressed the tracking consensus of MASs with general higher-order nonlinear node dynamics and switching topologies. For general systems, quadratic Lyapunov functions are commonly employed. However, they are not the optimal choice to tackle the positive consensus arising from the positivity of PMASs. In [31], an event-triggered filter was designed for positive systems using CLFs and LP. In [32], the shared CLFs among all subsystems in switched positive systems were investigated. An adaptive event-triggering mechanism and a new distributed filter framework were introduced into positive Markov systems in [33] using LP and stochastic CLFs to enhance output accuracy. Existing results verified that CLFs and LP are more suitable for dealing with the issues of positive systems than other approaches, such as quadratic Lyapunov functions and linear matrix inequalities. Thus, it is interesting to present a novel SPMAS consensus framework based on CLFs and LP.

The observer technique is extensively applied in smart techniques [34], water management [35], medical systems [36], and so on [37–39]. Its primary function is to estimate unknown variables of systems. Combined with controllers, the observer can enhance the performance and stability of systems. Owing to the unique characteristics of PMASs, distinctive considerations are required in the application of state observers. In [40], an observer-based control protocol was designed to address sufficient conditions for the consensus of PMASs. A fitting observer and a multistep algorithm were proposed in [41] to investigate the consensus of nonlinear PMASs. In [5], an observer-based controller was proposed, and two sufficient conditions for robust consensus were derived based on the positive system theory and graph theory. In [42], a distributed observer was introduced, and the consensus of PMAS was solved with the leader-following scenario by utilizing common Lyapunov functions. In [43], the positive system theory was applied to investigate the observer-based output-feedback protocol of PMASs; however, this study did not consider the impact of disturbances in systems. How to observe and reduce the effects of disturbances is a highly practical subject. For switched positive systems, a disturbance observer-based event-triggered control using LP and CLFs was designed in [44]. In [45], a controller was proposed based on disturbance observers to reduce the disturbances of each agent in MASs. Furthermore, Ref. [46] addressed the containment control problem of MASs with exogenous disturbances by adopting a disturbance observer-based control approach. For more studies on disturbance observer, please refer to [47–49]. To the best of the authors' knowledge, there is much scope for further exploration of double observers with state and disturbance observers in SMASs. This study intended to establish a novel framework for double observers on SPMASs.

This paper proposes state and disturbance observers for SMASs. Studies [14, 31–33] showed that LP and CLFs can effectively handle the issues of positive systems. Compared with [37, 42], the double observer framework proposed herein can achieve highly reliable control in complex environments. Inspired by [18, 42, 50, 51], an observer-based practical consensus with a single leader and multiple leaders was achieved. The contributions of this study are as follows. (i) A disturbance observer and a state observer were constructed for SPMASs. (ii) Two distinct observer-based feedback control protocols were proposed for a single leader and multiple leaders, respectively. (iii) All conditions on positivity and consensus were solved in terms of LP. The rest of this paper is organized as follows. Some preliminaries and problem formulation are provided in Section 2. Section 3 presents the main results. Section 4 presents an illustrative example, and the conclusion is summarized in Section 5.

Notation: $\mathcal{R}^n$ and $\mathcal{R}^{n\times n}$ represent the n-dimensional non-negative vectors and $n\times m$ matrix space, respectively. $\mathcal{N}^+$ denotes the set of non-negative real numbers. I_m denotes the m-dimensions identity matrix. For $A\in\mathcal{R}^{n\times n}$, a_{ij} represents the element in the ith row and the jth column of A. $A\succ 0$ ($\succeq 0$) means that $a_{ij}>0$ ($\geqslant 0$); $A\succ B$ and $A\succeq B$ too have similar meanings. $\mathbf{1}_m=(\underbrace{1,1,\ldots,1}_{m})^\top$, $\mathbf{1}_m^{(\iota)}=(\underbrace{0,0,\ldots,0}_{\iota-1},1,\underbrace{0,0,\ldots,0}_{m-\iota})^\top$, and $\mathbf{1}_{n\times n}$ symbolizes $n\times n$ matrix. The symbol $\otimes$ represents the Kronecker product. Further, blockdiag$\{\cdot\}$ is the block-diagonal matrix.

2 Preliminaries

Some lemmas and definitions for SPMASs are described in Subsection 2.1. The graph theory is introduced in Subsection 2.2 to explain communication relationships between agents.

2.1 SPMAS preliminaries

Consider the following switched multi-agent system:

$$\begin{aligned}&\dot{x}_i(t) = A_{i,\sigma_i(t)}x_i(t) + B_{i,\sigma_i(t)}u_i(t) + E_{i,\sigma_i(t)}w_i(t), i \in \mathcal{V},\\ &y_i(t) = C_{i,\sigma_i(t)}x_i(t) + D_{i,\sigma_i(t)}w_i(t),\end{aligned} \tag{1}$$

where $x_i(t) = (x_{1i}, x_{2i}, \ldots, x_{ni})^\top \in \mathcal{R}^n$, $u_i(t) \in \mathcal{R}^r$, $y_i(t) \in \mathcal{R}^o$, and $w_i(t) \in \mathcal{R}^s$ denote system states, control inputs, disturbances, and system outputs of each agent, respectively. Further, $\sigma_i(t)$ is the switching law of agents with $i \in \mathcal{V} = \{1, 2, \cdots, N\}$; it picks a value in a finite set $S = \{1, 2, \ldots, N\}$, where $N \in \mathcal{N}^+$. The system matrices satisfy the relation $A_{i,\sigma_i(t)} \in \mathcal{R}^{n\times n}$, $B_{i,\sigma_i(t)} \in \mathcal{R}^{n\times r}$, $E_{i,\sigma_i(t)} \in \mathcal{R}^{n\times s}$, $C_{i,\sigma_i(t)} \in \mathcal{R}^{o\times n}$, and $D_{i,\sigma_i(t)} \in \mathcal{R}^{o\times s}$. Throughout this paper, it is assumed that $A_{i,\sigma_i(t)}$ is Metzler, $B_{i,\sigma_i(t)} \succeq 0$, $E_{i,\sigma_i(t)} \succeq 0$, $C_{i,\sigma_i(t)} \succeq 0$, and $D_{i,\sigma_i(t)} \succeq 0$ for $\sigma_i(t) \in S$ in system (1).

The exogenous disturbance signal is constructed as follows:

$$\begin{aligned}&\dot{\xi}_i(t) = H_{i,\sigma_i(t)}\xi_i(t),\\ &w_i(t) = \mathit{\Gamma}_{i,\sigma_i(t)}\xi_i(t),\end{aligned} \tag{2}$$

where $\xi_i(t) \in \mathcal{R}^s$ is an additive signal of each agent, $\mathit{\Gamma}_{i,\sigma_i(t)} \in \mathcal{R}^{s\times s}$, $\mathit{\Gamma}_{i,\sigma_i(t)} \succeq 0$, and $H_{i,\sigma_i(t)} \in \mathcal{R}^{s\times s}$ is a Metzler matrix.

Definition 1 ([11, 12])**.** A control system is positive when all its states are non-negative for any non-negative initial conditions, inputs, and disturbance.

Lemma 1 ([11, 12])**.** A matrix A is Metzler if and only if there exists a constant δ such that $A + \delta I \succeq 0$.

Lemma 2 ([11, 12])**.** System (1) is positive if and only if A_ι is a Metzler matrix, $B_\iota \succeq 0$, $C_\iota \succeq 0$, $D_\iota \succeq 0$, and $E_\iota \succeq 0$ for $\iota \in S$.

Definition 2 ([7,8])**.** Given a switching signal $\sigma(t)$, let $N_{\sigma(t)}(t_2, t_1)$ be the times of the switching, where $0 \leqslant t_1 \leqslant t_2$. If $N_{\sigma(t)}(t_2, t_1) \leqslant N_0 + \dfrac{t_2 - t_1}{\tau^*}$ holds, then τ^* is an ADT of $\sigma(t)$, where $N_0 \geqslant 0$ and $\tau^* > 0$.

Lemma 3 ([11, 13])**.** For a positive system $\dot{x}(t) = Ax(t)$, the following conditions hold.

(i) The matrix A is a Hurwitz matrix.

(ii) There exists a vector $v \succ 0$ such that $A^\top v \prec 0$.

(iii) The system is stable.

Definition 3 ([4])**.** The leader-following consensus is achieved if the condition $\lim\limits_{t\to\infty} \|x_i(t) - \eta(t)\| = 0$ holds for $i = 1, \ldots, N$.

Definition 4. The practical consensus of PMASs is achieved if each agent is positive and the following conditions hold:

(i) $\lim\limits_{t\to\infty} \|x_i(t) - x^*\| < h$,

(ii) $\lim\limits_{t\to\infty} \|\eta_k(t) - \eta^*\| < \wp$, and

(iii) $\lim\limits_{t\to\infty} \|x_k(t) - \eta_k(t)\| < \Im$,

where $h > 0$, $\wp > 0$, $\Im > 0$, $x^* \succeq 0$, and $\eta^* \succeq 0$.

2.2 Communication topology

A switching signal $\sigma(t)$ is given with $\sigma(t)$ number of switching modes. The switching communication topology can be represented by a switching directed graph $\mathcal{G}_{\sigma(t)} = \{\mathcal{V}, \Psi_{\sigma(t)}\}$ in which each switching mode has a possible switching topology. $\mathcal{V} = \{1, 2, \ldots, N\}$ means that the vertex set of the graph $\mathcal{G}_{\sigma(t)}$. $\Psi_{\sigma(t)} \in \mathcal{V} \times \mathcal{V}$ is the edge set. It is assumed that the graph does not contain any self-loops. The joint graph of $\mathcal{G}_{\sigma(t)}$ over a certain time interval $[t_1^*, t_2^*)$, $0 \leqslant t_1^* \leqslant t_2^* \leqslant +\infty$ is defined as $\bigcup_{t\in[t_1^*,t_2^*)} \mathcal{G}_{\sigma(t)} = \{\mathcal{V}, \bigcup_{t\in[t_1^*,t_2^*)} \Psi_{\sigma(t)}\}$. Let $\mathcal{A}_{\sigma(t)} = [a_{ij}^{(\zeta(t))}] \in \mathcal{R}^{N\times N}$ be the switching adjacency matrix of $\mathcal{G}_{\sigma(t)}$ in which $a_{ij}^{(\zeta(t))} = 0$ if $(j, i) \notin \Psi_{\sigma(t)}$ and there is no edge directed from j to i, otherwise

$a_{ij}^{(\zeta(t))} = 1$. A Laplacian matrix of $\mathcal{G}_{\sigma(t)}$ is defined as $L_{\sigma(t)} = [l_{ij}^{(\zeta(t))}] = \mathcal{D}_{\sigma(t)} - \mathcal{A}_{\sigma(t)}$, where $\mathcal{D}_{\sigma(t)} = [d_{ij}(\zeta(t))]$ is the corresponding switching in-degree matrix with $[d_{ij}^{(\zeta(t))}] = 0$, $\forall i \neq j$ and $[d_{ii}^{(\zeta(t))}] = \sum_{j=1}^{N} a_{ij}^{(\zeta(t))}$.

3 Main results

This section consists of two subsections. Subsection 3.1 investigates the positive consensus for a single leader in the switching. Subsection 3.2 discusses the consensus of SPMAS with multiple leaders.

3.1 One Leader

The positive consensus of SPMASs (1) with one leader will be discussed in this subsection. The dynamic of the leader agent is described as follows:

$$\dot{\eta}(t) = S\eta(t), \tag{3}$$

where $\eta(t) \in \mathcal{R}^n$ is a state of a leader and $S \in \mathcal{R}^{n\times n}$ is a Metzler matrix.

The double observer-based switched protocol is designed for each agent as follows:

$$\begin{aligned} u_i(t) =& K_{1,i,\sigma_i(t)}\widehat{x}_i(t) + K_{2,i,\sigma_i(t)}\widehat{w}_i(t) + K_{3,i,\sigma_i(t)}y_i(t) + P_{i,\sigma_i(t)}\eta(t) \\ &+ K_{4,i,\sigma_i(t)}\textstyle\sum_{j=1}^{N} a_{ij}^{(\zeta(t))}(\widehat{x}_i(t) - \widehat{x}_j(t)), \quad i \in \mathcal{V}, \end{aligned} \tag{4}$$

where $\widehat{x}_i(t) \in \mathcal{R}^n$ and $\widehat{w}_i(t) \in \mathcal{R}^s$ are estimates of states and exogenous disturbances; $K_{1,i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$, $K_{2,i,\sigma_i(t)} \in \mathcal{R}^{r\times s}$, $K_{3,i,\sigma_i(t)} \in \mathcal{R}^{r\times o}$, $K_{4,i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$, and $P_{i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$ are gain matrices of the controller; the switching law of the topologies $\zeta(t)$ gets values in a finite set $S = \{1, 2, \ldots, N\}$, $N \in \mathcal{N}^+$.

A state observer is designed as follows:

$$\dot{\widehat{x}}_i(t) = G_{i,\sigma_i(t)}\widehat{x}_i(t) + B_{i,\sigma_i(t)}u_i(t) + J_{i,\sigma_i(t)}\widehat{w}_i(t) + M_{i,\sigma_i(t)}y_i(t), \tag{5}$$

where $G_{i,\sigma_i(t)} \in \mathcal{R}^{n\times n}$, $J_{i,\sigma_i(t)} \in \mathcal{R}^{n\times s}$, and $M_{i,\sigma_i(t)} \in \mathcal{R}^{n\times s}$ are observer gain matrices. Meanwhile, $G_{i,\sigma_i(t)}$ is Metzler. To measure the disturbances of system (2), a disturbance observer is constructed as follows:

$$\begin{aligned} &\dot{\widehat{\xi}}_i(t) = H_{i,\sigma_i(t)}\widehat{\xi}_i(t) + F_{i,\sigma_i(t)}\widehat{x}_i(t) + T_{i,\sigma_i(t)}y_i(t), \\ &\widehat{w}_i(t) = \varGamma_{i,\sigma_i(t)}\widehat{\xi}_i(t), \end{aligned} \tag{6}$$

where $\widehat{\xi}_i(t) \in R^s$ is the observer state, $H_{i,\sigma_i(t)} \in \mathcal{R}^{s\times n}$ is Metzler, and $F_{i,\sigma_i(t)} \in R^{s\times n}$ and $T_{i,\sigma_i(t)} \in R^{s\times o}$ are observer gain matrices. The error variables are defined as follows: $e_i(t) = \widehat{x}_i(t) - x_i(t)$, $\theta_i(t) = \widehat{\xi}_i(t) - \xi_i(t)$, and $\varepsilon_i(t) = x_i(t) - \eta(t)$.

Remark 1. A state observer is constructed to estimate the unmeasurable state variables in a system, while a disturbance observer is utilized to estimate external disturbances to enhance system robustness. Introduction of a disturbance observer can reduce system dependence on sensors, hardware costs, and design complexity. In practice, external disturbances and system states are somewhat correlated. From the forms in (5) and (6), it can be found that the state observer and the disturbance observer are interdependent and designed jointly. When the state and disturbance observers are designed separately, the coupling between them is disregarded. This can lead to a reduction in the accuracy of the estimation error. Joint observers can optimize system performance and improve the control effect of a system.

Thus, the closed-loop system is expressed as follows:

$$\begin{aligned} \dot{e}_i(t) =& G_{i,\sigma_i(t)}e_i(t) + J_{i,\sigma_i(t)}\varGamma_{i,\sigma_i(t)}\theta_i(t) + (G_{i,\sigma_i(t)} + M_{i,\sigma_i(t)}C_{i,\sigma_i(t)} - A_{i,\sigma_i(t)})\varepsilon_i(t) + (G_{i,\sigma_i(t)} \\ &+ M_{i,\sigma_i(t)}C_{i,\sigma_i(t)} - A_{i,\sigma_i(t)})\eta(t) + (J_{i,\sigma_i(t)} - E_{i,\sigma_i(t)} + M_{i,\sigma_i(t)}D_{i,\sigma_i(t)})\varGamma_{i,\sigma_i(t)}\xi_i(t), \\ \dot{\theta}_i(t) =& F_{i,\sigma_i(t)}e_i(t) + H_{i,\sigma_i(t)}\theta_i(t) + (F_{i,\sigma_i(t)} + T_{i,\sigma_i(t)}C_{i,\sigma_i(t)})\varepsilon_i(t) + (F_{i,\sigma_i(t)} + T_{i,\sigma_i(t)}C_{i,\sigma_i(t)})\eta(t) \\ &+ T_{i,\sigma_i(t)}D_{i,\sigma_i(t)}w_i(t), \\ \dot{\varepsilon}_i(t) =& B_{i,\sigma_i(t)}K_{1,i,\sigma_i(t)}e_i(t) + B_{i,\sigma_i(t)}K_{2,i,\sigma_i(t)}\varGamma_{i,\sigma_i(t)}\theta_i(t) + (A_{i,\sigma_i(t)} + B_{i,\sigma_i(t)}K_{1i,\sigma_i(t)} + B_{i,\sigma_i(t)} \\ &\times K_{3,i,\sigma_i(t)}C_{i,\sigma_i(t)})\varepsilon_i(t) + (A_{i,\sigma_i(t)} + B_{i,\sigma_i(t)}K_{1i,\sigma_i(t)} + B_{i,\sigma_i(t)}K_{3,i,\sigma_i(t)}C_{i,\sigma_i(t)} + B_{i,\sigma_i(t)} \\ &\times P_{i,\sigma_i(t)} - S_{i,\sigma_i(t)})\eta(t) + (B_{i,\sigma_i(t)}K_{2,i,\sigma_i(t)} + B_{i,\sigma_i(t)}K_{3,i,\sigma_i(t)}D_{i,\sigma_i(t)} + E_{i,\sigma_i(t)})w_i(t) \\ &+ B_{i,\sigma_i(t)}K_{4,i,\sigma_i(t)}\textstyle\sum_{j=1}^{N} a_{ij}^{(\zeta(t))}(\widehat{x}_j(t) - \widehat{x}_i(t)), \\ \dot{\eta}(t) =& S\eta(t). \end{aligned} \tag{7}$$

Let $X_i(t)=(e_i^\top(t),\ \theta_i^\top(t),\ \varepsilon_i^\top(t),\ \eta^\top(t))^\top$. Then,

$$\dot X_i(t)=\begin{pmatrix} G_{i,\sigma_i(t)} & J_{i,\sigma_i(t)}\Gamma_{i,\sigma_i(t)} & \begin{pmatrix}G_{i,\sigma_i(t)}-A_{i,\sigma_i(t)}\\ +M_{i,\sigma_i(t)}C_{i,\sigma_i(t)}\end{pmatrix} & \begin{pmatrix}G_{i,\sigma_i(t)}-A_{i,\sigma_i(t)}\\ +M_{i,\sigma_i(t)}C_{i,\sigma_i(t)}\end{pmatrix}\\ F_{i,\sigma_i(t)} & H_{i,\sigma_i(t)} & \begin{pmatrix}F_{i,\sigma_i(t)}\\ +T_{i,\sigma_i(t)}C_{i,\sigma_i(t)}\end{pmatrix} & \begin{pmatrix}F_{i,\sigma_i(t)}\\ +T_{i,\sigma_i(t)}C_{i,\sigma_i(t)}\end{pmatrix}\\ B_{i,\sigma_i(t)}K_{1,i,\sigma_i(t)} & B_{i,\sigma_i(t)}K_{2,i,\sigma_i(t)}\Gamma_{i,\sigma_i(t)} & \mathcal{A}_1 & \mathcal{B}_1\\ 0 & 0 & 0 & S\end{pmatrix}X_i(t)$$
$$+\begin{pmatrix}J_{i,\sigma_i(t)}-E_{i,\sigma_i(t)}+M_{i,\sigma_i(t)}D_{i,\sigma_i(t)}\\ T_{i,\sigma_i(t)}D_{i,\sigma_i(t)}\\ \begin{pmatrix}B_{i,\sigma_i(t)}K_{2,i,\sigma_i(t)}+E_{i,\sigma_i(t)}\\ +B_{i,\sigma_i(t)}K_{3,i,\sigma_i(t)}D_{i,\sigma_i(t)}\end{pmatrix}\\ 0\end{pmatrix}w_i(t)+\begin{pmatrix}0\\ 0\\ \begin{pmatrix}B_{i,\sigma_i(t)}K_{4,i,\sigma_i(t)}\\ \times\sum_{j=1}^N a_{ij}^{(\zeta(t))}(\widehat{x}_i(t)-\widehat{x}_j(t))\end{pmatrix}\\ 0\end{pmatrix}, \tag{8}$$

where $\mathcal{A}_1=A_{i,\sigma_i(t)}+B_{i,\sigma_i(t)}K_{1,i,\sigma_i(t)}+B_{i,\sigma_i(t)}K_{3,i,\sigma_i(t)}C_{i,\sigma_i(t)}$ and $\mathcal{B}_1=A_{i,\sigma_i(t)}-S_{i,\sigma_i(t)}+B_{i,\sigma_i(t)}K_{1,i,\sigma_i(t)}+B_{i,\sigma_i(t)}K_{3,i,\sigma_i(t)}C_{i,\sigma_i(t)}+B_{i,\sigma_i(t)}P_{i,\sigma_i(t)}$.

Let $e(t)=(e_1^\top(t),\ e_2^\top(t),\dots,\ e_m^\top(t))^\top$, $\theta(t)=(\theta_1^\top(t),\ \theta_2^\top(t),\dots,\ \theta_m^\top(t))^\top$, $\varepsilon(t)=(\varepsilon_1^\top(t),\ \varepsilon_2^\top(t),\dots,\varepsilon_m^\top(t))^\top$, and $\overline{\eta}(t)=\mathbf{1}_m\otimes\eta(t)$. Then,

$$\begin{aligned}\dot e(t)&=\widetilde G_pe(t)+\widetilde J_p\widetilde\Gamma_p\theta(t)+(\widetilde G_p+\widetilde M_p\widetilde C_p-\widetilde A_p)\varepsilon(t)+(\widetilde G_p+\widetilde M_p\widetilde C_p-\widetilde A_p)\eta(t)\\&\quad+(\widetilde J_p-\widetilde E_p+\widetilde M_p\widetilde D_p)w(t),\\ \dot\theta(t)&=\widetilde F_pe(t)+\widetilde H_p\theta(t)+(\widetilde F_p+\widetilde T_p\widetilde C_p)\varepsilon(t)+(\widetilde F+\widetilde T\widetilde C_p)\eta(t)+\widetilde T_p\widetilde D_pw(t),\\ \dot\varepsilon(t)&=(\widetilde B_p\widetilde K_{1,p}+\mathcal{L}_p\widetilde B_p\widetilde K_{4,p})e(t)+(\widetilde B_p\widetilde K_2\widetilde\Gamma_p)\theta(t)+(\widetilde A_p+\widetilde B_p\widetilde K_{1,p}+\widetilde B_p\widetilde K_{3,p}\widetilde C_p\\&\quad+\mathcal{L}_p\widetilde B_p\widetilde K_{4,p})\varepsilon(t)+(\widetilde A_p+\widetilde B_p\widetilde K_{1,p}+\widetilde B_p\widetilde K_{3,p}\widetilde C_p+\widetilde B_p\widetilde P-I_m\otimes S+\mathcal{L}_p\widetilde B_p\widetilde K_{4,p})\eta(t)\\&\quad+(\widetilde B_p\widetilde K_2+\widetilde B_p\widetilde K_{3,p}\widetilde D_p+\widetilde E_p)w(t),\\ \dot{\overline\eta}(t)&=(I_m\otimes S)\overline\eta(t).\end{aligned}\tag{9}$$

Let $X(t)=(X_1^\top(t),\ X_2^\top(t),\dots,\ X_m^\top(t))^\top$. Then,

$$\dot X(t)=\begin{pmatrix}\widetilde G_p & \widetilde J_p\widetilde\Gamma_p & \widetilde G_p-\widetilde A_p+\widetilde M_p\widetilde C_p & \widetilde G_p-\widetilde A_p+\widetilde M_p\widetilde C_p\\ \widetilde F_p & \widetilde H_p & \widetilde F_p+\widetilde T_p\widetilde C_p & \widetilde F_p+\widetilde T_p\widetilde C_p\\ \mathcal{C}_1 & \widetilde B_p\widetilde K_{2,p}\widetilde\Gamma_p & \mathcal{A}_2 & \mathcal{B}_2\\ 0 & 0 & 0 & I_n\otimes S\end{pmatrix}X(t)+\begin{pmatrix}\widetilde J_p-\widetilde E_p+\widetilde M_p\widetilde D_p\\ \widetilde T_p\widetilde D_p\\ \begin{pmatrix}\widetilde B_p\widetilde K_{2,p}+\widetilde E_p\\ +\widetilde B_p\widetilde K_{3,p}\widetilde D_p\end{pmatrix}\\ 0\end{pmatrix}w(t),\tag{10}$$

where

$$\begin{aligned}&\widetilde A_p+\widetilde B_p\widetilde K_{1,p}+\widetilde B_p\widetilde K_{3,p}\widetilde C_p+\mathcal{L}_p\widetilde B_p\widetilde K_{4,p}\\&=\begin{pmatrix}\begin{pmatrix}A_{1,p_1}+B_{1,p_1}K_{1,1,p_1}\\ +B_{1,p_1}K_{3,1,p_1}C_{1,p_1}\\ +\sum_{j\in N_1}a_{1j}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1}\end{pmatrix} & -a_{12}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1} & \cdots & -a_{1m}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{m1}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} & -a_{m2}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} & \cdots & \begin{pmatrix}A_{m,p_m}+B_{m,p_m}K_{1,m,p_m}\\ +B_{m,p_m}K_{3,m,p_m}C_{m,p_m}\\ +\sum_{j\in N_m}a_{mj}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m}\end{pmatrix}\end{pmatrix},\end{aligned}$$

$$\widetilde{A}_{i,p_i} - I_m \otimes S + \widetilde{B}_{i,p_i}\widetilde{K}_{1,i,p_i} + \widetilde{B}_{i,p_i}\widetilde{K}_{3,i,p_i}\widetilde{C}_{i,p_i} + \widetilde{B}_{i,p_i}\widetilde{P} + \mathcal{L}_p\widetilde{B}_{i,p_i}\widetilde{K}_{4,i,p_i}$$
$$= \begin{pmatrix} \begin{pmatrix} A_{1,p_1} - S + B_{1,p_1}K_{1,1,p_1} \\ +B_{1,p_1}K_{3,1,p_1}C_{1,p_1} + B_{1,p_1}P_{1,p_1} \\ +\sum_{j\in N_1} a_{1j}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1} \end{pmatrix} & \cdots & -a_{1j}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1} \\ \vdots & \ddots & \vdots \\ -a_{m1}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} & \cdots & \begin{pmatrix} A_{m,p_m} - S + B_{m,p_m}K_{1,m,p_m} \\ +B_{m,p_m}K_{3,m,p_m}C_{m,p_m} + B_{m,p_m}P_{m,p_m} \\ +\sum_{j\in N_m} a_{mj}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} \end{pmatrix} \end{pmatrix},$$

$$\widetilde{B}_p\widetilde{K}_{1,p} + \mathcal{L}_p\widetilde{B}_p\widetilde{K}_{4,p}$$
$$= \begin{pmatrix} \begin{pmatrix} B_{1,p_1}K_{1,1,p_1} \\ +\sum_{j\in N_1} a_{1j}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1} \end{pmatrix} & -a_{12}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1} & \cdots & -a_{1m}^{(\zeta(t))}B_{1,p_1}K_{4,1,p_1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} & -a_{m2}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} & \cdots & \begin{pmatrix} B_{m,p_m}K_{1,m,p_m} \\ +\sum_{j\in N_m} a_{mj}^{(\zeta(t))}B_{m,p_m}K_{4,m,p_m} \end{pmatrix} \end{pmatrix},$$

and $\widetilde{E}_p = \text{blockdiag}(E_{1,p_1},\ldots,E_{m,p_m})$, $\widetilde{G}_p = \text{blockdiag}(G_{1,p_1},\ldots,G_{m,p_m})$, $\widetilde{J}_p\widetilde{\Gamma}_p = \text{blockdiag}(J_{1,p_1}\Gamma_{1,p_1}, J_{2p}\Gamma_{2p},\ldots, J_{m,p_m}\Gamma_{m,p_m})$, $\widetilde{G}_p - \widetilde{A}_p + \widetilde{M}_p\widetilde{C}_p = \text{blockdiag}(G_{1,p_1} - A_{1,p_1} + M_{1,p_1}C_{1,p_1},\ldots,G_{m,p_m} - A_{m,p_m} + M_{m,p_m}C_{m,p_m})$, $\widetilde{F}_p + \widetilde{T}_p\widetilde{C}_p = \text{blockdiag}(F_{1,p_1} + T_{1,p_1}C_{1,p_1},\ldots,F_{m,p_m} + T_{m,p_m}C_{m,p_m})$, $\widetilde{H}_p = \text{blockdiag}(H_{1,p_1},\ldots, H_{m,p_m})$, $\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\Gamma}_p = \text{blockdiag}(B_{1,p_1}K_{2,1,p_1}\Gamma_{1,p_1},\ldots,B_{m,p_m}K_{2,m,p_m}\Gamma_{m,p_m})$.

Remark 2. To mitigate the effects of disturbances on system stability, gain-method-based controllers were designed in previous studies [52–54]. However, compared to gain-method-based control, disturbance-observer-based control has three significant advantages. (i) Gain-method-based control can be regarded as passive antidisturbance control. In contrast, disturbance observer-based control is an active and efficient approach to tackle disturbances and enhance the robust stability of the closed-loop system. Consequently, disturbance-observer-based control is considered an active antidisturbance control method. (ii) Although the gain-method-based control can suppress disturbances, it is not as effective as the disturbance-observer-based control in terms of accuracy and performance. Owing to its design features, the disturbance observer can estimate disturbances within the system more accurately and can compensate for them effectively. Thus, high control performance is achieved. (iii) The disturbance observer is designed to provide robust stability for the system. Conversely, gain methods can have an adverse effect on system stability when inappropriate gain parameters are applied. Therefore, in this study, a disturbance observer was designed to address the disturbances.

Theorem 1. If there exist constants $m>0$, $\alpha_1<0$, $\alpha_2<0$, $\lambda_i>1$, $\mu_1>0$, $\delta_1>0$, $\delta_2>0$, $\delta_3>0$, $\delta_4>0$, $\delta_5>0$, $\underline{\gamma}_1>0$, $\overline{\gamma}_1>0$, $\underline{\gamma}_2>0$, $\overline{\gamma}_2>0$, $\underline{\gamma}_3>0$, $\overline{\gamma}_3>0$, $\mathcal{R}^n$ vectors $\mathfrak{P}_{i,p_i}^{-(\iota)}\preceq 0$, $\mathfrak{P}_{i,p_i}^{+(\iota)}\succeq 0$, $\mathfrak{K}_{i,p_i}^{-(\iota)}\preceq 0$, $\mathfrak{K}_{i,p_i}^{+(\iota)}\succeq 0$, $\mathfrak{N}_{i,p_i}^{(\iota)}\preceq 0$, $\mathfrak{F}_{i,p_i}^{(\iota)}\succeq 0$, $\mathfrak{G}_{i,p_i}^{(\iota)}\succeq 0$, $v_1^{(ip_i)}\succeq 0$, $v_3^{(ip_i)}\succeq 0$, $v_4^{(p)}\succeq 0$, $\mathcal{R}^o$ vectors $\mathfrak{Y}_{i,p_i}^{-(\iota)}\preceq 0$, $\mathfrak{Y}_{i,p_i}^{+(\iota)}\succeq 0$, $\mathfrak{M}_{i,p_i}^{+(\iota)}\succeq 0$, $\mathfrak{M}_{i,p_i}^{-(\iota)}\preceq 0$, $\mathfrak{T}_{i,p_i}^{+(\iota)}\succeq 0$, $\mathfrak{T}_{i,p_i}^{-(\iota)}\preceq 0$, and $\mathcal{R}^s$ vectors $\mathfrak{Z}_{i,p_i}^{(\iota)}\succeq 0$, $\mathfrak{I}_{i,p_i}^{(\iota)}\succeq 0$, $\mathfrak{H}_{i,p_i}^{(\iota)}\succeq 0$, $\Gamma_{i,p_i}^{(\iota)}\succeq 0$, $v_2^{(ip_i)}\succeq 0$ such that

$$\begin{aligned}&\tfrac{1}{\overline{\gamma}_3}B_{i,p_i}\textstyle\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Z}_{i,p_i}^{(\iota)\top} + \tfrac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{+(\iota)\top}D_{i,p_i}\\ &\quad+\tfrac{1}{\underline{\gamma}_3}B_{i,p_i}\textstyle\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{-(\iota)\top}D_{i,p_i} + E_{i,p_i} = 0,\end{aligned} \tag{11a}$$

$$\begin{aligned}&\tfrac{1}{\underline{\gamma}_3}B_{i,p_i}\textstyle\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Z}_{i,p_i}^{(\iota)\top} + \tfrac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{+(\iota)\top}D_{i,p_i}\\ &\quad+\tfrac{1}{\overline{\gamma}_3}B_{i,p_i}\textstyle\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{-(\iota)\top}D_{i,p_i} + E_{i,p_i} = 0,\end{aligned} \tag{11b}$$

$$\tfrac{1}{\overline{\gamma}_1}\textstyle\sum_{\iota=1}^n \mathbf{1}_n^{(\iota)}\mathfrak{I}_{i,p_i}^{(\iota)\top} + \tfrac{1}{\overline{\gamma}_1}\sum_{\iota=1}^n \mathbf{1}_n^{(\iota)}\mathfrak{M}_{i,p_i}^{+(\iota)\top}D_{i,p_i} + \tfrac{1}{\underline{\gamma}_1}\sum_{\iota=1}^n \mathbf{1}_n^{(\iota)}\mathfrak{M}_{i,p_i}^{-(\iota)\top}D_{i,p_i} - E_{i,p_i} = 0, \tag{11c}$$

$$\tfrac{1}{\underline{\gamma}_1}\textstyle\sum_{\iota=1}^n \mathbf{1}_n^{(\iota)}\mathfrak{I}_{i,p_i}^{(\iota)\top} + \tfrac{1}{\underline{\gamma}_1}\sum_{\iota=1}^n \mathbf{1}_n^{(\iota)}\mathfrak{M}_{i,p_i}^{+(\iota)\top}D_{i,p_i} + \tfrac{1}{\overline{\gamma}_1}\sum_{\iota=1}^n \mathbf{1}_n^{(\iota)}\mathfrak{M}_{i,p_i}^{-(\iota)\top}D_{i,p_i} - E_{i,p_i} = 0, \tag{11d}$$

$$\tfrac{1}{\overline{\gamma}_2}\textstyle\sum_{\iota=1}^s \mathbf{1}_s^{(\iota)}\mathfrak{T}_{i,p_i}^{+(\iota)\top}D_{i,p_i} + \tfrac{1}{\underline{\gamma}_2}\sum_{\iota=1}^s \mathbf{1}_s^{(\iota)}\mathfrak{T}_{i,p_i}^{-(\iota)\top}D_{i,p_i} = 0, \tag{11e}$$

$$\tfrac{1}{\underline{\gamma}_2}\textstyle\sum_{\iota=1}^s \mathbf{1}_s^{(\iota)}\mathfrak{T}_{i,p_i}^{+(\iota)\top}D_{i,p_i} + \tfrac{1}{\overline{\gamma}_2}\sum_{\iota=1}^s \mathbf{1}_s^{(\iota)}\mathfrak{T}_{i,p_i}^{-(\iota)\top}D_{i,p_i} = 0, \tag{11f}$$

and

$$\frac{1}{\overline{\gamma}_1}\sum_{\iota=1}^{n}\mathbf{1}_n^{(\iota)}\mathfrak{G}_{i,p_i}^{(\iota)\top}+(\delta_2-\frac{m\delta_1}{\underline{\gamma}_1})I_n\succeq 0, \tag{12a}$$

$$\begin{aligned}&\frac{1}{\overline{\gamma}_1}\sum_{\iota=1}^{n}\mathbf{1}_n^{(\iota)}\mathfrak{G}_{i,p_i}^{(\iota)\top}-\frac{m\delta_1}{\underline{\gamma}_1}I_n-A_{i,p_i}+\frac{1}{\overline{\gamma}_1}\sum_{\iota=1}^{n}\mathbf{1}_n^{(\iota)}\mathfrak{M}_{i,p_i}^{+(\iota)\top}C_{i,p_i}\\&+\frac{1}{\underline{\gamma}_1}\sum_{\iota=1}^{n}\mathbf{1}_n^{(\iota)}\mathfrak{M}_{i,p_i}^{-(\iota)\top}C_{i,p_i}\succeq 0,\end{aligned} \tag{12b}$$

$$\frac{1}{\overline{\gamma}_2}\sum_{\iota=1}^{s}\mathbf{1}_s^{(\iota)}\mathfrak{H}_{i,p_i}^{(\iota)\top}+(\delta_4-\frac{m\delta_3}{\underline{\gamma}_2})I_s\succeq 0, \tag{12c}$$

$$\frac{1}{\overline{\gamma}_2}\sum_{\iota=1}^{s}\mathbf{1}_s^{(\iota)}\mathfrak{F}_{i,p_i}^{(\iota)\top}+\frac{1}{\overline{\gamma}_2}\sum_{\iota=1}^{s}\mathbf{1}_s^{(\iota)}\mathfrak{T}_{i,p_i}^{+(\iota)\top}C_{i,p_i}+\frac{1}{\underline{\gamma}_2}\sum_{\iota=1}^{s}\mathbf{1}_s^{(\iota)}\mathfrak{T}_{i,p_i}^{-(\iota)\top}C_{i,p_i}\succeq 0, \tag{12d}$$

$$\begin{aligned}&\frac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}^{(\iota)}\mathfrak{K}_{i,p_i}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}^{(\iota)}\mathfrak{K}_{i,p_i}^{-(\iota)\top}\\&+\frac{1}{\underline{\gamma}_3}\sum_{j\in N_i}a_{ij}^{(\zeta(t))}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{N}_{i,p_i}^{(\iota)\top}\succeq 0,\end{aligned} \tag{12e}$$

$$\begin{aligned}&A_{i,p_i}+\frac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{i,p_i}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{i,p_i}^{-(\iota)\top}+\frac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{+(\iota)\top}C_{i,p_i}\\&+\frac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{-(\iota)\top}C_{i,p_i}+\frac{1}{\underline{\gamma}_3}\sum_{j\in N_i}a_{ij}^{(\zeta(t))}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{N}_{i,p_i}^{(\iota)\top}+\delta_5 I_n\succeq 0,\end{aligned} \tag{12f}$$

$$\begin{aligned}&A_{i,p_i}-S+\frac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{P}_{i,p_i}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{P}_{i,p_i}^{-(\iota)\top}+\frac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{i,p_i}^{+(\iota)\top}\\&+\frac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{i,p_i}^{-(\iota)\top}+\frac{1}{\overline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{+(\iota)\top}C_{i,p_i}+\frac{1}{\underline{\gamma}_3}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{-(\iota)\top}C_{i,p_i}\\&+\frac{1}{\underline{\gamma}_3}\sum_{j\in N_i}a_{ij}^{(\zeta(t))}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{N}_{i,p_i}^{(\iota)\top}\succeq 0,\end{aligned} \tag{12g}$$

$$\begin{aligned}&\sum_{\iota=1}^{n}\mathfrak{G}_{i,p_i}^{(\iota)}+\alpha_1 v_1^{(ip_i)}+\sum_{\iota=1}^{r}\mathfrak{F}_{i,p_i}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{K}_{i,p_i}^{(\iota)}\\&+\sum_{j\in N_i}a_{ij}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{i,p_i}^{(\iota)}-\sum_{j\in N_i}a_{ji}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\prec 0,\end{aligned} \tag{12h}$$

$$(\mu_1-\alpha_1)\underline{\gamma}_1-m\delta_1\leqslant 0, \tag{12i}$$

$$\varGamma_{i,p_i}^{\top}\sum_{\iota=1}^{n}\mathfrak{I}_{i,p_i}^{(\iota)}+\sum_{\iota=1}^{s}\mathfrak{H}_{i,p_i}^{(\iota)}+\varGamma_{i,p_i}^{\top}\sum_{\iota=1}^{r}\mathfrak{Z}_{i,p_i}^{(\iota)}+\alpha_2 v_2^{(ip_i)}\prec 0, \tag{12j}$$

$$(\mu_1-\alpha_2)\underline{\gamma}_2-m\delta_3\leqslant 0, \tag{12k}$$

$$\begin{aligned}&\sum_{\iota=1}^{n}\mathfrak{G}_{i,p_i}^{(\iota)}+(\alpha_1-\mu_1)v_1^{(ip_i)}+\mu v_3^{(ip_i)}+A_{i,p_i}^{\top}v_3^{(ip_i)}+\sum_{\iota=1}^{s}\mathfrak{F}_{i,p_i}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{K}_{i,p_i}^{(\iota)}\\&-A_{i,p_i}^{\top}v_1^{(ip_i)}+C_{i,p_i}^{\top}\sum_{\iota=1}^{n}\mathfrak{M}_{i,p_i}^{(\iota)}+C_{i,p_i}^{\top}\sum_{\iota=1}^{s}\mathfrak{T}_{i,p_i}^{(\iota)}+C_{i,p_i}^{\top}\sum_{\iota=1}^{r}\mathfrak{Y}_{i,p_i}^{(\iota)}\\&+\sum_{j\in N_i}a_{ij}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{i,p_i}^{(\iota)}-\sum_{j\in N_i}a_{ji}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\prec 0,\end{aligned} \tag{12l}$$

$$\begin{aligned}&\sum_{\iota=1}^{r}\mathfrak{G}_{i,p_i}^{(\iota)}+\alpha_1 v_1^{(ip_i)}-\mu_1 v_1^{(ip_i)}+\mu_1 v_4^{(p)}-A_{i,p_i}^{\top}v_1^{(ip_i)}+C_{i,p_i}^{\top}\sum_{\iota=1}^{m}\mathfrak{M}_{i,p_i}^{(\iota)}+\sum_{\iota=1}^{s}\mathfrak{F}_{i,p_i}^{(\iota)}\\&+C_{i,p_i}^{\top}\sum_{\iota=1}^{s}\mathfrak{T}_{i,p_i}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{K}_{i,p_i}^{+(\iota)}+\sum_{\iota=1}^{r}\mathfrak{K}_{i,p_i}^{-(\iota)}+C_{i,p_i}^{\top}\sum_{\iota=1}^{r}\mathfrak{Y}_{i,p_i}^{+(\iota)}\\&+C_{i,p_i}^{\top}\sum_{\iota=1}^{r}\mathfrak{Y}_{i,p_i}^{-(\iota)}+S^{\top}v_4^{(p)}+\sum_{\iota=1}^{r}\mathfrak{P}_{i,p_i}^{(\iota)}+A_{i,p_i}^{\top}v_3^{(ip_i)}-S^{\top}v_3^{(ip_i)}\\&+\sum_{j\in N_i}a_{ij}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{i,p_i}^{(\iota)}-\sum_{j\in N_i}a_{ji}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\prec 0,\end{aligned} \tag{12m}$$

$$v_3^{(ip_i)}\preceq v_4^{(p)},v_4^{(p)}\preceq v_1^{(ip_i)},v_3^{(ip_i)}\preceq v_1^{(ip_i)}, \tag{12n}$$

$$\begin{aligned}&\underline{\gamma}_1\leqslant\mathbf{1}_n^{(\iota)\top}v_1^{(ip_i)}\leqslant\overline{\gamma}_1,\iota=1,2,\cdots,r,\underline{\gamma}_2\leqslant\mathbf{1}_s^{(\iota)\top}v_2^{(ip_i)}\leqslant\overline{\gamma}_2,\iota=1,2,\cdots,s,\\&\underline{\gamma}_3\leqslant\mathbf{1}_r^{(\iota)\top}B_{i,p_i}^{\top}v_3^{(ip_i)}\leqslant\overline{\gamma}_3,\iota=1,2,\cdots,m,\end{aligned} \tag{12o}$$

$$v^{(ip_i)}\preceq\lambda_i v^{(iq_i)}, \tag{12p}$$

hold $\forall(p,q)\in\mathcal{S},\ p\neq q$ and $i=1,2,\cdots,n$, then system (1) with the ADT that satisfies

$$\tau_i\geqslant\frac{\ln\lambda_i}{\mu_1} \tag{13}$$

is positive and reaches consensus under control protocol (4) and double observers (5) and (6). Further, $K_{1,i,p_i} = K^+_{1,i,p_i} + K^-_{1,i,p_i}$, $K_{3,i,p_i} = K^+_{3,i,p_i} + K^-_{3,i,p_i}$, $P_{i,p_i} = P^+_{i,p_i} + P^-_{i,p_i}$, $T_{i,p_i} = T^+_{i,p_i} + T^-_{i,p_i}$, and $M_{i,p_i} = M^+_{i,p_i} + M^-_{i,p_i}$ satisfy

$$\begin{aligned}
K^+_{1,i,p_i} &= \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{K}_{i,p_i}^{+(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}}, K^-_{1,i,p_i} = \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{K}_{i,p_i}^{-(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}}, K_{2,i,p_i} = \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Z}_{i,p_i}^{(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}},\\
K^+_{3,i,p_i} &= \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Y}_{i,p_i}^{+(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}}, K^-_{3,i,p_i} = \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Y}_{i,p_i}^{-(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}}, K_{4,i,p_i} = \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{N}_{i,p_i}^{(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}},\\
P^+_{i,p_i} &= \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{P}_{i,p_i}^{+(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}}, P^-_{i,p_i} = \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{P}_{i,p_i}^{-(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} B_{i,p_i}^\top v_3^{(ip_i)}}, G_{i,p_i} = \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{G}_{i,p_i}^{(\iota)\top} - \delta_1 I_n}{\mathbf{1}_n^{(\iota)\top} v_1^{(ip_i)}},\\
H_{i,p_i} &= \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{H}_{i,p_i}^{(\iota)\top} - \delta_3 I_s}{\mathbf{1}_s^{(\iota)\top} v_2^{(ip_i)}}, M^+_{i,p_i} = \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{+(\iota)\top}}{\mathbf{1}_n^{(\iota)\top} v_1^{(ip_i)}}, M^-_{i,p_i} = \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{-(\iota)\top}}{\mathbf{1}_n^{(\iota)\top} v_1^{(ip_i)}},\\
F_{i,p_i} &= \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{F}_{i,p_i}^{(\iota)\top}}{\mathbf{1}_s^{(\iota)\top} v_2^{(ip_i)}}, T^+_{i,p_i} = \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{+(\iota)\top}}{\mathbf{1}_s^{(\iota)\top} v_2^{(ip_i)}}, T^-_{i,p_i} = \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{-(\iota)\top}}{\mathbf{1}_s^{(\iota)\top} v_2^{(ip_i)}}, J_{i,p_i} = \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{J}_{i,p_i}^{(\iota)\top}}{\mathbf{1}_n^{(\iota)\top} v_1^{(ip_i)}}.
\end{aligned} \tag{14}$$

Proof. First, the positivity of the systems is addressed. Using (12n) and (12o), Eq. (14) can be rewritten as follows:

$$\begin{aligned}
K^+_{1,i,p_i} &\succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{K}_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_3}, K^-_{1,i,p_i} \succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{K}_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_3}, K_{2,i,p_i} \succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Z}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_3}, K^+_{3,i,p_i} \succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Y}_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_3},\\
K^-_{3,i,p_i} &\succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Y}_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_3}, K_{4,i,p_i} \succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{N}_{i,p_i}^{(\iota)\top}}{\underline{\gamma}_3}, P^+_{i,p_i} \succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{P}_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_3}, P^-_{i,p_i} \succeq \sum_{\iota=1}^r \frac{\mathbf{1}_r^{(\iota)} \mathfrak{P}_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_3},\\
H_{i,p_i} &\succeq \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{H}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_2} - \frac{\delta_3}{\underline{\gamma}_2} I_s, F_{i,p_i} \succeq \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{F}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_2}, M^+_{i,p_i} \succeq \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_1}, M^-_{i,p_i} \succeq \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_1},\\
G_{i,p_i} &\succeq \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{G}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_1} - \frac{\delta_1}{\underline{\gamma}_1} I_n, T^+_{i,p_i} \succeq \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_2}, T^-_{i,p_i} \succeq \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_2}, J_{i,p_i} \succeq \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{J}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_1}.
\end{aligned} \tag{15}$$

From (11) and (15), we have $0 = \frac{1}{\overline{\gamma}_3} B_{i,p_i} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \mathfrak{Z}_{i,p_i}^{(\iota)\top} + \frac{1}{\overline{\gamma}_3} B_{i,p_i} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \mathfrak{Y}_{i,p_i}^{+(\iota)\top} D_{i,p_i} + \frac{1}{\underline{\gamma}_3} B_{i,p_i} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \times \mathfrak{Y}_{i,p_i}^{-(\iota)\top} D_{i,p_i} + E_{i,p_i} \preceq B_{i,p_i} K_{2,i,p_i} + B_{i,p_i} K_{3,i,p_i} D_{i,p_i} + E_{i,p_i} \preceq \frac{1}{\underline{\gamma}_3} B_{i,p_i} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \mathfrak{Z}_{i,p_i}^{(\iota)\top} + \frac{1}{\underline{\gamma}_3} B_{i,p_i} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \times \mathfrak{Y}_{i,p_i}^{+(\iota)\top} D_{i,p_i} + \frac{1}{\overline{\gamma}_3} B_{i,p_i} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \mathfrak{Y}_{i,p_i}^{-(\iota)\top} D_{i,p_i} + E_{i,p_i} = 0$, $0 = \frac{1}{\overline{\gamma}_1} \sum_{\iota=1}^m \mathbf{1}_m^{(\iota)} \mathfrak{J}_{i,p_i}^{(\iota)\top} + \frac{1}{\overline{\gamma}_1} \sum_{\iota=1}^n \mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{+(\iota)\top} D_{i,p_i} + \frac{1}{\underline{\gamma}_1} \sum_{\iota=1}^n \mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{-(\iota)\top} D_{i,p_i} - E_{i,p_i} \preceq J_{i,p_i} - E_{i,p_i} + M_{i,p_i} D_{i,p_i} \preceq \frac{1}{\underline{\gamma}_1} \sum_{\iota=1}^n \mathbf{1}_n^{(\iota)} \mathfrak{J}_{i,p_i}^{(\iota)\top} + \frac{1}{\underline{\gamma}_1} \sum_{\iota=1}^n \mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{+(\iota)\top} \times D_{i,p_i} + \frac{1}{\overline{\gamma}_1} \sum_{\iota=1}^n \mathbf{1}_n^{(\iota)} \mathfrak{M}_{i,p_i}^{-(\iota)\top} D_{i,p_i} - E_{i,p_i} = 0$, and $0 = \frac{1}{\overline{\gamma}_2} \sum_{\iota=1}^s \mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{+(\iota)\top} D_{i,p_i} + \frac{1}{\underline{\gamma}_2} \sum_{\iota=1}^s \mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{-(\iota)\top} D_{i,p_i} \preceq T_{i,p_i} D_{i,p_i} \preceq \frac{1}{\underline{\gamma}_2} \sum_{\iota=1}^s \mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{+(\iota)\top} D_{i,p_i} + \frac{1}{\overline{\gamma}_2} \sum_{\iota=1}^s \mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{-(\iota)\top} D_{i,p_i} = 0$. Therefore, $B_{i,p_i} K_{2,i,p_i} + B_{i,p_i} K_{3,i,p_i} D_{i,p_i} + E_{i,p_i} = 0$, $J_{i,p_i} - E_{i,p_i} + M_{i,p_i} D_{i,p_i} = 0$, $T_{i,p_i} D_{i,p_i} = 0$ is easily obtained. System (9) can be transformed as follows:

$$\begin{aligned}
\dot{e}(t) &= \widetilde{G}_p e(t) + \widetilde{J}_p \widetilde{\varGamma}_p \theta(t) + (\widetilde{G}_p + \widetilde{M}_p \widetilde{C}_p - \widetilde{A}_p)\varepsilon(t) + (\widetilde{G}_p + \widetilde{M}_p \widetilde{C}_p - \widetilde{A}_p)\eta(t),\\
\dot{\theta}(t) &= \widetilde{F}_p e(t) + \widetilde{H}_p \theta(t) + (\widetilde{F}_p + \widetilde{T}_p \widetilde{C}_p)\varepsilon(t) + (\widetilde{F}_p + \widetilde{T}_p \widetilde{C}_p)\eta(t),\\
\dot{\varepsilon}(t) &= (\widetilde{B}_p \widetilde{K}_{1,p} + \mathcal{L}_p \widetilde{B}_p \widetilde{K}_{4,p}) e(t) + (\widetilde{B}_p \widetilde{K}_{2,p} \widetilde{\varGamma}_p)\theta(t) + (\widetilde{A}_p + \widetilde{B}_p \widetilde{K}_{1,p} + \widetilde{B}_p \widetilde{K}_{3,p} \widetilde{C}_p\\
&\quad + \mathcal{L}_p \widetilde{B}_p \widetilde{K}_{4,p})\varepsilon(t) + (\widetilde{A}_p + \widetilde{B}_p \widetilde{K}_{1,p} + \widetilde{B}_p \widetilde{K}_{3,p} \widetilde{C}_p + \widetilde{B}_p \widetilde{P}_p - (I_m \otimes S) + \mathcal{L}_p \widetilde{B}_p \widetilde{K}_{4,p})\eta(t),\\
\dot{\overline{\eta}}(t) &= (I_m \otimes S)\overline{\eta}(t).
\end{aligned} \tag{16}$$

Then,

$$\dot{X}(t) = \begin{pmatrix} \widetilde{G}_p & \widetilde{J}_p \widetilde{\varGamma}_p & \widetilde{G}_p - \widetilde{A}_p + \widetilde{M}_p \widetilde{C}_p & \widetilde{G}_p - \widetilde{A}_p + \widetilde{M}_p \widetilde{C}_p \\ \widetilde{F}_p & \widetilde{H}_p & \widetilde{F}_p + \widetilde{T}_p \widetilde{C}_p & \widetilde{F}_p + \widetilde{T}_p \widetilde{C}_p \\ \mathcal{C}_1 & \widetilde{B}_p \widetilde{K}_2 \widetilde{\varGamma}_p & \mathcal{A}_1 & \mathcal{B}_1 \\ 0 & 0 & 0 & I_m \otimes S \end{pmatrix} X(t). \tag{17}$$

By (12a), we have $G_{i,p_i} + \delta_2 I_n = \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{G}_{i,p_i}^{(\iota)\top}}{\mathbf{1}_n^{(\iota)\top} v_1^{(i,p_i)}} - \sum_{\iota=1}^n \frac{\delta_1 I_n}{\mathbf{1}_n^{(\iota)\top} v_1^{(i,p_i)}} + \delta_2 I_n \succeq \frac{1}{\overline{\gamma}_1} \sum_{\iota=1}^n \mathbf{1}_n^{(\iota)} \mathfrak{G}_{i,p_i}^{(\iota)\top} - (\frac{\delta_1}{\underline{\gamma}_1} - \delta_2) I_n \succeq 0$. Thus, $\widetilde{G}_p$ is Metzler by Lemma 1. From $J_{i,p_i} \succeq 0$ and $\varGamma_{i,p_i} \succeq 0$, $\widetilde{J}_p \widetilde{\varGamma}_p \succeq 0$ is easily obtained. From (12b), it follows that $G_{i,p_i} - A_{i,p_i} + M_{i,p_i} C_{i,p_i} \succeq \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} \mathfrak{G}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_1} - \sum_{\iota=1}^n \frac{\delta_1 I_n}{\underline{\gamma}_1} - A_{i,p_i} + \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} M_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_1} C_{i,p_i} + \sum_{\iota=1}^n \frac{\mathbf{1}_n^{(\iota)} M_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_1} C_{i,p_i} \succeq 0$. Further, $\widetilde{G}_p - \widetilde{A}_p + \widetilde{M}_p \widetilde{C}_p \succeq 0$. Similar to the derivation of $\widetilde{G}_p$, the matrix $\widetilde{H}_p$ is also Metzler by (12c). Using (12d), one can deduce that

$$F_{i,p_i} + T_{i,p_i} C_{i,p_i} \succeq \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{F}_{i,p_i}^{(\iota)\top}}{\overline{\gamma}_2} + \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{+(\iota)\top}}{\overline{\gamma}_2} C_{i,p_i} + \sum_{\iota=1}^s \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}_{i,p_i}^{-(\iota)\top}}{\underline{\gamma}_2} C_{i,p_i} \succeq 0.$$

Then, one can derive that $\widetilde{F}_p + \widetilde{T}_p\widetilde{C}_p \succeq 0$. From $\mathfrak{N}^{\iota}_{i,p_i} \preceq 0$ and the graph theory, we have $-a^{(\zeta(t))}_{ij} B_{i,p_i} K_{4,i,p_i} \succeq 0$. Using (12e), the following is obtained:

$$\begin{aligned} B_{i,p_i}K_{1,i,p_i} + \textstyle\sum_{j\in N_i} a^{(\zeta(t))}_{ij} B_{i,p_i}K_{4,i,p_i} \succeq B_{i,p_i}\sum_{\iota=1}^{r} \frac{\mathbf{1}^{(\iota)}\mathfrak{K}^{+(\iota)\top}_{i,p_i}}{\overline{\gamma}_3} + B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}\mathfrak{K}^{-(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} \\ + \textstyle\sum_{j\in N_i} a^{(\zeta(t))}_{ij} B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{N}^{(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} \succeq 0. \end{aligned}$$

Further, it can be obtained that $\widetilde{B}_p\widetilde{K}_{1,p} + \mathcal{L}_p\widetilde{B}_p\widetilde{K}_{4,p} \succeq 0$. From $B_{i,p_i} \succeq 0$, $\varGamma^{(\iota)}_{i,p_i} \succeq 0$, and $\mathfrak{Z}^{(\iota)}_{i,p_i} \succeq 0$, it is deduced that $\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p \succeq 0$. From (12f), we have

$$\begin{aligned} &A_{i,p_i} + B_{i,p_i}K_{1,i,p_i} + B_{i,p_i}K_{3,i,p_i}C_{i,p_i} + \textstyle\sum_{j\in N_i} a^{(\zeta(t))}_{ij} B_{i,p_i}K_{4,i,p_i} + \delta_5 I_n \\ &\succeq A_{i,p_i} + B_{i,p_i}\textstyle\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{K}^{+(\iota)\top}_{i,p_i}}{\overline{\gamma}_3} + B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{K}^{-(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} + B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{Y}^{+(\iota)\top}_{i,p_i}}{\overline{\gamma}_3}C_{i,p_i} \\ &\quad + B_{i,p_i}\textstyle\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{Y}^{-(\iota)\top}_{i,p_i}}{\underline{\gamma}_3}C_{i,p_i} + \sum_{j\in N_i} a^{(\zeta(t))}_{ij} B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{N}^{(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} + \delta_5 I_n \succeq 0. \end{aligned}$$

Therefore, it can be deduced that $\widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \mathcal{L}_p\widetilde{B}_p\widetilde{K}_{4,p}$ is Metzler. From (12g), we have

$$\begin{aligned} &A_{i,p_i} - S_{i,p_i} + B_{i,p_i}P_{i,p_i} + B_{i,p_i}K_{1,i,p_i} + B_{i,p_i}K_{3,i,p_i}C_{i,p_i} + \textstyle\sum_{j\in N_1} a^{(\zeta(t))}_{ij} B_{i,p_i}K_{4,i,p_i} \\ &\succeq A_{i,p_i} - S_{i,p_i} + B_{i,p_i}\textstyle\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{K}^{+(\iota)\top}_{i,p_i}}{\overline{\gamma}_3} + B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{K}^{-(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} + B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{Y}^{+(\iota)\top}_{i,p_i}}{\overline{\gamma}_3}C_{i,p_i} \\ &\quad + B_{i,p_i}\textstyle\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{Y}^{-(\iota)\top}_{i,p_i}}{\underline{\gamma}_3}C_{i,p_i} + \sum_{j\in N_i} a^{(\zeta(t))}_{ij} B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{N}^{(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} + B_{i,p_i}\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{P}^{+(\iota)\top}_{i,p_i}}{\overline{\gamma}_3} \\ &\quad + B_{i,p_i}\textstyle\sum_{\iota=1}^{r}\frac{\mathbf{1}^{(\iota)}_r\mathfrak{P}^{-(\iota)\top}_{i,p_i}}{\underline{\gamma}_3} \succeq 0. \end{aligned}$$

Then, we get that $\widetilde{A}_p - (I_m\otimes S) + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{B}_p\widetilde{P} + \mathcal{L}_p\widetilde{B}_p\widetilde{K}_{4,p} \succeq 0$. Using Lemma 1, it is not hard to obtain that $I_m \otimes S$ is Metzler. By Definition 1, system (7) achieves positivity.

Choose a multiple CLF $V_{\sigma(t)}(X(t)) = X^\top(t)v^{(\sigma(t))} = e^\top(t)v_1^{(\sigma(t))} + \theta^\top(t)v_2^{(\sigma(t))} + \varepsilon^\top(t)v_3^{(\sigma(t))} + \overline{\eta}^\top(t)\overline{v}_4^{(\sigma(t))}$, where $v^{(\sigma(t))} = (v_1^{(\sigma(t))\top}, v_2^{(\sigma(t))\top}, v_3^{(\sigma(t))\top}, \overline{v}_4^{(\sigma(t))})^\top$ with $v_\iota^{(\sigma(t))} = (v_\iota^{(1,\sigma_1(t))\top}, v_\iota^{(2,\sigma_2(t))\top}, \ldots, v_\iota^{(m,\sigma_m(t))\top})^\top$, $\iota = \{1,2,3\}$ and $\overline{v}_4^{(\sigma(t))} = \mathbf{1}_m \otimes v_4^{(\sigma(t))}$. Combined with (10), we get

$$\begin{aligned} \dot{V}_{i,p_i}(X(t)) = {}& e^\top(t)(\widetilde{G}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{K}_{1,p}^\top\widetilde{B}_p^\top v_3^{(p)} + \widetilde{K}_{4,p}^\top\widetilde{B}_p^\top\mathcal{L}_p^\top v_3^{(p)}) + \theta^\top(t)(\widetilde{\varGamma}_p^\top\widetilde{J}_p^\top v_1^{(p)} \\ &+ \widetilde{H}_p^\top v_2^{(p)} + \widetilde{\varGamma}_p^\top\widetilde{K}_{2,p}^\top\widetilde{B}_p^\top v_3^{(p)}) + \varepsilon^\top(t)(\widetilde{G}_p^\top v_1^{(p)} - \widetilde{A}_p^\top v_1^{(p)} + \widetilde{C}_p^\top\widetilde{M}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} \\ &+ \widetilde{C}_p^\top\widetilde{T}_p^\top v_2^{(p)} + \widetilde{A}_p^\top v_3^{(p)} + \widetilde{K}_{1,p}^\top\widetilde{B}_p^\top v_3^{(p)} + \widetilde{C}_p^\top\widetilde{B}_p^\top\widetilde{K}_{3,p}^\top v_3^{(p)} + \widetilde{K}_{4,p}^\top\widetilde{B}_p^\top\mathcal{L}_p^\top v_3^{(p)}) \\ &+ \overline{\eta}^\top(t)(\widetilde{G}_p^\top v_1^{(p)} - \widetilde{A}_p^\top v_1^{(p)} + \widetilde{C}_p^\top\widetilde{M}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{C}_p^\top\widetilde{T}_p^\top v_2^{(p)} + \widetilde{A}_p^\top v_3^{(p)} \\ &+ \widetilde{K}_{1,p}^\top\widetilde{B}_p^\top v_3^{(p)} + \widetilde{C}_p^\top\widetilde{K}_{3,p}^\top v_3^{(p)} - (I_m\otimes S^\top)v_3^{(p)} + \widetilde{P}_p^\top\widetilde{B}_p^\top v_3^{(p)} \\ &+ \widetilde{K}_{4,p}^\top\widetilde{B}_p^\top\mathcal{L}_p^\top v_3^{(p)} + (I_m\otimes S^\top)\overline{v}_4^{(p)}). \end{aligned} \tag{18}$$

Additionally, we have

$$\begin{aligned} &\widetilde{G}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{K}_{1,p}^\top\widetilde{B}_p^\top v_3^{(p)} + \widetilde{K}_4^\top\widetilde{B}_p^\top\mathcal{L}_p^\top v_3^{(p)} \\ &= \begin{pmatrix} \begin{pmatrix} G_{1,p_1}^\top v_1^{(1p_1)} + K_{1,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} + F_{1,p_1}^\top v_2^{(1p_1)} + \sum_{j\in N_1} a^{(\zeta(t))}_{1j} K_{4,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} \\ - \sum_{j\in N_1} a^{(\zeta(t))}_{j1} K_{4,jp_1}^\top B_{jp_1}^\top v_3^{(jp_1)} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} G_{m,p_m}^\top v_1^{(mp_m)} + K_{1,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} + F_{m,p_m}^\top v_2^{(mp_m)} \\ + \sum_{j\in N_m} a^{(\zeta(t))}_{mj} K_{4,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} - \sum_{j\in N_m} a^{(\zeta(t))}_{jm} K_{4,jp_m}^\top B_{jp_m}^\top v_3^{(jp_m)} \end{pmatrix} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
&\widetilde{\varGamma}_p^\top \widetilde{J}_p^\top v_1^{(p)} + \widetilde{H}_p^\top v_2^{(p)} + \widetilde{\varGamma}_p^\top \widetilde{K}_{2,p}^\top \widetilde{B}_p^\top v_3^{(p)} \\
&\quad = \begin{pmatrix} \left(\varGamma_{1,p_1}^\top J_{1,p_1}^\top v_1^{(1p_1)} + H_{1,p_1}^\top v_2^{(1p_1)} + \varGamma_{1,p_1}^\top K_{2,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)}\right) \\ \vdots \\ \left(\varGamma_{m,p_m}^\top J_{m,p_m}^\top v_1^{(mp_m)} + H_{m,p_m}^\top v_2^{(mp_m)} + \varGamma_{m,p_m}^\top K_{2,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)}\right) \end{pmatrix}, \\
&\widetilde{G}_p^\top v_1^{(p)} - \widetilde{A}_p^\top v_1^{(p)} + \widetilde{C}_p^\top \widetilde{M}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{C}_p^\top \widetilde{T}_p^\top v_2^{(p)} + \widetilde{A}_p^\top v_3^{(p)} + \widetilde{K}_{1,p}^\top \widetilde{B}_p^\top v_3^{(p)} + \widetilde{C}_p^\top \widetilde{K}_{3,p}^\top \widetilde{B}_p^\top v_3^{(p)} \\
&\quad + \widetilde{K}_{4,p}^\top \widetilde{B}_p^\top \mathcal{L}_p^\top v_3^{(p)} \\
&\quad = \begin{pmatrix} \begin{pmatrix} G_{1,p_1}^\top v_1^{(1p_1)} - A_{1,p_1}^\top v_1^{(1p_1)} + C_{1,p_1}^\top M_{1,p_1}^\top v_1^{(1p_1)} + F_{1,p_1}^\top v_2^{(1p_1)} + C_{1,p_1}^\top T_{(1p_1)}^\top v_2^{(1p_1)} \\ + A_{1,p_1}^\top v_3^{(1p_1)} + K_{1,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} + C_{3p}^\top K_{3,1,p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} \\ + \sum_{j\in N_1} a_{1j}^{(\zeta(t))} K_{4,1,p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} - \sum_{j\in N_1} a_{j1}^{(\zeta(t))} K_{4,jp}^\top B_{j,p_j}^\top v_3^{(jp_j)} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} G_{m,p_m}^\top v_1^{(mp_m)} - A_{m,p_m}^\top v_1^{(mp_m)} + C_{m,p_m}^\top M_{m,p_m}^\top v_1^{(mp_m)} + F_{m,p_m}^\top v_2^{(mp_m)} \\ + C_{m,p_m}^\top T_{(mp_m)}^\top v_2^{(mp_m)} + A_{m,p_m}^\top v_3^{(mp_m)} + K_{1,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} \\ + C_{m,p_m}^\top K_{3,m,p_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} + \sum_{j\in N_m} a_{mj}^{(\zeta(t))} K_{4,m,p_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} \\ - \sum_{j\in N_m} a_{jm}^{(\zeta(t))} K_{4,jp}^\top B_{j,p_j}^\top v_3^{(jp_j)} \end{pmatrix} \end{pmatrix}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\widetilde{G}_p^\top v_1^{(p)} - \widetilde{A}_p^\top v_1^{(p)} + \widetilde{C}_p^\top \widetilde{M}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{C}_p^\top \widetilde{T}_p^\top v_2^{(p)} + \widetilde{A}_p^\top v_3^{(p)} + \widetilde{K}_{1,p}^\top \widetilde{B}_p^\top v_3^{(p)} \\
&\quad + \widetilde{C}_p^\top \widetilde{K}_{3,p}^\top v_3^{(p)} - \widetilde{S}^\top v_3^{(p)} + \widetilde{P}_p^\top \widetilde{B}_p^\top v_3^{(p)} + \widetilde{K}_{4,p}^\top \widetilde{B}_p^\top \mathcal{L}_p^\top v_3^{(p)} + (I_m \otimes S^\top)\overline{v}_4^{(p)} \\
&\quad = \begin{pmatrix} \begin{pmatrix} G_{1,p_1}^\top v_1^{(1p_1)} + C_{1,p_1}^\top M_{1,p_1}^\top v_1^{(1p_1)} + F_{1,p_1}^\top v_2^{(1p_1)} + C_{1,p_1}^\top T_{1,p_1}^\top v_2^{(1p_1)} + A_{1,p_1}^\top v_3^{(1p_1)} \\ - S^\top v_3^{(1p_1)} + K_{1,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} + C_{3p}^\top K_{3,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} + S^\top v_4^{(p)} + P_{1,p_1}^\top B_{1,p_1}^\top v_3^{1,p_1} \\ - A_{1,p_1}^\top v_1^{(1p_1)} + \sum_{j\in N_1} a_{1j}^{(\zeta(t))} K_{4,1p_1}^\top B_{1,p_1}^\top v_3^{(1p_1)} - \sum_{j\in N_1} a_{j1}^{(\zeta(t))} K_{4,jp}^\top B_{j,p_j}^\top v_3^{(jp_j)} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} G_{m,p_m}^\top v_1^{(mp_m)} + C_{m,p_m}^\top M_{m,p_m}^\top v_1^{(mp_m)} + F_{m,p_m}^\top v_2^{(mp_m)} + C_{m,p_m}^\top T_{m,p_m}^\top v_2^{(mp_m)} \\ + A_{m,p_m}^\top v_3^{(mp_m)} - S^\top v_3^{(mp_m)} + K_{1,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} C_{m,p_m}^\top K_{3,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} \\ + S^\top v_4^{(p)} + P_{m,p_m}^\top B_{m,p_m}^\top v_3^{ip_i} - A_{m,p_m}^\top v_1^{(mp_m)} + \sum_{j\in N_m} a_{mj}^{(\zeta(t))} K_{4,mp_m}^\top B_{m,p_m}^\top v_3^{(mp_m)} \\ - \sum_{j\in N_1} a_{jm}^{(\zeta(t))} K_{4,jp}^\top B_{j,p_j}^\top v_3^{(jp_j)} \end{pmatrix} \end{pmatrix}.
\end{aligned} \tag{20}$$

From (12h), (12i), and (14), we have

$$\begin{aligned}
&G_{i,p_i}^\top v_1^{(ip_i)} + F_{i,p_i}^\top v_2^{(ip_i)} + K_{1,i,p_i}^\top B_{i,p_i}^\top v_3^{(ip_i)} + K_{4,i,p_i}^\top B_{i,p_i}^\top \mathcal{L}_p^\top v_3^{(ip_i)} + \mu v_1^{(ip_i)} - \sum_{j\in N_i} a_{ji}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{j,p_j}^{(\iota)} \\
&\quad \prec \sum_{\iota=1}^n \mathfrak{G}_{i,p_i}^{(\iota)} + \alpha_1 v_1^{(ip_i)} + \sum_{\iota=1}^s \mathfrak{F}_{i,p_i}^{(\iota)} + \sum_{\iota=1}^r \mathfrak{K}_{i,p_i}^{(\iota)} + \sum_{j\in N_i} a_{ij}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{i,p_i}^{(\iota)} - \sum_{j\in N_i} a_{ji}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{j,p_j}^{(\iota)} \\
&\quad \prec 0,
\end{aligned}$$

that is, $\widetilde{G}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{K}_{1,p}^\top \widetilde{B}_p^\top v_3^{(p)} + \widetilde{K}_{4,p}^\top \widetilde{B}_p^\top \widetilde{\mathcal{L}}_p^\top v_3^{(p)} \prec -\mu_1 v_1^{(p)}$. From (12j), (12k), and (14), the following holds:

$$\begin{aligned}
&\varGamma_{i,p_i}^\top J_{i,p_i}^\top v_1^{(ip_i)} + H_{i,p_i}^\top v_2^{(ip_i)} + \varGamma_{i,p_i}^\top K_{2,i,p_i}^\top B_{i,p_i}^\top v_3^{(ip_i)} + \mu_1 v_2^{(ip_i)} \\
&\quad \prec \varGamma_{i,p_i}^\top \sum_{\iota=1}^n \mathfrak{I}_{i,p_i}^{(\iota)} + \sum_{\iota=1}^s \mathfrak{H}_{i,p_i}^{(\iota)} + \varGamma_{i,p_i}^\top \sum_{\iota=1}^r \mathfrak{Z}_{i,p_i}^{(\iota)} + \alpha_2 v_2^{(ip_i)} \\
&\quad \prec 0.
\end{aligned}$$

Thus, $\widetilde{\varGamma_p}^\top \widetilde{J}_p^\top v_1^{(p)} + \widetilde{H}_p^\top v_2^{(p)} + \widetilde{\varGamma}_p^\top \widetilde{K}_{2,p}^\top \widetilde{B}_p^\top v_3^{(p)} \prec -\mu v_2^{(p)}$. From (12i), (12l), and (14), we have

$$\begin{aligned}
&G_{i,p_i}^\top v_1^{(ip_i)} - A_{i,p_i}^\top v_1^{(ip_i)} + C_{i,p_i}^\top M_{i,p_i}^\top v_1^{(ip_i)} + F_{i,p_i}^\top v_2^{(ip_i)} + C_{i,p_i}^\top T_{i,p_i}^\top v_2^{(ip_i)} + A_{i,p_i}^\top v_3^{(ip_i)} \\
&\quad + K_{1ip}^\top B_{i,p_i}^\top v_3^{(ip_i)} + C_{i,p_i}^\top K_{3,i,p_i}^\top v_3^{(ip_i)} + \widetilde{K}_4^\top \widetilde{B}_{i,p_i}^\top \mathcal{L}_p^\top v_3^{(ip_i)} + \mu_1 v_3^{(ip_i)} \\
&\quad \prec \sum_{\iota=1}^n \mathfrak{G}_{i,p_i}^{(\iota)} + \alpha_1 v_1^{(ip_i)} - \mu_1 v_1^{(ip_i)} + \mu_1 v_3^{(ip_i)} + \sum_{\iota=1}^s \mathfrak{F}_{i,p_i}^{(\iota)} + \sum_{\iota=1}^r \mathfrak{K}_{i,p_i}^{(\iota)} - A_{i,p_i}^\top v_1^{(ip_i)} \\
&\quad + C_{i,p_i}^\top \sum_{\iota=1}^n \mathfrak{M}_{i,p_i}^{(\iota)} + C_{i,p_i}^\top \sum_{\iota=1}^s \mathfrak{T}_{i,p_i}^{(\iota)} + A_{i,p_i}^\top v_3^{(ip_i)} + C_{i,p_i}^\top \sum_{\iota=1}^r \mathfrak{Y}_{i,p_i}^{(\iota)} \\
&\quad + \sum_{j\in N_i} a_{ij}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{i,p_i}^{(\iota)} - \sum_{j\in N_i} a_{ji}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{j,p_j}^{(\iota)} \\
&\quad \prec 0.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}&\widetilde{G}_p^\top v_1^{(p)} - \widetilde{A}_p^\top v_1^{(p)} + \widetilde{C}_p^\top \widetilde{M}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{C}_p^\top \widetilde{T}_p^\top v_2^{(p)} + \widetilde{A}_p^\top v_3^{(p)} \\ &+\widetilde{K}_{1,p_1}^\top \widetilde{B}_p^\top v_3^{(p)} + \widetilde{C}_p^\top \widetilde{K}_{3p}^\top \widetilde{B}_p^\top v_3^{(p)} + \widetilde{K}_4^\top \widetilde{B}_p^\top \widetilde{\mathcal{L}}_p^\top v_3^{(p)} \prec -\mu_1 v_3^{(p)}.\end{aligned}$$

Using (12i), (12m), and (14) gives

$$\begin{aligned}&G_{i,p_i}^\top v_1^{(ip_i)} - A_{i,p_i}^\top v_1^{(ip_i)} + C_{i,p_i}^\top M_{i,p_i}^\top v_1^{(ip_i)} + F_{i,p_i}^\top v_2^{(ip_i)} + C_{i,p_i}^\top T_{i,p_i}^\top v_2^{(ip_i)} + A_{i,p_i}^\top v_3^{(ip_i)} + K_{1ip}^\top B_{i,p_i}^\top v_3^{(ip_i)} \\ &\quad +C^\top K_{3ip}^\top v_3^{(ip_i)} - S^\top v_3^{(ip_i)} + P_{i,p_i}^\top B_{i,p_i}^\top v_3^{(ip_i)} + S^\top v_4^{(p)} + \mu_1 v_4^{(p)} + \widetilde{K}_4^\top \widetilde{B}_{i,p_i}^\top \mathcal{L}_p^\top v_3^{(ip_i)} \\ &\quad = \sum_{\iota=1}^n \mathfrak{G}_{i,p_i}^{(\iota)} + \alpha_1 v_1^{(ip_i)} - \mu_1 v_1^{(ip_i)} + \mu_1 v_4^{(p)} + \sum_{\iota=1}^s \mathfrak{F}_{i,p_i}^{(\iota)} + \sum_{\iota=1}^r \mathfrak{K}_{i,p_i}^{(\iota)} - A_{i,p_i}^\top v_1^{(ip_i)} + C_{i,p_i}^\top \sum_{\iota=1}^n \mathfrak{M}_{i,p_i}^{(\iota)} \\ &\quad + C_{i,p_i}^\top \sum_{\iota=1}^s \mathfrak{T}_{i,p_i}^{(\iota)} - S^\top v_3^{(ip_i)} + A_{i,p_i}^\top v_3^{(ip_i)} + C_{i,p_i}^\top \sum_{\iota=1}^r \mathfrak{Y}_{i,p_i}^{(\iota)} + S^\top v_4^{(p)} + \sum_{\iota=1}^r \mathfrak{P}_{i,p_i}^{(\iota)} \\ &\quad + \sum_{j\in N_i} a_{ij}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{i,p_i}^{(\iota)} - \sum_{j\in N_i} a_{ji}^{(\zeta(t))} \sum_{\iota=1}^r \mathfrak{N}_{j,p_j}^{(\iota)} \\ &\quad \prec 0,\end{aligned}$$

that is, $\widetilde{G}_p^\top v_1^{(p)} - \widetilde{A}_p^\top v_1^{(p)} + \widetilde{C}_p^\top \widetilde{M}_p^\top v_1^{(p)} + \widetilde{F}_p^\top v_2^{(p)} + \widetilde{C}_p^\top \widetilde{T}_p^\top v_2^{(p)} + \widetilde{A}_p^\top v_3^{(p)} + \widetilde{K}_{1,p}^\top \widetilde{B}_p^\top v_3^{(p)} + \widetilde{C}_p^\top \widetilde{K}_{3,p}^\top v_3^{(p)} - \widetilde{S} v_3^{(p)} + \widetilde{P}_p^\top \widetilde{B}_p^\top v_3^{(p)} + (\mathbf{1}_m \otimes S^\top)\overline{v}_4^{((p))} + \widetilde{K}_4^\top \widetilde{B}_p^\top \widetilde{\mathcal{L}}_p^\top v_3^{(p)} \prec -\mu_1 v_4^{(p)}$.

For each agent, the proof of the positive consensus can be given. Then,

$$\begin{aligned}\dot{V}_{\sigma(t)}(X(t)) &\leqslant e^\top(t)(-\mu_1)v_1^{(p)} + \theta^\top(t)(-\mu_1)v_2^{(p)} + \varepsilon^\top(t)(-\mu_1)v_3^{(p)} + \overline{\eta}^\top(t)(-\mu_1)\overline{v}_4^{(p)} \\ &\leqslant -\mu_1 V_{\sigma(t)}(X(t)).\end{aligned} \tag{21}$$

Integrating both sides of (21) gives $V_{\sigma_i(t)}(X(t)) \leqslant e^{-\mu_1(t-t_k)} V_{\sigma_i(t_k)}(X(t_k)), t \in [t_k, t_{k+1})$. By repeating the above steps, the following is derived:

$$V_{\sigma(t)}(X(t)) \leqslant \prod_{i=1}^n \lambda_i e^{-\mu_1(t-t_{k-1})} V_{\sigma(t_{k-2})}(X(t_{k-1})) \leqslant \cdots \leqslant \prod_{i=1}^n \lambda_i^k e^{-\mu_1(t-t_{k_0})} V_{\sigma(t_{k_0})}(X(t_{k_0})).$$

From (13) and $\lambda_i > 1$, we have

$$V_{\sigma(t)}(X(t)) \leqslant e^{(N_0 + \frac{t-t_{k_0}}{\tau_i})\sum_{i=1}^n \ln \lambda_i} e^{-\mu_1(t-t_{k_0})} V_{\sigma(t_{k_0})}(X(t_{k_0})) \leqslant \rho e^{-\phi(t-t_0)} V_{\sigma(t_0)}(X(t_0)),$$

where $\rho = e^{N_0 \sum_{i=1}^n \ln \lambda_i}$ and $\phi = \mu_1 - \frac{\sum_{i=1}^n \ln \lambda_i}{\tau_i}$. Therefore, the proof of leader-following consensus of the system can be provided using Definition 3. By Lemma 2 and Definition 4, system (1) is positive and achieves the consensus.

Remark 3. Controller (4) consists of two switching signals: $\sigma_i(t)$ for agents and $\zeta(t)$ for communication topologies. Here, two points should be pointed out. First, signal $\sigma_i(t)$ describes the state switching of the ith agent at time t. This signal aims to reach a quick response from the agents to changes in the external environment. Second, signal $\zeta(t)$ indicates the transformation of a communication topology at time t, which will affect the consensus of a system. These two signals work together to enable controller (4) to adapt to dynamic changes in states and communication topologies of SPMASs.

Remark 4. In [31, 33, 44], the denominator in the control gain matrix is designed as follows: $\mathbf{1}_n^\top B^\top v$. To ensure system stability, an upper bound on the gain matrix variable is given. The introduction of this upper bound clearly narrows the feasible region of solutions of the corresponding conditions. In Theorem 1, the denominator of the control and observer gain matrices is defined as $\mathbf{1}_n^{(\iota)\top} B^\top v$. This approach aims to use different constants for different components of the numerators. In this strategy, the upper bound restriction is removed. Therefore, the gain design in Theorem 1 reduces the conservatism in [31, 33, 44]. Additionally, condition (12o) is used to derive relations (15). Since these relations (15) involve scaling issues and bounds on vectors, a matrix decomposition technique is introduced to decompose the controller gain matrix into nonpositive and non-negative components, that is, $K_{1,i,p} = K_{1,i,p}^+ + K_{1,i,p}^-$ and $K_{3,i,p} = K_{3,i,p}^+ + K_{3,i,p}^-$. In such a framework, the positivity and consensus of MASs can be easily handled.

Remark 5. This paper investigates SPMASs with heterogeneous properties. These systems have two characteristics. (i) Different agents have different agent dynamics. (ii) Each agent has a different switching signal. At different time intervals, the agents switch under different switching signals. Thus, control protocol (4) is designed to account for the switching of each agent. This design can effectively handle the heterogeneity of switching signals.

A flowchart is given in Figure 1 to further explain the main design in Theorem 1.

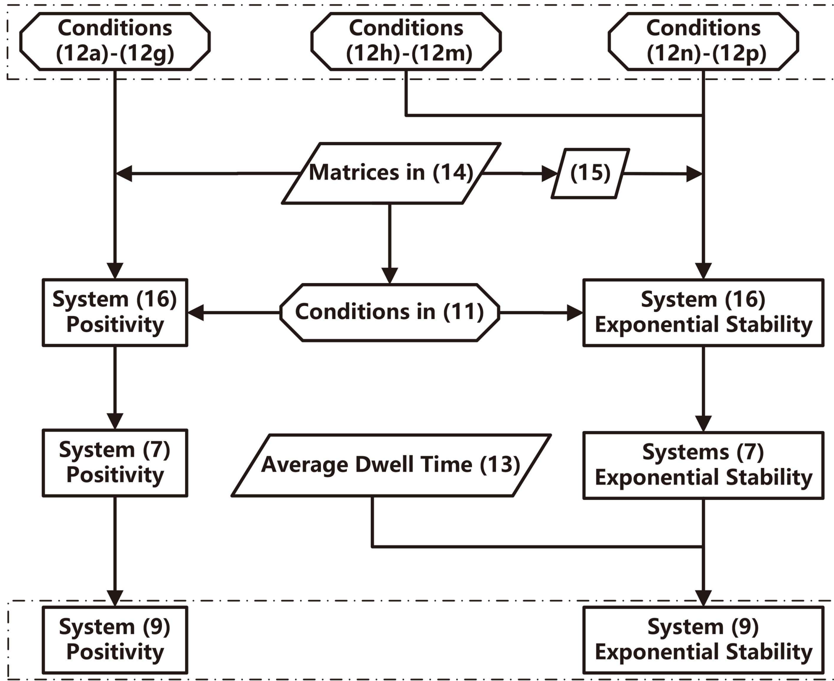

Figure 1 Flowchart for Theorem 1.

3.2 Multileaders

Subsection 3.1 introduces the leader-following consensus for SPMASs with a single leader. This subsection discusses the consensus of SPMASs with multiple leaders. The leaders' dynamics is shown as follows:

$$\dot{\eta}_k(t) = S\eta_k(t) + W_1\overline{u}_k(t), \tag{22}$$

where $k \in [1, M]$, $\eta_k(t) \in \mathcal{R}^n$ and $\overline{u}_k(t) \in \mathcal{R}^r$ are the state and control input of the leader system, respectively; $S \in \mathcal{R}^{n\times n}$ is Metzler; and $W_1 \in \mathcal{R}^{n\times r}$. The control protocol of the leader is designed as follows:

$$\overline{u}_k(t) = P_1 \textstyle\sum_{q=1}^M \widehat{a}_{kq}(\eta_k(t) - \eta_q(t)) + P_2\eta_o^*, \tag{23}$$

where $\eta_o^* \succ 0$ is a vector of constants, gain control matrix $P_1 \in \mathcal{R}^{r\times n}$, $P_2 \in \mathcal{R}^{r\times n}$, and $P_2\eta_o^* \in \mathcal{R}^{r\times 1}$ is considered to be an auxiliary item. Using (22) and (23), we get

$$\dot{\eta}_k(t) = S\eta_k(t) + W_1P_1 \textstyle\sum_{q=1}^M \widehat{a}_{kq}(\eta_k(t) - \eta_q(t)) + W_1P_2\eta_o^*. \tag{24}$$

A control protocol for system (1) is designed as follows:

$$\begin{aligned} u_i(t) =\ & K_{1,i,\sigma_i(t)}\widehat{x}_i(t) + d_{k,\sigma_i(t)}(\widehat{x}_k(t) - \eta_k(t)) + K_{2,i,\sigma_i(t)}\widehat{w}_i(t) + K_{3,i,\sigma_i(t)}y_i(t) \\ & + K_{4,i,\sigma_i(t)} \textstyle\sum_{j=1}^M a_{ij}^{(\zeta(t))}(\widehat{x}_i(t) - \widehat{x}_j(t)) + P_{3,i,\sigma_i(t)}x^*, \end{aligned} \tag{25}$$

where $u_i(t) \in \mathcal{R}^r$, $\widehat{x}_i(t) \in \mathcal{R}^n$, $\widehat{w}_i(t) \in \mathcal{R}^s$, and $y_i(t) \in \mathcal{R}^o$ are the control protocol, state estimations, disturbance estimations, and outputs, respectively; $K_{1,i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$, $d_{i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$, $K_{2,i,\sigma_i(t)} \in \mathcal{R}^{s\times n}$, $K_{3,i,\sigma_i(t)} \in \mathcal{R}^{s\times o}$, $K_{4,i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$, and $P_{3,i,\sigma_i(t)} \in \mathcal{R}^{r\times n}$ are gain matrices.

Error variables are defined as follows: $\chi_i(t) = x_i(t) - x^*$, $e_i(t) = \widehat{x}_i(t) - x_i(t)$, $\theta_i(t) = \widehat{\xi}_i(t) - \xi_i(t)$, and $\varepsilon_k(t) = x_k(t) - \eta_k(t)$, where $\chi_i(t)$, $e_i(t)$, $\theta_i(t)$, and $\varepsilon_k(t)$ are a tracking error, a state-observer error, a disturbance-observer error, and errors between agents, respectively.

Remark 6. Control protocol (23) guarantees the consensus of leader system (22). It is necessary to prove that system (22) is positive and achieves practical consensus first. Otherwise, leaders cannot converge to 0. This will lead to the fact that the follower cannot be sure of successfully tracking the leader. Controller (25) should be taken into account in two directions: $k \in [1, M]$ and $f \in [M+1, N]$. There are N followers and M leaders in total. In this study, the communication between the first M leaders and followers is assumed to be one-to-one, as emphasized in the tracking error $\varepsilon_k(t)$. Then, the remaining $N-M+1$ followers communicate with the first M followers.

In Theorem 2, we first handle the positivity and consensus of leader system (24).

Theorem 2. If there exist constants $\mu_2>0$, $\delta_{21}>0$, $\beta>0$, $\eta^*>0$, $\underline{\gamma}_{21}>0$, $\overline{\gamma}_{21}>0$, and $\mathcal{R}^n$ vectors $\mathfrak{P}^{(\iota)}\prec 0$, $\mathfrak{Q}^{+(\iota)}\succ 0$, $\mathfrak{Q}^{-(\iota)}\prec 0$, $v\succ 0$ such that

$$S+\frac{\ell_{\max}}{\underline{\gamma}_{22}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{P}^{(\iota)\top}+\delta_{21}I_M\succeq 0, \tag{26a}$$

$$\begin{aligned}&\frac{\ell_{\max}}{\underline{\gamma}_{21}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{P}^{(\iota)\top}+S+(\beta-1)\\ &\quad\times\Big(\frac{1}{\overline{\gamma}_{21}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Q}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{21}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Q}^{-(\iota)\top}\Big)\succeq 0,\end{aligned} \tag{26b}$$

$$\beta\Big(\frac{1}{\overline{\gamma}_{21}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Q}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{21}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{Q}^{-(\iota)\top}\Big)+\frac{\overline{\ell}_{\max}}{\overline{\gamma}_{21}}W_1\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)}\mathfrak{P}_1^{(\iota)\top}\succeq 0, \tag{26c}$$

$$\mathbf{1}_M\otimes S^\top v+\mathcal{L}_\eta^\top\mathbf{1}_M\otimes\sum_{\iota=1}^r\mathfrak{P}^{(\iota)}+\mu_2\mathbf{1}_M\otimes v\preceq 0, \tag{26d}$$

$$\eta^{*\top}\Big(\mathbf{1}_M\otimes S^\top v+\mathcal{L}_\eta^\top\mathbf{1}_M\otimes\sum_{\iota=1}^r\mathfrak{P}^{(\iota)}+(\overline{\beta}\mathbf{1}_M^\top-I_M)\mathbf{1}_M\otimes\Big(\sum_{\iota=1}^r\mathfrak{Q}^{(\iota)}\Big)\Big)-\eta^*\preceq 0, \tag{26e}$$

$$\underline{\gamma}_{21}\leqslant \mathbf{1}_r^{(\iota)\top}W_1^\top v^{(ip_i)}\leqslant\overline{\gamma}_{21},\ \iota=1,2,\cdots,r \tag{26f}$$

hold, then system (22) is positive and reaches practical consensus under control protocol (25) with $P_2=P_2^++P_2^-$ satisfying the following:

$$P_1=\sum_{\iota=1}^r\frac{\mathbf{1}_r^{(\iota)}\mathfrak{P}^{(\iota)\top}}{\mathbf{1}_r^{(\iota)\top}W_1^\top v^{(ip_i)}},\ P_2^+=\sum_{\iota=1}^r\frac{\mathbf{1}_r^{(\iota)}\mathfrak{Q}^{+(\iota)\top}}{\mathbf{1}_r^{(\iota)\top}W_1^\top v^{(ip_i)}},\ P_2^-=\sum_{\iota=1}^r\frac{\mathbf{1}_r^{(\iota)}\mathfrak{Q}^{-(\iota)\top}}{\mathbf{1}_r^{(\iota)\top}W_1^\top v^{(ip_i)}}, \tag{27}$$

where $\overline{\beta}=(\underbrace{\beta,\beta,\ldots,\beta}_{M})^\top$, $l_{\max}=\max_{k\in[1,M]}\{\widehat{a}_{ii}\}$, and $\overline{l}_{\max}=\max_{k\in[1,M]}\{\sum_{j\in\mathcal{N}_k}\widehat{a}_{jk}\}$.

Proof. Let $\eta(t)=(\eta_1(t),\eta_2(t),\ldots,\eta_m(t))$. Then,

$$\dot{\eta}(t)=(I_M\otimes S+\mathcal{L}_\eta\otimes W_1P_1)\eta(t)+(I_M\otimes W_1P_2)\eta^*, \tag{28}$$

where $\eta^*=(\underbrace{\eta_o^*,\eta_o^*,\ldots,\eta_o^*}_{N})^\top$ and $\mathcal{L}_\eta$ is a Laplacian matrix denoting the communication topology between agents with

$$\mathcal{L}_\eta=\begin{pmatrix}\sum_{j\in N_1}\widehat{a}_{1j} & -\widehat{a}_{12} & \cdots & -\widehat{a}_{1M}\\ -\widehat{a}_{21} & \sum_{j\in N_2}\widehat{a}_{2j} & \cdots & -\widehat{a}_{2M}\\ \vdots & \vdots & \ddots & \vdots\\ -\widehat{a}_{M1} & -\widehat{a}_{M2} & \cdots & \sum_{j\in N_M}\widehat{a}_{Mj}\end{pmatrix}. \tag{29}$$

The following variables are defined:

$$\psi_i(t)=\sum_{\iota=1}^m\beta_\iota\eta_i(t)-\eta_i(t)-\eta_o^*. \tag{30}$$

Let $\psi(t)=(\psi_1^\top,\psi_2^\top,\ldots,\psi_M^\top)^\top$. We have $\psi(t)=(\mathbf{1}_M\overline{\beta}^\top\otimes I_n-I_{nM})\eta(t)-\eta^*$. Therefore, it is deduced that

$$\dot{\psi}(t)=(I_M\otimes S+\mathcal{L}_\eta\otimes W_1P_1)\psi(t)+(I_M\otimes S+\mathcal{L}_\eta\otimes W_1P_1+(\mathbf{1}_M\overline{\beta}^\top-I_M)\otimes W_1P_2)\eta^*. \tag{31}$$

By (26a), (26f), and (27), it follows that

$$\begin{aligned}&I_M\otimes S+\mathcal{L}_\eta\otimes W_1P_1+\delta_{21}I_{nM}\\ &=\begin{pmatrix}\sum_{j\in N_1}\widehat{a}_{1j}W_1P_1+S+\delta_{21}I_n & -\widehat{a}_{12}W_1P_1 & \cdots & -\widehat{a}_{1M}W_1P_1\\ -\widehat{a}_{21}W_1P_1 & \sum_{j\in N_2}\widehat{a}_{2j}W_1P_1+S+\delta_{21}I_n & \cdots & -\widehat{a}_{2M}W_1P_1\\ \vdots & \vdots & \ddots & \vdots\\ -\widehat{a}_{M1}W_1P_1 & -\widehat{a}_{M2}W_1P_1 & \cdots & \sum_{j\in N_M}\widehat{a}_{Mj}W_1P_1+S+\delta_{21}I_n\end{pmatrix}\succeq 0.\end{aligned}$$

From Lemma 1, it is easy to see that $I_M \otimes S + \mathcal{L}_\eta \otimes W_1 P_1$ is Metzler. By (26b), (26f), (26c), and (27), we have $\sum_{j\in N_i} \widehat{a}_{ij} W_1 P_1 + S + (\beta - 1) W_1 P_2 \succeq 0$, $\beta W_1 P_2 - \widehat{a}_{ij} W_1 P_1 \succeq 0$, where $i \neq j$. Thus,

$$
\begin{aligned}
& I_M \otimes S + \mathcal{L}_\eta \otimes W_1 P_1 + (\mathbf{1}_M \overline{\beta}^\top - I_M) \otimes W_1 P_2 \\
&= \begin{pmatrix}
\begin{pmatrix} \sum_{j\in N_1} \widehat{a}_{1j} W_1 P_1 \\ +S + (\beta-1) W_1 P_2 \end{pmatrix} & \beta W_1 P_2 - \widehat{a}_{12} W_1 P_1 & \cdots & \beta W_1 P_2 - \widehat{a}_{1M} W_1 P_1 \\
\beta W_1 P_2 - \widehat{a}_{21} W_1 P_1 & \begin{pmatrix} \sum_{j\in N_2} \widehat{a}_{2j} W_1 P_1 \\ +S + (\beta-1) W_1 P_2 \end{pmatrix} & \cdots & \beta W_1 P_2 - \widehat{a}_{2M} W_1 P_1 \\
\vdots & \vdots & \ddots & \vdots \\
\beta W_1 P_2 - \widehat{a}_{M1} W_1 P_1 & \beta W_1 P_2 - \widehat{a}_{M2} W_1 P_1 & \cdots & \begin{pmatrix} \sum_{j\in N_M} \widehat{a}_{Mj} W_1 P_1 \\ +S + (\beta-1) W_1 P_2 \end{pmatrix}
\end{pmatrix} \succeq 0.
\end{aligned}
$$

Therefore, we can deduce that $\dot{\eta}(t) = (I_M \otimes S + \mathcal{L}_\eta \otimes W_1 P_1)\eta(t) \succeq 0$, indicating that leader system (22) is positive by Definition 1 and Lemma 2.

Next, the issue of practical consensus will be addressed. Choose a CLF $V = \psi^\top(t)(\mathbf{1}_M \otimes v)$. Then, $V_{\sigma(t)} = \psi^\top(t)(\mathbf{1}_M \otimes S^\top v + \mathcal{L}_\eta^\top \mathbf{1}_M \otimes P_1^\top W_1^\top v) + \eta^{*\top}(\mathbf{1}_M \otimes S^\top v + \mathcal{L}_\eta^\top \mathbf{1}_M \otimes P_1^\top W_1^\top v + (\beta \mathbf{1}_M^\top - I_M)\mathbf{1}_M \otimes P_2^\top W_1^\top v)$. Using (26d) and (27), we get $\mathbf{1}_M \otimes S^\top v + \mathcal{L}_\eta^\top \mathbf{1}_M \otimes P_1^\top W_1^\top v + \mu_2 \mathbf{1}_M \otimes v_1 \preceq 0$. From (26e), it holds that $\eta^{*\top}(\mathbf{1}_M \otimes S^\top v + \mathcal{L}_\eta^\top \mathbf{1}_m \otimes P_1^\top W_1^\top v + (\beta \mathbf{1}_M^\top - I_M)\mathbf{1}_M \otimes P_2^\top W_1^\top v) - \eta_o^* \preceq 0$. Thus, $\dot{V}(t) < -\mu_2 \psi^\top(t) v + \eta_o^*$. It follows that $V(t) < e^{-\mu_2} V(t) + \eta_o^* \int_{t_0}^t e^{-\mu_2(t-\tau)}\, d\tau$. By Definition 4, the practical consensus of leader system (22) is guaranteed.

Remark 7. The introduction of vector η_o^* facilitates the stability of each agent in leader system (22) to a set of equilibrium points. The final convergence values of the states are associated with the value of η_o^*. Without the introduction of η_o^*, the final convergence depends on the states of the systems, and the value is zero. Moreover, vector η_o^* can be designed with different values by following the practical requirements. It can be concluded that the design of different gain matrices will lead to disparate convergence values.

In Theorem 2, some constants need to be known. Thus, their choice is important for Theorem 2. Algorithm 1 can be used to determine these constants.

Algorithm 1 Searching the parameters of Theorem 2.

Input: Leader system matrices S, W, and parameters m_i, $\underline{\mu}_2$, $\overline{\mu}_2$, $\underline{\underline{\gamma}}_{21}$, $\overline{\underline{\gamma}_{21}}$, $\underline{\overline{\gamma}_{21}}$, $\overline{\overline{\gamma}}_{21}$, $\underline{\delta}_{21}$, $\overline{\delta}_{21}$, $\underline{\beta}$, $\overline{\beta}$;

Output: $\mu_2, \underline{\gamma}_{21}, \overline{\gamma}_{21}, \delta_{21}, \beta$;

  Define a set $\Omega = \emptyset$ and initialize parameters $\mu_2 = \underline{\mu}_2$, $\underline{\gamma}_{21} = \underline{\underline{\gamma}}_{21}$, $\overline{\gamma}_{21} = \underline{\overline{\gamma}_{21}}$, $\delta_{21} = \underline{\delta}_{21}$, and $\beta = \underline{\beta}$;

  while $\mu_2 \leqslant \overline{\mu}_2$ **do**

    repeat

      repeat

        repeat

          repeat

            if Eqs. (26a)–(26f) are feasible, **then** save $\mu_2, \underline{\gamma}_{21}, \overline{\gamma}_{21}, \delta_{21}$, and β to Ω

            end

            $\mu_2 \leftarrow \mu_2 + m_5$;

          until $\underline{\gamma}_{21} > \overline{\underline{\gamma}_{21}}$; $\underline{\gamma}_{21} \leftarrow \underline{\gamma}_{21} + m_4$;

        until $\overline{\gamma}_{21} > \overline{\overline{\gamma}}_{21}$; $\overline{\gamma}_{21} \leftarrow \overline{\gamma}_{21} + m_3$;

      until $\delta_{21} > \overline{\delta}_{21}$; $\delta_{21} \leftarrow \delta_{21} + m_2$;

    until $\beta > \overline{\beta}$; $\beta \leftarrow \beta + m_1$;

  end

Theorem 3. If there exist constants $\underline{\gamma}_1 > 0$, $\underline{\gamma}_2 > 0$, $\underline{\gamma}_3 > 0$, $\overline{\gamma}_1 > 0$, $\overline{\gamma}_2 > 0$, $\overline{\gamma}_3 > 0$, $\underline{\gamma}_{31} > 0$, $\underline{\gamma}_{32} > 0$, $\underline{\gamma}_{33} > 0$, $\overline{\gamma}_{31} > 0$, $\overline{\gamma}_{32} > 0$, $\overline{\gamma}_{33} > 0$, $\varrho > 0$, $M > 0$, $\alpha_{31} < 0$, $\alpha_{32} < 0$, $\lambda_i > 1$, $\mu_3 > 0$, $\delta_{32} > 0$, $\delta_{33} > 0$, $\delta_4 > 0$, $\delta_5 > 0$, $\varpi > 0$, $\mathcal{R}^n$ vectors $\mathfrak{Q}^{+(\iota)} \succeq 0$, $\mathfrak{Q}^{-(\iota)} \preceq 0$, $\mathfrak{d}_{i,p_i}^{(\iota)} \succeq 0$, $\mathfrak{P}^{(\iota)} \preceq 0$, $\mathfrak{P}_{i,p_i}^{-(\iota)} \preceq 0$, $\mathfrak{P}_{i,p_i}^{+(\iota)} \succeq 0$, $\mathfrak{K}_{i,p_i}^{-(\iota)} \preceq 0$, $\mathfrak{K}_{i,p_i}^{+(\iota)} \succeq 0$, $\mathfrak{N}_{i,p_i}^{(\iota)} \preceq 0$, $\mathfrak{F}_{i,p_i}^{(\iota)} \succeq 0$, $\mathfrak{G}_{i,p_i}^{(\iota)} \succeq 0$, $v_1^{(ip_i)} \succeq 0$, $v_3^{(ip_i)} \succeq 0$, $v_4^{(ip_i)} \succeq 0$, $\mathcal{R}^o$ vectors $\mathfrak{Y}_{i,p_i}^{-(\iota)} \preceq 0$, $\mathfrak{Y}_{i,p_i}^{+(\iota)} \succeq 0$, $\mathfrak{M}_{i,p_i}^{+(\iota)} \succeq 0$, $\mathfrak{M}_{i,p_i}^{-(\iota)} \preceq 0$, $\mathfrak{T}_{i,p_i}^{+(\iota)} \succeq 0$, $\mathfrak{T}_{i,p_i}^{-(\iota)} \preceq 0$, and $\mathcal{R}^s$ vectors $\mathfrak{Z}_{i,p_i}^{(\iota)} \succeq 0$, $\mathfrak{J}_{i,p_i}^{(\iota)} \succeq 0$, $\mathfrak{H}_{i,p_i}^{(\iota)} \succeq 0$, $\varGamma_{i,p_i}^{(\iota)} \succeq 0$, $v_2^{(ip_i)} \succeq 0$ such that the conditions (11), (12a), (12b), (12c), (12d), (12e), (12f), (12o), and

$$\varrho x^* - \eta_o^* = 0, \tag{32a}$$

$$\tfrac{1}{\overline{\gamma}_{31}} B_{k,p_k} \textstyle\sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \mathfrak{d}_{k,p_k}^{+(\iota)\top} + \tfrac{1}{\underline{\gamma}_{31}} B_{k,p_k} \sum_{\iota=1}^r \mathbf{1}_r^{(\iota)} \mathfrak{d}_{k,p_k}^{-(\iota)\top} \succeq 0, \tag{32b}$$

$$\begin{aligned}&A_{i,p_i}+\frac{1}{\overline{\gamma}_{31}}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{i,p_i}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{31}}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{i,p_i}^{-(\iota)\top}+\frac{1}{\overline{\gamma}_{31}}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{+(\iota)\top}C_{i,p_i}\\&+\frac{1}{\underline{\gamma}_{31}}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{i,p_i}^{-(\iota)\top}C_{i,p_i}+\frac{1}{\overline{\gamma}_{31}}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{R}_{i,p_i}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{31}}B_{i,p_i}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{R}_{i,p_i}^{-(\iota)\top}\succeq 0,\end{aligned}\tag{32c}$$

$$\begin{aligned}&A_{k,p_k}+\frac{1}{\overline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{k,p_k}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{k,p_k}^{-(\iota)\top}+\frac{1}{\overline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{k,p_k}^{+(\iota)\top}C_{k,p_k}\\&-S+\frac{1}{\underline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{k,p_k}^{-(\iota)\top}C_{k,p_k}+\frac{1}{\overline{\gamma}_{33}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{R}_{k,p_k}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{33}}B_{p}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{R}_{k,p_k}^{-(\iota)\top}\\&+\frac{\varrho}{\overline{\gamma}_{33}}W_1\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Q}^{+(\iota)\top}+\frac{\varrho}{\underline{\gamma}_{33}}W_1\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Q}^{-(\iota)\top}\succeq 0,\end{aligned}\tag{32d}$$

$$\begin{aligned}&\frac{1}{\underline{\gamma}_{31}}\sum_{j\in N_k}a_{kj}^{(\zeta(t))}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{N}_{k,p_k}^{(\iota)\top}+A_{k,p_k}+\frac{1}{\overline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{k,p_k}^{+(\iota)\top}\\&+\frac{1}{\underline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{K}_{k,p_k}^{-(\iota)\top}-S+\frac{1}{\underline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{k,p_k}^{+(\iota)\top}C_{k,p_k}\\&+\frac{1}{\underline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{Y}_{k,p_k}^{-(\iota)\top}C_{k,p_k}+\frac{1}{\underline{\gamma}_{32}}\sum_{j\in N_k}\widehat{a}_{kj}W_1\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{P}^{(\iota)\top}\succeq 0,\end{aligned}\tag{32e}$$

$$\begin{aligned}&S+\frac{1}{\overline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{d}_{k,p_k}^{+(\iota)\top}+\frac{1}{\underline{\gamma}_{31}}B_{k,p_k}\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{d}_{k,p_k}^{-(\iota)\top}\\&+\frac{1}{\underline{\gamma}_{32}}\sum_{j\in N_k}\widehat{a}_{kj}W_1\sum_{\iota=1}^{r}\mathbf{1}_r^{(\iota)}\mathfrak{P}^{(\iota)\top}+\delta_{31}I_N\succeq 0,\end{aligned}\tag{32f}$$

$$\begin{aligned}&A_{k,p_k}^{\top}v_1^{(kp_k)}+A_{k,p_k}^{\top}v_2^{(kp_k)}+A_{k,p_k}^{\top}v_4^{(kp_k)}-S^{\top}v_4^{(kp_k)}+\sum_{\iota=1}^{r}\mathfrak{G}_{k,p_k}^{(\iota)}+\alpha_{31}v_1^{(kp_k)}+\sum_{\iota=1}^{r}\mathfrak{F}_{k,p_k}^{(\iota)}\\&+C_{k,p_k}^{\top}\sum_{\iota=1}^{n}\mathfrak{M}_{k,p_k}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{K}_{k,p_k}^{(\iota)}+\sum_{j\in N_k}a_{kj}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{k,p_k}^{(\iota)}-\sum_{j\in N_k}a_{jk}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\\&-\sum_{j\in N_k}\widehat{a}_{kj}\sum_{\iota=1}^{r}\mathfrak{P}_{k,p_k}^{(\iota)}+\sum_{j\in N_k}\widehat{a}_{jk}\sum_{\iota=1}^{r}\mathfrak{P}_{j,p_j}^{(\iota)}\prec 0,\end{aligned}\tag{32g}$$

$$(\mu_3-\alpha_{31})\underline{\gamma}_1-M\delta_{32}\leqslant 0,\tag{32h}$$

$$\begin{aligned}&A_{f,p_f}^{\top}v_1^{(fp_f)}+A_{f,p_f}^{\top}v_2^{(fp_f)}+C_{f,p_f}^{\top}\sum_{\iota=1}^{n}\mathfrak{M}_{f,p_f}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{G}_{f,p_f}^{(\iota)}+\alpha_{31}v_1^{(fp_f)}+\sum_{\iota=1}^{r}\mathfrak{F}_{f,p_f}^{(\iota)}\\&+\sum_{\iota=1}^{r}\mathfrak{K}_{f,p_f}^{(\iota)}+\sum_{j\in N_f}a_{fj}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{f,p_f}^{(\iota)}-\sum_{j\in N_f}a_{jf}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\prec 0,\end{aligned}\tag{32i}$$

$$\begin{aligned}&\sum_{\iota=1}^{r}\mathfrak{K}_{k,p_k}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{d}_{k,p_k}^{(\iota)}+\sum_{j\in N_k}a_{ij}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{k,p_k}^{(\iota)}-\sum_{j\in N_k}a_{jk}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\\&+\sum_{\iota=1}^{r}\mathfrak{G}_{k,p_k}^{(\iota)}+\alpha_{31}v_2^{(kp_k)}+\sum_{\iota=1}^{r}\mathfrak{F}_{k,p_k}^{(\iota)}\preceq 0,\end{aligned}\tag{32j}$$

$$\begin{aligned}&\sum_{\iota=1}^{r}\mathfrak{K}_{f,p_f}^{(\iota)}+\sum_{j\in N_f}a_{fj}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{f,p_f}^{(\iota)}-\sum_{j\in N_f}a_{jf}^{(\zeta(t))}\sum_{\iota=1}^{r}\mathfrak{N}_{j,p_j}^{(\iota)}\\&+\sum_{\iota=1}^{r}\mathfrak{G}_{f,p_f}^{(\iota)}+\alpha_{31}v_2^{(fp_f)}+\sum_{\iota=1}^{r}\mathfrak{F}_{f,p_f}^{(\iota)}\preceq 0,\end{aligned}\tag{32k}$$

$$\mathit{\Gamma}_{i,p_i}^{\top}\sum_{\iota=1}^{n}\mathfrak{I}_{i,p_i}^{(\iota)}+\sum_{\iota=1}^{s}\mathfrak{H}_{i,p_i}^{(\iota)}+\mathit{\Gamma}_{i,p_i}^{\top}\sum_{\iota=1}^{r}\mathfrak{Z}_{i,p_i}^{(\iota)}+\alpha_{32}v_3^{(ip_i)}\prec 0,\tag{32l}$$

$$(\mu_3-\alpha_{32})\underline{\gamma}_{33}-M\delta_{33}\leqslant 0,\tag{32m}$$

$$\sum_{\iota=1}^{r}\mathfrak{d}_{k,p_k}+S^{\top}v_4^{(kp_k)}+\sum_{j\in N_k}\widehat{a}_{kj}\sum_{\iota=1}^{r}\mathfrak{P}^{(\iota)}-\sum_{j\in N_k}\widehat{a}_{jk}\sum_{\iota=1}^{r}\mathfrak{P}^{(\iota)}+\mu_3v_4^{(kp_k)}\preceq 0,\tag{32n}$$

$$\begin{aligned}&X^{*\top}(A_{k,p_k}^{\top}v_1^{(kp_k)}+\sum_{\iota=1}^{n}\mathfrak{R}_{k,p_k}^{(\iota)}+A_{k,p_k}^{\top}v_2^{(kp_k)}+A_{k,p_k}^{\top}v_4^{(kp_k)}-S^{\top}v_4^{(kp_k)}+\varrho\sum_{\iota=1}^{r}\mathfrak{Q}_{k,p_k}^{(\iota)}\\&+\sum_{\iota=1}^{r}\mathfrak{G}_{k,p_k}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{F}_{k,p_k}^{(\iota)}+C_{k,p_k}^{\top}\sum_{\iota=1}^{n}\mathfrak{T}_{k,p_k}^{(\iota)}+C_{k,p_k}^{\top}\sum_{\iota=1}^{n}\mathfrak{M}_{k,p_k}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{K}_{k,p_k}^{(\iota)})-\varpi\prec 0,\end{aligned}\tag{32o}$$

$$\overline{X}^{*\top}(A_{f,p_f}^{\top}v_1^{(fp_f)}-A_{f,p_f}^{\top}v_2^{(fp_f)}+\sum_{\iota=1}^{r}\mathfrak{G}_{f,p_f}^{(\iota)}+\sum_{\iota=1}^{r}\mathfrak{F}_{f,p_f}^{(\iota)}+C_{f,p_f}^{\top}\sum_{\iota=1}^{n}\mathfrak{M}_{f,p_f}^{(\iota)})-\varpi\prec 0,\tag{32p}$$

$$v_4^{(ip_i)}\prec v_1^{(ip_i)},\tag{32q}$$

$$\begin{aligned}&\underline{\gamma}_{31}\leqslant\mathbf{1}_r^{(\iota)\top}B_{i,p_i}^{\top}v_1^{(ip_i)}\leqslant\overline{\gamma}_{31},\underline{\gamma}_{34}\leqslant\mathbf{1}_r^{(\iota)\top}W_1^{\top}v_1^{(ip_i)}\leqslant\overline{\gamma}_{34},\underline{\gamma}_{32}\leqslant\mathbf{1}_n^{(\iota)\top}v_2^{(ip_i)}\leqslant\overline{\gamma}_{22},\\&\underline{\gamma}_{33}\leqslant\mathbf{1}_s^{(\iota)\top}v_3^{(ip_i)}\leqslant\overline{\gamma}_{33},\underline{\gamma}_{35}\leqslant\mathbf{1}_n^{(\iota)\top}v_1^{(ip_i)}\leqslant\overline{\gamma}_{35},\end{aligned}\tag{32r}$$

$$v^{(ip_i)}\preceq\lambda_i v^{(iq)}\tag{32s}$$

hold for $i\in[1,N]$, $k\in[M+1,N]$, $f\in[M+1,N]$, $\forall(p,q)\in\mathcal{S},\ p\neq q$, and $i=1,2,\ldots,n$, then system (1) is positive and reaches practical consensus with ADT (13) under double observers (5) and (6) and control protocol (25) with

$K_{1,i,p_i} = K^{+}_{1,i,p_i} + K^{-}_{1,i,p_i}$, $K_{3,i,p_i} = K^{+}_{3,i,p_i} + K^{-}_{3,i,p_i}$, $P_2 = P_2^{+} + P_2^{-}$, $T_{i,p_i} = T^{+}_{i,p_i} + T^{-}_{i,p_i}$, $P_{3,i,p_i} = P^{+}_{3,i,p_i} + P^{-}_{3,i,p_i}$, and $M_{i,p_i} = M^{+}_{i,p_i} + M^{-}_{i,p_i}$ satisfying

$$\begin{gathered}
K^{+}_{1,i,p} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{K}^{+(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}}, K^{-}_{1,i,p} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{K}^{-(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}}, K_{2,i,p} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Z}^{(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}},\\
K^{+}_{3,i,p} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Y}^{-(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}}, K^{-}_{3,i,p} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Y}^{-(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}}, K_{4,i,p} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{N}^{(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}},\\
d_{i,p_i} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{d}^{(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}}, P_1 = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{P}^{(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} W_1^{\top} v_4^{(ip_i)}}, P_2^{+} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Q}^{+(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} W_1^{\top} v_4^{(ip_i)}},\\
P_2^{-} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{Q}^{-(\iota)\top}}{\mathbf{1}_r^{(\iota)\top} W_1^{\top} v_4^{(ip_i)}}, P^{+}_{3,i,p_i} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{R}^{+(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}}, P^{-}_{3,i,p_i} = \sum_{\iota=1}^{r} \frac{\mathbf{1}_r^{(\iota)} \mathfrak{R}^{-(\iota)\top}_{i,p_i}}{\mathbf{1}_r^{(\iota)\top} B^{\top}_{i,p_i} v_1^{(ip_i)}},\\
G_{i,p_i} = \sum_{\iota=1}^{n} \frac{\mathbf{1}_n^{(\iota)} \mathfrak{G}^{(\iota)\top}_{i,p_i} - \delta_{32} I_n}{\mathbf{1}_n^{(\iota)\top} v_2^{(ip_i)}}, H_{i,p_i} = \sum_{\iota=1}^{s} \frac{\mathbf{1}_s^{(\iota)} \mathfrak{H}^{(\iota)\top}_{i,p_i} - \delta_{33} I_s}{\mathbf{1}_s^{(\iota)\top} v_3^{(ip_i)}}, M^{+}_{i,p_i} = \sum_{\iota=1}^{n} \frac{\mathbf{1}_n^{(\iota)} \mathfrak{M}^{+(\iota)\top}_{i,p_i}}{\mathbf{1}_n^{(\iota)\top} v_2^{(ip_i)}},\\
M^{-}_{i,p_i} = \sum_{\iota=1}^{n} \frac{\mathbf{1}_n^{(\iota)} \mathfrak{M}^{-(\iota)\top}_{i,p_i}}{\mathbf{1}_n^{(\iota)\top} v_2^{(ip_i)}}, F_{i,p_i} = \sum_{\iota=1}^{s} \frac{\mathbf{1}_s^{(\iota)} \mathfrak{F}^{(\iota)\top}_{i,p_i}}{\mathbf{1}_s^{(\iota)\top} v_3^{(ip_i)}}, T^{+}_{i,p_i} = \sum_{\iota=1}^{s} \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}^{+(\iota)\top}_{i,p_i}}{\mathbf{1}_s^{(\iota)\top} v_3^{(ip_i)}},\\
T^{-}_{i,p_i} = \sum_{\iota=1}^{s} \frac{\mathbf{1}_s^{(\iota)} \mathfrak{T}^{-(\iota)\top}_{i,p_i}}{\mathbf{1}_s^{(\iota)\top} v_3^{(ip_i)}}, J_{i,p_i} = \sum_{\iota=1}^{n} \frac{\mathbf{1}_n^{(\iota)} \mathfrak{J}^{(\iota)\top}_{i,p_i}}{\mathbf{1}_n^{(\iota)\top} v_2^{(ip_i)}}.
\end{gathered} \tag{33}$$

Proof. From (11), we can deduce that $B_{i,p_i} K_{2ip} \mathit{\Gamma}_{i,p_i} + B_{i,p_i} K_{3ip} D_{i,p_i} \mathit{\Gamma}_{i,p_i} + E_{i,p_i} \mathit{\Gamma}_{i,p_i} = 0$, $J_{i,p_i} \mathit{\Gamma}_{i,p_i} + M_{i,p_i} D_{i,p_i} \times \mathit{\Gamma}_{i,p_i} - E_{i,p_i} \mathit{\Gamma}_{i,p_i} = 0$, and $T_{i,p_i} D_{i,p_i} = 0$. The system is transformed into two parts as follows. (i) When $k \in [1, M]$,

$$\begin{aligned}
\begin{pmatrix} \dot{\chi}_k(t) \\ \dot{e}_k(t) \\ \dot{\theta}_k(t) \\ \dot{\varepsilon}_k(t) \end{pmatrix} = {} & \begin{pmatrix}
\begin{pmatrix} A_{k,p_k} + B_{k,p_k} K_{1,k,p_k} \\ + B_{k,p_k} K_{3,k,p_k} C_{k,p_k} \end{pmatrix} & \begin{pmatrix} B_{k,p_k} K_{1,k,p_k} \\ + B_{k,p_k} d_{k,p_k} \end{pmatrix} & B_{k,p_k} K_{2,k,p_k} \mathit{\Gamma}_{k,p_k} & B_{k,p_k} d_{k,p_k} \\
G_{k,p_k} + M_{k,p_k} C_{k,p_k} - A_{k,p_k} & G_{k,p_k} & J_{k,p_k} \mathit{\Gamma}_{k,p_k} & 0 \\
T_{k,p_k} C_{k,p_k} + F_{k,p_k} & F_{k,p_k} & H_{k,p_k} & 0 \\
\begin{pmatrix} A_{k,p_k} - S + B_{k,p_k} K_{1,k,p_k} \\ + B_{k,p_k} K_{3,k,p_k} C_{k,p_k} \end{pmatrix} & \begin{pmatrix} B_{k,p_k} K_{1,k,p_k} \\ + B_{k,p_k} d_{k,p_k} \end{pmatrix} & B_{k,p_k} K_{2,k,p_k} \mathit{\Gamma}_{i,p_i} & S + B_{k,p_k} d_{k,p_k}
\end{pmatrix} \\
& \times \begin{pmatrix} \chi_k(t) \\ e_k(t) \\ \theta_k(t) \\ \varepsilon_k(t) \end{pmatrix} + \begin{pmatrix}
(A_{k,p_k} + B_{k,p_k} K_{1,k,p_k} + B_{k,p_k} K_{3,k,p_k} C_{k,p_k} + B_{k,p_k} P_{3,k,p_k}) x^* \\
(G_{k,p_k} + M_{k,p_k} C_{k,p_k} - A_{k,p_k}) x^* \\
(T_{k,p_k} C_{k,p_k} + F_{k,p_k}) x^* \\
\begin{pmatrix} (A_{k,p_k} - S + B_{k,p_k} K_{1,k,p_k} + B_{k,p_k} K_{3,k,p_k} C_{k,p_k} \\ + B_{k,p_k} P_{3,k,p_k}) x^* + W_1 P_2 \eta_o^* \end{pmatrix}
\end{pmatrix} \\
& + \begin{pmatrix}
B_{k,p_k} K_{4,k,p_k} \sum_{j=1}^{M} a_{ij}^{(\zeta(t))} (x_k - x_j) + B_{k,p_k} K_{4,k,p_k} \sum_{j=1}^{M} a_{kj}^{(\zeta(t))} (e_k - e_j) \\
0 \\
0 \\
\begin{pmatrix} B_{k,p_k} K_{4kp} \sum_{j=1}^{M} a_{ij}^{(\zeta(t))} (x_k - x_j) - W_1 P_1 \sum_{q=1}^{M} \widehat{a}_{qk} (x_q - x_k) \\ + B_{k,p_k} K_{4kp} \sum_{j=1}^{M} a_{ij}^{(\zeta(t))} (e_k - e_j) - W_1 P_1 \sum_{q=1}^{M} \widehat{a}_{qk} (\varepsilon_k - \varepsilon_q) \end{pmatrix}
\end{pmatrix}.
\end{aligned} \tag{34}$$

Further, (ii) when $f \in [M+1, N]$,

$$\begin{aligned}
\begin{pmatrix} \dot{\chi}_f(t) \\ \dot{e}_f(t) \\ \dot{\theta}_f(t) \end{pmatrix} = {} & \begin{pmatrix}
\begin{pmatrix} A_{f,p_f} + B_{f,p_f} K_{1,f,p_f} \\ + B_{f,p_f} K_{3,f,p_f} C_{f,p_f} \end{pmatrix} & B_{f,p_f} K_{1,f,p_f} & B_{f,p_f} K_{2,f,p_f} \mathit{\Gamma}_{f,p_f} \\
G_{f,p_f} + M_{f,p_f} C_{f,p_f} - A_{f,p_f} & G_{f,p_f} & J_{f,p_f} \mathit{\Gamma}_{f,p_f} \\
T_{f,p_f} C_{f,p_f} + F_{f,p_f} & F_{f,p_f} & H_{f,p_f}
\end{pmatrix} \begin{pmatrix} \chi_f(t) \\ e_f(t) \\ \theta_f(t) \end{pmatrix} \\
& + \begin{pmatrix}
(A_{f,p_f} + B_{f,p_f} K_{1,f,p_f} + B_{f,p_f} K_{3,f,p_f} C_{f,p_f} + B_{f,p_f} P_{3,f,p_f}) x^* \\
(G_{f,p_f} + M_{f,p_f} C_{f,p_f} - A_{f,p_f}) x^* \\
(T_{f,p_f} C_{f,p_f} + F_{f,p_f}) x^*
\end{pmatrix}
\end{aligned}$$

$$
+\begin{pmatrix}\begin{pmatrix}B_{f,p_f}K_{4,f,p_f}\sum_{j=M+1}^{N}a_{fj}^{(\zeta(t))}(x_i-x_j)\\ +B_{f,p_f}K_{4fp}\sum_{j=M+1}^{N}a_{fj}^{(\zeta)(t)}(e_f-e_j)\end{pmatrix}\\ 0\\ 0\end{pmatrix}. \tag{35}
$$

Let $\chi(t)=(\chi_1(t)^\top,\chi_2(t)^\top,\ldots,\chi_N(t)^\top)^\top$, $e(t)=(e_1(t)^\top,e_2(t)^\top,\ldots,e_N(t)^\top)^\top$, $\theta(t)=(\theta_1(t)^\top,\theta_2(t)^\top,\ldots,\theta_N(t)^\top)^\top$, $\varepsilon(t)=(\varepsilon_1(t)^\top,\varepsilon_2(t)^\top,\ldots,\varepsilon_M(t)^\top)^\top$. Therefore, we can deduce that

$$
\begin{aligned}
\dot{\chi}(t)&=(\widetilde{A}_p+\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})\chi(t)+(\widetilde{B}_p\widetilde{K}_{1,p}+\overline{\widetilde{B}_p\widetilde{d}_p}+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})e(t)\\
&\quad+(\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p)\theta(t)+(\widetilde{\mathbf{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p)\varepsilon(t)+(\widetilde{A}_p+\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{B}_p\widetilde{P}_{3,p})X^*,\\
\dot{e}(t)&=(\widetilde{G}_p+\widetilde{M}_p\widetilde{C}_p-\widetilde{A}_p)\chi(t)+(\widetilde{G}_p)e(t)+(\widetilde{J}_p\widetilde{\varGamma}_p)\theta(t)+(\widetilde{G}_p+\widetilde{M}_p\widetilde{C}_p-\widetilde{A}_p)X^*,\\
\dot{\theta}(t)&=(\widetilde{T}_p\widetilde{C}_p+\widetilde{F}_p)\chi(t)+\widetilde{F}_pe(t)+\widetilde{H}_p\theta(t)+(\widetilde{T}_p\widetilde{C}_p+\widetilde{F}_p)X^*,\\
\dot{\varepsilon}(t)&=(\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p-\widetilde{\mathbf{\Omega}}_MI_m\otimes S+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p-\widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta\otimes W_1P_1)\\
&\quad\times\chi(t)+(\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{d}_p+\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})e(t)+(\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\varGamma_p)\theta(t)+(I_m\otimes S\\
&\quad+\widetilde{B}_pd_p+\mathcal{L}_\eta\otimes W_1P_1)\varepsilon(t)+(\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p-\widetilde{\mathbf{\Omega}}_MI_m\otimes S-\widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta\otimes W_1P_1\\
&\quad+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{P}_{3,p})\overline{X}^*+I_M\otimes(W_1P_2)\overline{\eta}^*,
\end{aligned}\tag{36}
$$

where $X^*=(\underbrace{x^*,x^*,\ldots,x^*}_{N})^\top$, $\overline{X}^*=(\underbrace{x^*,x^*,\ldots,x^*}_{M})^\top$, and $\overline{\eta}^*=(\underbrace{\eta_o^*,\eta_o^*,\ldots,\eta_o^*}_{M})^\top$. The closed-loop system is rewritten as follows:

$$
\begin{aligned}
\begin{pmatrix}\dot{\chi}(t)\\ \dot{e}(t)\\ \dot{\theta}(t)\\ \dot{\varepsilon}(t)\end{pmatrix}&=\begin{pmatrix}\mathcal{A}_3 & \begin{pmatrix}\widetilde{B}_p\widetilde{K}_{1,p}+\overline{\widetilde{B}_p\widetilde{d}_p}\\ +\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}\end{pmatrix} & \widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p & \widetilde{\mathbf{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p\\ \widetilde{G}_p-\widetilde{A}_p+\widetilde{M}_p\widetilde{C}_p & \widetilde{G}_p & \widetilde{J}_p\widetilde{\varGamma}_p & 0\\ \widetilde{T}_p\widetilde{C}_p+\widetilde{F}_p & \widetilde{F}_p & \widetilde{H}_p & 0\\ \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{A}}_2 & \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p & \begin{pmatrix}I_m\otimes S+\widetilde{B}_p\widetilde{d}_p\\ +\mathcal{L}_\eta\otimes W_1P_1\end{pmatrix}\end{pmatrix}\begin{pmatrix}\chi(t)\\ e(t)\\ \theta(t)\\ \varepsilon(t)\end{pmatrix}\\
&\quad+\begin{pmatrix}\widetilde{A}_p+\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{B}_p\widetilde{P}_{3,p}\\ \widetilde{G}_p+\widetilde{M}_p\widetilde{C}_p-\widetilde{A}_p\\ \widetilde{T}_p\widetilde{C}_p+\widetilde{F}_p\\ \begin{pmatrix}\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p-\widetilde{\mathbf{\Omega}}_MI_m\otimes S+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p}\\ +\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{P}_{3,p}+\widetilde{\mathbf{\Omega}}_M(\varrho I_M\otimes(W_1P_2))\end{pmatrix}\end{pmatrix}X^*,
\end{aligned}\tag{37}
$$

where $\mathcal{A}_3=\widetilde{A}_p+\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}$, $\widetilde{\mathcal{A}}_1=\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p-\widetilde{\mathbf{\Omega}}_M(I_m\otimes S)+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p-\widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta\otimes W_1P_1$, $\widetilde{\mathcal{A}}_2=\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{d}_p+\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}$, and the matrices $\widetilde{\mathbf{\Omega}}_M$, $\widetilde{\mathbf{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p$, and $\overline{\widetilde{B}_p\widetilde{d}_p}$ are given as follows:

$$
\widetilde{\mathbf{\Omega}}_M=\begin{pmatrix}1&\cdots&0&\cdots&0\\ 0&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ 0&\cdots&1&\cdots&0\end{pmatrix},\widetilde{\mathbf{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p=\begin{pmatrix}B_1d_1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&B_Md_M\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{pmatrix},\overline{\widetilde{B}_p\widetilde{d}_p}=\begin{pmatrix}B_1d_1&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ 0&\cdots&B_Md_M&\cdots&0\\ 0&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\ddots&\vdots\\ 0&\cdots&0&\cdots&0\end{pmatrix}.
$$

Let $X(t)=(\chi(t)^\top,e(t)^\top,\theta(t)^\top,\varepsilon(t)^\top)^\top$. System (37) is transformed into $\dot{X}(t)=\mathbb{A}X(t)+\mathbb{B}X^*$. By (12a), (12b), (12c), (12d), and (12f), it is easy to see that $\widetilde{G}_p$ and $\widetilde{H}_p$ are Metzler. Then, $\widetilde{G}_p+\widetilde{M}_p\widetilde{C}_p-\widetilde{A}_p\succeq 0$, $\widetilde{F}_p+\widetilde{T}_p\widetilde{C}_p\succeq 0$ and $\widetilde{A}_p+\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{B}_p\widetilde{K}_{3p}\widetilde{C}_p+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}\succeq 0$. From $J_{i,p_i}\succeq 0$ and $\varGamma_{i,p_i}\succeq 0$, it follows that $\widetilde{J}_p\widetilde{\varGamma}_p\succeq 0$. Further, Eq. (32b) gives $\widetilde{\mathbf{\Omega}}_M^\top \widetilde{B}_p \widetilde{d}_p \succeq 0$. From (32c), we have $\widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{B}_p\widetilde{P}_{3,p} \succeq 0$. Using (32d), we get $\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_m \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{P}_{3,p} + \varrho I_M \otimes (W_1P_2) \succeq 0$. Therefore, we can deduce that

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \\ \dot{\theta}(t) \\ \dot{\varepsilon}(t) \end{pmatrix} \succeq \begin{pmatrix} \mathcal{A}_1 & \begin{pmatrix} \widetilde{B}_p\widetilde{K}_{1,p} + \overline{\widetilde{B}_p\widetilde{d}_p} \\ +\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p} \end{pmatrix} & \widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p & \widetilde{\mathbf{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p \\ \widetilde{G}_p + \widetilde{M}_p\widetilde{C}_p - \widetilde{A}_p & \widetilde{G}_p & \widetilde{J}_p\widetilde{\varGamma}_p & 0 \\ \widetilde{T}_p\widetilde{C}_p + \widetilde{F}_p & \widetilde{F}_p & \widetilde{H}_p & 0 \\ \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{A}}_2 & \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p & \begin{pmatrix} I_m \otimes S + \widetilde{B}_p\widetilde{d}_p \\ +\mathcal{L}_\eta \otimes W_1P_1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \\ \theta(t) \\ \varepsilon(t) \end{pmatrix}, \tag{38}$$

where $\mathcal{A}_3 = \widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}$, $\widetilde{\mathcal{A}}_1 = \widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_M \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p - \widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta \otimes W_1P_1$, and $\widetilde{\mathcal{A}}_2 = \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{d}_p + \widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}$. By $\mathfrak{Z}_{i,p_i}^{(\iota)} \succeq 0$, $\mathfrak{d}_{i,p_i}^{(\iota)} \succeq 0$, and (12e), it follows that $\widetilde{B}_p\widetilde{K}_{1,p} + \overline{\widetilde{B}_p\widetilde{d}_p} + \widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p} \succeq 0$ and $\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p \succeq 0$. From (32e), we get $\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{4,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_M \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p - \widetilde{\mathbf{\Omega}}_M(\mathcal{L}_\eta \otimes W_1P_1) \succeq 0$. Using (32f), we can deduce that $I_M \otimes S + \widetilde{B}_p\widetilde{d}_p + \mathcal{L}_\eta \otimes W_1P_1 + \delta_{34}I_M \succeq 0$, where $I_M \otimes S + \widetilde{B}_p\widetilde{d}_p + \mathcal{L}_\eta \otimes W_1P_1$ is a Metzler matrix. Thus, the system (36) is positive.

Next, we consider the consensus of the system. Choose a multiple CLF: $V_p(X(t)) = X(t)^\top v^{(p)} = \chi(t)^\top v_1^{(p)} + e(t)^\top v_2^{(p)} + \theta(t)^\top v_3^{(p)} + \varepsilon(t)^\top v_4^{(p)}$, where $v^{(\sigma(t))} = (v_1^{(\sigma(t))\top}, v_2^{(\sigma(t))\top}, v_3^{(\sigma(t))\top}, v_4^{(\sigma(t))})^\top$ with $v_\iota^{(\sigma(t))} = (v_\iota^{(1,\sigma_1(t))\top}, v_\iota^{(2,\sigma_2(t))\top}, \ldots, v_\iota^{(m,\sigma_m(t))\top})^\top$, $\iota = \{1,2,3,4\}$. Then,

$$\begin{aligned} \dot{V}_p(X(t)) = {} & \chi(t)^\top((\widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_1^{(p)} + (\widetilde{G}_p + \widetilde{M}_p\widetilde{C}_p - \widetilde{A}_p)^\top v_2^{(p)} \\ & + (\widetilde{T}_p\widetilde{C}_p + \widetilde{F}_p)^\top v_3^{(p)} + (\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_M \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} \\ & + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p - \widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta \otimes W_1P_1)^\top v_4^{(p)}) + e(t)^\top((\widetilde{B}_p\widetilde{K}_{1,p} + \overline{\widetilde{B}_p\widetilde{d}_p} + \widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_1^{(p)} \\ & + (\widetilde{G}_p)^\top v_2^{(p)} + (\widetilde{F}_p)^\top v_3^{(p)} + (\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{d}_p + \widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_4^{(p)}) \\ & + \theta(t)^\top((\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p)^\top v_1^{(p)} + (\widetilde{J}_p\widetilde{\varGamma}_p)^\top v_2^{(p)} + \widetilde{H}_p^\top v_3^{(p)} + (\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\varGamma}_p)^\top v_4^{(p)}) \\ & + \varepsilon(t)^\top((\widetilde{\mathbf{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p)^\top v_1^{(p)} + (I_M \otimes S + \widetilde{B}_p d_p + \mathcal{L}_\eta \otimes W_1P_1)^\top \overline{v}_4^{(p)}) + X^{*\top}((\widetilde{A}_p \\ & + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{B}_p\widetilde{P}_{3,p})^\top v_1^{(p)} + (\widetilde{G}_p + \widetilde{M}_p\widetilde{C}_p - \widetilde{A}_p)^\top v_2^{(p)} + (\widetilde{T}_p\widetilde{C}_p \\ & + \widetilde{F}_p)^\top v_3^{(p)} + (\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_M \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3p}\widetilde{C}_p + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{P}_{3,p})^\top \\ & + (\widetilde{\mathbf{\Omega}}_M\varrho I_M \otimes (W_1P_2))^\top v_4^{(p)}). \end{aligned} \tag{39}$$

First, we derive that

$$\begin{aligned} & (\widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_1^{(p)} + (\widetilde{G}_p + \widetilde{M}_p\widetilde{C}_p - \widetilde{A}_p)^\top v_2^{(p)} + (\widetilde{T}_p\widetilde{C}_p + \widetilde{F}_p)^\top v_3^{(p)} \\ & \quad +(\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_M \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p - \widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta \otimes W_1P_1)^\top v_4^{(p)} \end{aligned}$$

$$
=\begin{pmatrix}
\begin{pmatrix}
A_{1,p_1}^\top v_1^{(1p_1)}+K_{1,1,p_1}^\top B_{1,p_1}^\top v_1^{(1p_1)}+C_{3,p}^\top K_{3,1,p_1}^\top B_{1,p_1}^\top v_1^{(1p_1)}\\
+\sum_{j\in N_1} a_{1j}^{(\zeta(t))}K_{4,1,p_1}^\top B_{1,p_1}^\top v_1^{(1p_1)}-\sum_{j\in N_1} a_{j1}^{(\zeta(t))}K_{4,j,p_J}^\top B_{j,p_j}^\top v_1^{(jp_j)}+G_{1,p_1}^\top v_2^{(1p_1)}\\
-A_{1,p_1}^\top v_2^{(1p_1)}+C_{1,p_1}^\top M_{1,p_1}^\top v_2^{(1p_1)}+F_{1,p_1}^\top v_3^{(1p_1)}+C_{1,p_1}^\top T_{1,p_1}^\top v_3^{(1p_1)}\\
+A_{1,p_1}^\top v_4^{(1p_1)}+K_{1,1,p_1}^\top B_{1,p_1}^\top v_4^{(1p_1)}+C_{1,p_1}^\top K_{3,1,p_1}^\top B_{1,p_1}^\top v_4^{(1p_1)}\\
+\sum_{j\in N_1} a_{1j}^{(\zeta(t))}K_{4,1,p_1}^\top B_{1,p_1}^\top v_4^{(1p_1)}-\sum_{j\in N_1} a_{j1}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_4^{(jp_j)}\\
-Sv_4^{(1p_1)}-(\sum_{j\in N_1} a_{1j}^{(\zeta(t))}P_1^\top W_1^\top v_4^{(1p_1)}-\sum_{j\in N_1} a_{j1}^{(\zeta(t))}P_1^\top W_1^\top v_4^{(jp_j)})
\end{pmatrix}\\
\vdots\\
\begin{pmatrix}
A_{M,p_M}^\top v_1^{(Mp_M)}+K_{1,M,p_M}^\top B_{M,p_M}^\top v_1^{(Mp_M)}+C_{M,p_M}^\top K_{3,M,p_M}^\top B_{M,p_M}^\top v_1^{(Mp_M)}\\
+\sum_{j\in N_M} a_{Mj}^{(\zeta(t))}K_{4,Mp}^\top B_{M,p_M}^\top v_1^{(Mp_M)}-\sum_{j\in N_M} a_{jM}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\\
+G_{M,p_M}^\top v_2^{(Mp_M)}-A_{M,p_M}^\top v_2^{(Mp_M)}+C_{M,p_M}^\top M_{M,p_M}^\top v_2^{(Mp_M)}+F_{M,p_M}^\top v_3^{(Mp_M)}\\
+C_{M,p_M}^\top T_{M,p_M}^\top v_3^{(Mp_M)}+A_{M,p_M}^\top v_4^{(Mp_M)}+K_{1,M,p_M}^\top B_{M,p_M}^\top v_4^{(Mp_M)}\\
+C_{M,p_M}^\top K_{3,M,p_M}^\top B_{M,p_M}^\top v_4^{(Mp_M)}+\sum_{j\in N_M} a_{Mj}^{(\zeta(t))}K_{4,M,p_M}^\top B_{M,p_M}^\top v_4^{(Mp_M)}\\
-\sum_{j\in N_M} a_{j1}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_4^{(jp_j)}-Sv_4^{(Mp_M)}\\
-(\sum_{j\in N_M} a_{Mj}^{(\zeta(t))}P_1^\top W_1^\top v_4^{(Mp_M)}-\sum_{j\in N_M} a_{jM}^{(\zeta(t))}P_1^\top W_1^\top v_4^{(jp_j)})
\end{pmatrix}\\
\vdots\\
\begin{pmatrix}
A_{N,p_N}^\top v_1^{(Np_N)}+K_{1,N,p_N}^\top B_{N,p_N}^\top v_1^{(Np_N)}+C_{N,p_N}^\top K_{3,N,p_N}^\top B_{N,p_N}^\top v_1^{(Np_N)}\\
+\sum_{j\in N_N} a_{N}^{(\zeta(t))}K_{4,N,p_N}^\top B_{N,p_N}^\top v_1^{(Np_N)}-\sum_{j\in N_N} a_{jN}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\\
+G_{N,p_N}^\top v_2^{(Np_N)}-A_{N,p_N}^\top v_2^{(Np_N)}+C_{N,p_N}^\top M_{N,p_N}^\top v_2^{(Np_N)}\\
+F_{N,p_N}^\top v_3^{(Np_N)}+C_{N,p_N}^\top T_{N,p_N}^\top v_3^{(Np_N)}
\end{pmatrix}
\end{pmatrix}. \tag{40}
$$

Using (32g), (32h), and (32i), we get

$$
\begin{aligned}
&A_{i,p_i}^\top v_1^{(ip_i)}+K_{1,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+C_{3p}^\top K_{3,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+\sum_{j\in N_i} a_{ij}^{(\zeta(t))}K_{4,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}\\
&\quad-\sum_{j\in N_i} a_{ji}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}+G_{i,p_i}^\top v_2^{(ip_i)}-A_{i,p_i}^\top v_2^{(ip_i)}+C_{i,p_i}^\top M_{i,p_i}^\top v_2^{(ip_i)}+F_{i,p_i}^\top v_3^{(ip_i)}+C_{i,p_i}^\top\\
&\quad\times T_{i,p_i}^\top v_3^{(ip_i)}+A_{i,p_i}^\top v_4^{(ip_i)}+K_{1,i,p_i}^\top B_{i,p_i}^\top v_4^{(ip_i)}+C_{i,p_i}^\top K_{3,i,p_i}^\top B_{i,p_i}^\top v_4^{(ip_i)}+\sum_{j\in N_i} a_{ij}^{(\zeta(t))}K_{4,i,p_i}^\top B_{i,p_i}^\top v_4^{(ip_i)}\\
&\quad-\sum_{j\in N_i} a_{ji}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_4^{(jp_j)}-Sv_4^{(ip_i)}-(\sum_{j\in N_i} a_{ij}^{(\zeta(t))}P_1^\top W_1^\top v_4^{(ip_i)}-\sum_{j\in N_i} a_{ji}^{(\zeta(t))}P_1^\top W_1^\top v_4^{(jp_j)})\\
&\quad\preceq -\mu_3 v_1^{(ip_i)},
\end{aligned}
$$

where $i\in[1,M]$ and

$$
\begin{aligned}
&A_{i,p_i}^\top v_1^{(ip_i)}+K_{1,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+C_{i,p_i}^\top K_{3,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+\sum_{j\in N_m} a_{mj}^{(\zeta(t))}K_{4,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}\\
&\quad-\sum_{j\in N_1} a_{jm}^{(\zeta(t))}K_{4,jp}^\top B_{j,p_j}^\top v_1^{(jp_j)}+G_{i,p_i}^\top v_2^{(ip_i)}-A_{i,p_i}^\top v_2^{(ip_i)}+C_{i,p_i}^\top M_{i,p_i}^\top v_2^{(ip_i)}+F_{i,p_i}^\top v_3^{(ip_i)}\\
&\quad+C_{i,p_i}^\top T_{i,p_i}^\top v_3^{(ip_i)}\preceq -\mu_3 v_1^{(ip_i)},
\end{aligned}
$$

where $i\in[M+1,N]$. Thus, $(\widetilde{A}_p+\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_1^{(p)}+(\widetilde{G}_p+\widetilde{M}_p\widetilde{C}_p-\widetilde{A}_p)^\top v_2^{(p)}+(\widetilde{T}_p\widetilde{C}_p+\widetilde{F}_p)^\top v_3^{(p)}+(\widetilde{\mathbf{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p-\widetilde{\mathbf{\Omega}}_M I_M\otimes S+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p-\widetilde{\mathbf{\Omega}}_M\mathcal{L}_\eta\otimes W_1P_1)^\top v_4^{(p)}+\mu_3 v_1^{(p)}\preceq 0$.

Moreover, it follows that

$$\begin{aligned}&(\widetilde{B}_p\widetilde{K}_{1,p}+\overline{\widetilde{B}_p\widetilde{d}_p}+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_1^{(p)}+\widetilde{G}_p^\top v_2^{(p)}+\widetilde{F}_p^\top v_3^{(p)}+(\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\boldsymbol{\Omega}}_M\widetilde{B}_p\widetilde{d}_p+\widetilde{\boldsymbol{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_4^{(p)}\\ &=\begin{pmatrix}\begin{pmatrix}d_{1,1p}^\top B_{1,p_1}^\top v_1^{(1p_1)}+\sum_{j\in N_1}a_{1j}^{\sigma(t)}K_{4,1,p_1}^\top B_{1,p_1}^\top v_1^{(1p_1)}-\sum_{j\in N_1}a_{j1}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\\ K_{1,1p}^\top B_{1,p_1}^\top v_1^{(1p_1)}+G_{1,p_1}^\top v_2^{(1p_1)}+F_{1,p_1}^\top v_3^{(1p_1)}+K_{1,1p}^\top B_{1,p_1}^\top v_4^{(1p_1)}+d_{1,1p}^\top B_{1,p_1}^\top v_4^{(1p_1)}\\ +\sum_{j\in N_1}a_{1j}^{(\zeta(t))}K_{4,1,p_1}^\top B_{1,p_1}^\top v_4^{(1p_1)}-\sum_{j\in N_1}a_{j1}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_4^{(jp_j)}\end{pmatrix}\\ \vdots\\ \begin{pmatrix}d_{1,Mp}^\top B_{M,p_M}^\top v_1^{(Mp_M)}+\sum_{j\in N_M}a_{Mj}^{(\zeta(t))}K_{4,M,p_M}^\top B_{M,p_M}^\top v_1^{(Mp_M)}-\sum_{j\in N_M}a_{jM}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\\ K_{1,Mp}^\top B_{M,p_M}^\top v_1^{(Mp_M)}+G_{M,p_M}^\top v_2^{(Mp_M)}+F_{M,p_M}^\top v_3^{(Mp_M)}+K_{1,Mp}^\top B_{M,p_M}^\top v_4^{(Mp_M)}+d_{1,Mp}^\top B_{M,p_M}^\top\\ \times v_4^{(Mp_M)}+\sum_{j\in N_M}a_{Mj}^{(\zeta(t))}K_{4,M,p_M}^\top B_{M,p_M}^\top v_4^{(Mp_M)}-\sum_{j\in N_M}a_{jM}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_4^{(jp_j)}\end{pmatrix}\\ \vdots\\ \begin{pmatrix}K_{1,N,p_N}^\top B_{N,p_N}^\top v_1^{(Np_N)}++G_{N,p_N}^\top v_2^{(Np_N)}+F_{N,p_N}^\top v_3^{(Np_N)}+K_{1,N,p_N}^\top B_{N,p_N}^\top v_4^{(Np_N)}\\ +\sum_{j\in N_N}a_{Nj}^{(\zeta(t))}K_{4,N,p_N}^\top B_{N,p_N}^\top v_1^{(Np_N)}-\sum_{j\in N_N}a_{jN}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\end{pmatrix}\end{pmatrix}.\end{aligned}\tag{41}$$

From (32j) and (32k), we have

$$\begin{aligned}&K_{1,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+d_{1,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+\sum_{j\in N_i}a_{ij}^{(\zeta(t))}K_{4,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}-\sum_{j\in N_i}a_{ji}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\\ &\quad+G_{i,p_i}^\top v_2^{(ip_i)}+F_{i,p_i}^\top v_3^{(ip_i)}+K_{1,i,p_i}^\top B_{i,p_i}^\top v_4^{(p)}+d_{1,i,p_i}^\top B_{i,p_i}^\top v_4^{(p)}+\sum_{j\in N_i}a_{ij}^{(\zeta(t))}K_{4,i,p_i}^\top B_{i,p_i}^\top v_4^{(ip_i)}\\ &\quad-\sum_{j\in N_i}a_{ji}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_4^{(jp_j)}\preceq-\mu_3 v_2^{(ip_i)},\end{aligned}$$

where $i\in[1,M]$ and

$$\begin{aligned}&K_{1,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+\sum_{j\in N_i}a_{ij}^{(\zeta(t))}K_{4,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}-\sum_{j\in i_j}a_{ji}^{(\zeta(t))}K_{4,j,p_j}^\top B_{j,p_j}^\top v_1^{(jp_j)}\\ &\quad+G_{i,p_i}^\top v_2^{(ip_i)}+F_{i,p_i}^\top v_3^{(ip_i)}+K_{1,i,p_i}^\top B_{i,p_i}^\top v_4^{(ip_i)}\preceq-\mu_3 v_2^{(ip_i)},\end{aligned}$$

where $i\in[M+1,N]$. Thus, $(\widetilde{B}_p\widetilde{K}_{1,p}+\overline{\widetilde{B}_p\widetilde{d}_p}+\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_1^{(p)}+\widetilde{G}_p^\top v_2^{(p)}+\widetilde{F}_p^\top v_3^{(p)}+(\widetilde{B}_p\widetilde{K}_{1,p}+\widetilde{\boldsymbol{\Omega}}_M\widetilde{B}_p\widetilde{d}_p+\widetilde{\boldsymbol{\Omega}}_M\widetilde{\mathcal{L}}_p\widetilde{B}_p\widetilde{K}_{4,p})^\top v_4^{(p)}\preceq-\mu_3 v_2^{(p)}$. Then,

$$\begin{aligned}&(\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\Gamma}_p)^\top v_1^{(p)}+(\widetilde{J}_p\widetilde{\Gamma}_p)^\top v_2^{(p)}+\widetilde{H}_p^\top v_3^{(p)}+(\widetilde{\boldsymbol{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\Gamma}_p)^\top\overline{v}_4^{(p)}\\ &=\begin{pmatrix}\Gamma_{1,p_1}^\top K_{2,1,p_1}^\top B_{1,p_1}^\top v_1^{(1p_1)}+\Gamma_{1,p_1}^\top J_{1,p_1}^\top v_2^{(1p_1)}+H_{1,p_1}^\top v_3^{(1p_1)}+\Gamma_{1,p_1}^\top K_{2,1p}^\top B_{1,p_1}^\top v_4^{(1p_1)}\\ \vdots\\ \Gamma_{M,p_M}^\top K_{2,M,p_M}^\top B_{M,p_M}^\top v_1^{(Mp_M)}+\Gamma_{2p}^\top J_{2p}^\top v_2^{(Mp_M)}+H_{M,p_M}^\top v_3^{(Mp_M)}+\Gamma_{M,p_M}^\top K_{2,Mp}^\top B_{M,p_M}^\top v_4^{(Mp_M)}\\ \vdots\\ \Gamma_{N,p_N}^\top K_{2,N,p_N}^\top B_{N,p_N}^\top v_1^{(Np_N)}+\Gamma_{N,p_N}^\top J_{N,p_N}^\top v_2^{(Np_N)}+H_{N,p_N}^\top v_3^{(Np_N)}\end{pmatrix}.\end{aligned}\tag{42}$$

From (32l) and (32m), we get $\Gamma_{i,p_i}^\top K_{2,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+\Gamma_{i,p_i}^\top J_{i,p_i}^\top v_2^{(ip_i)}+H_{i,p_i}^\top v_3^{(ip_i)}+\Gamma_{i,p_i}^\top K_{2,i,p_i}^\top B_{i,p_i}^\top v_4^{(ip_i)}\preceq-\mu_3 v_3^{(ip_i)}$ for $i\in[1,M]$ and $\Gamma_{i,p_i}^\top K_{2,i,p_i}^\top B_{i,p_i}^\top v_1^{(ip_i)}+\Gamma_{i,p_i}^\top J_{i,p_i}^\top v_2^{(ip_i)}+H_{i,p_i}^\top v_3^{(ip_i)}\preceq-\mu_3 v_3^{(ip_i)}$ for $i\in[M+1,N]$. Thus, $(\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\Gamma}_p)^\top v_1^{(p)}+(\widetilde{J}_p\widetilde{\Gamma}_p)^\top v_2^{(p)}+\widetilde{H}_p^\top v_3^{(p)}+(\widetilde{\boldsymbol{\Omega}}_M\widetilde{B}_p\widetilde{K}_{2,p}\widetilde{\Gamma}_p)^\top v_4^{(p)}\preceq-\mu_3 v_3^{(p)}$. Additionally, we have

$$\begin{aligned}&(\widetilde{\boldsymbol{\Omega}}_M^\top\widetilde{B}_p\widetilde{d}_p)^\top v_1^{(p)}+(I_m\otimes S+\widetilde{B}_p d_p+\mathcal{L}_\eta\otimes W_1P_1)^\top v_4^{(p)}\\ &=\begin{pmatrix}\begin{pmatrix}d_{1,p_1}^\top B_{1,p_1}^\top v_1^{(1p_1)}+Sv_4^{(1p_1)}+d_{1,1p}^\top B_{1,p_1}^\top v_4^{(1p_1)}\\ +\sum_{j\in N_1}\widehat{a}_{1j}P_1^\top W_1^\top v_4^{(1p_1)}-\sum_{j\in N_1}\widehat{a}_{j1}P_1^\top W_1^\top v_4^{(jp_j)}\end{pmatrix}\\ \vdots\\ \begin{pmatrix}d_{M,p_M}^\top B_{M,p_M}^\top v_1^{(Mp_M)}+Sv_4^{(Mp_M)}+d_{1,Mp}^\top B_{M,p_M}^\top v_4^{(Mp_M)}\\ +\sum_{j\in N_M}\widehat{a}_{Mj}P_1^\top W_1^\top v_4^{(Mp_M)}-\sum_{j\in N_M}\widehat{a}_{jM}P_1^\top W_1^\top v_4^{(jp_j)}\end{pmatrix}\end{pmatrix}.\end{aligned}\tag{43}$$

By (32n), we get $d_{k,p_k}^{\top} B_{k,p_k}^{\top} v_1^{(kp_k)} + S v_4^{(kp_k)} + d_{k,p_k}^{\top} B_{k,p_k}^{\top} v_4^{(kp_k)} + \sum_{j\in N_M} \widehat{a}_{kj} P_1^{\top} W_1^{\top} v_4^{(ip_i)} - \sum_{j\in N_M} \widehat{a}_{jk} P_1^{\top} \times W_1^{\top} v_4^{(jp_j)} \preceq -\mu_3 v_4^{(p)}$, where $i \in [1, M]$. Finally, it follows that

$$
\begin{aligned}
&(\widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p_1} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{B}_p\widetilde{P}_{3,p})^{\top} v_1^{(p)} + (\widetilde{G}_p + \widetilde{M}_p\widetilde{C}_p - \widetilde{A}_p)^{\top} v_2^{(p)} + (\widetilde{T}_p\widetilde{C}_p + \widetilde{F}_p)^{\top} v_3^{(p)} \\
&\quad + (\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_M \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p_1} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3p}\widetilde{C}_p + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{P}_{3p})^{\top} + (\widetilde{\mathbf{\Omega}}_M \varrho I_M \otimes (W_1P_2))^{\top} v_4^{(p)} \\
&= \begin{pmatrix}
\begin{pmatrix}
A_{1,p_1}^{\top} v_1^{(1p_1)} + K_{1,1,p_1}^{\top} B_{1,p_1}^{\top} v_1^{(1p_1)} + C_{1,p_1}^{\top} K_{3,1,p_1}^{\top} B_{1,p_1}^{\top} v_1^{(1p_1)} + P_{3,1,p_1}^{\top} B_{1,p_1}^{\top} v_1^{(1p_1)} \\
+G_{1,p_1}^{\top} v_2^{(1p_1)} - A_{1,p_1}^{\top} v_2^{(1p_1)} + C_{1,p_1}^{\top} M_{1,p_1}^{\top} v_2^{(1p_1)} + F_{1,p_1}^{\top} v_3^{(1p_1)} + C_{1,p_1}^{\top} T_{1,p_1}^{\top} v_3^{(1p_1)} \\
+A_{1,p_1}^{\top} v_4^{(1p_1)} + K_{1,1,p_1}^{\top} B_{1,p_1}^{\top} v_4^{(1p_1)} + C_{1,p_1}^{\top} K_{3,1,p_1}^{\top} B_{1,p_1}^{\top} v_4^{(1p_1)} + P_{3,p_1}^{\top} B_{p_1}^{\top} v_4^{(1p_1)} \\
-S v_4^{(1p_1)} - \varrho W_1 P_2 v_4^{(1p_1)}
\end{pmatrix} \\
\vdots \\
\begin{pmatrix}
A_{M,p_M}^{\top} v_1^{(Mp_M)} + K_{1,M,p_M}^{\top} B_{M,p_M}^{\top} v_1^{(Mp_M)} + C_{M,p_M}^{\top} K_{3,M,p_M}^{\top} \\
\times B_{M,p_M}^{\top} v_1^{(Mp_M)} + P_{3,M,p_M}^{\top} B_{M,p_M}^{\top} v_1^{(Mp_M)} + G_{M,p_M}^{\top} v_2^{(Mp_M)} - A_{M,p_M}^{\top} v_2^{(Mp_M)} \\
+C_{M,p_M}^{\top} M_{M,p_M}^{\top} v_2^{(Mp_M)} + F_{M,p_M}^{\top} v_3^{(Mp_M)} + C_{M,p_M}^{\top} T_{M,p_M}^{\top} v_3^{(Mp_M)} \\
+A_{M,p_M}^{\top} v_4^{(Mp_M)} + K_{1,Mp}^{\top} B_{M,p_M}^{\top} v_4^{(Mp_M)} + C_{M,p_M}^{\top} K_{3,M,p_M}^{\top} B_{M,p_M}^{\top} v_4^{(Mp_M)} \\
+P_{3,M,p_M}^{\top} B_p^{\top} v_4^{(Mp_M)} - S v_4^{(Mp_M)} - \varrho W_1 P_2 v_4^{(Mp_M)}
\end{pmatrix} \\
\vdots \\
\begin{pmatrix}
A_{N,p_N}^{\top} v_1^{(Np_N)} + K_{1,N,p_N}^{\top} B_{N,p_N}^{\top} v_1^{(Np_N)} + C_{N,p_N}^{\top} K_{3,N,p_N}^{\top} \\
\times B_{N,p_N}^{\top} v_1^{(Np_N)} + P_{3,N,p_N}^{\top} B_{N,p_N}^{\top} v_1^{(Np_N)} + G_{N,p_N}^{\top} v_2^{(Np_N)} - A_{N,p_N}^{\top} v_2^{(Np_N)} \\
+C_{N,p_N}^{\top} M_{N,p_N}^{\top} v_2^{(Np_N)} + F_{N,p_N}^{\top} v_3^{(Np_N)} + C_{N,p_N}^{\top} T_{N,p_N}^{\top} v_3^{(Np_N)}
\end{pmatrix}
\end{pmatrix}.
\end{aligned} \tag{44}
$$

From (32o) and (32p), the following holds:

$$
\begin{aligned}
&\overline{X}^{*\top}(A_{i,p_i}^{\top} v_1^{(ip_i)} + K_{1,i,p_i}^{\top} B_{i,p_i}^{\top} v_1^{(ip_i)} + C_{3p}^{\top} K_{3,i,p_i}^{\top} B_{i,p_i}^{\top} v_1^{(ip_i)} + P_{3,i,p_i}^{\top} B_{i,p_i}^{\top} v_1^{(ip_i)} + G_{i,p_i}^{\top} v_2^{(ip_i)} - A_{i,p_i}^{\top} v_2^{(ip_i)} \\
&\quad + C_{i,p_i}^{\top} M_{i,p_i}^{\top} v_2^{(ip_i)} + F_{i,p_i}^{\top} v_3^{(ip_i)} + C_{i,p_i}^{\top} T_{i,p_i}^{\top} v_3^{(ip_i)} + A_{i,p_i}^{\top} v_4^{(ip_i)} + K_{1,i,p_i}^{\top} B_{i,p_i}^{\top} v_4^{(ip_i)} + C_{3p}^{\top} K_{3,i,p_i}^{\top} B_{i,p_i}^{\top} v_4^{(ip_i)} \\
&\quad + P_{3,i,p_i}^{\top} B_p^{\top} v_4^{(ip_i)} - S v_4^{(ip_i)} - \varrho W_1 P_2 v_4^{(ip_i)}) \preceq \varpi
\end{aligned}
$$

for $i \in [1, M]$ and

$$
\begin{aligned}
&\underline{X}^{*\top}(A_{i,p_i}^{\top} v_1^{(ip_i)} + K_{1,i,p_i}^{\top} B_{i,p_i}^{\top} v_1^{(ip_i)} + C_{i,p_i}^{\top} K_{3,i,p_i}^{\top} B_{i,p_i}^{\top} v_1^{(ip_i)} + P_{3,i,p_i}^{\top} B_{i,p_i}^{\top} v_1^{(ip_i)} \\
&\quad + G_{i,p_i}^{\top} v_2^{(ip_i)} - A_{i,p_i}^{\top} v_2^{(ip_i)} + C_{i,p_i}^{\top} M_{i,p_i}^{\top} v_2^{(ip_i)} + F_{i,p_i}^{\top} v_3^{(ip_i)} + C_{i,p_i}^{\top} T_{i,p_i}^{\top} v_3^{(ip_i)}) \preceq \varpi
\end{aligned}
$$

for $i \in [M+1, N]$, where $\underline{X}^* = (\underbrace{x^*, x^*, \ldots, x^*}_{N-M})$. Therefore, $X^{*\top}((\widetilde{A}_p + \widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{B}_p\widetilde{P}_{3,p})^{\top} v_1^{(p)} + (\widetilde{G}_p + \widetilde{M}_p\widetilde{C}_p - \widetilde{A}_p)^{\top} v_2^{(p)} + (\widetilde{T}_p\widetilde{C}_p + \widetilde{F}_p)^{\top} v_3^{(p)} + (\widetilde{\mathbf{\Omega}}_M\widetilde{A}_p - \widetilde{\mathbf{\Omega}}_M I_m \otimes S + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{1,p} + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{K}_{3,p}\widetilde{C}_p + \widetilde{\mathbf{\Omega}}_M\widetilde{B}_p\widetilde{P}_{3,p})^{\top} + (\widetilde{\mathbf{\Omega}}_M \varrho I_M \otimes (W_1P_2))^{\top} v_4^{(p)}) - \varpi \preceq 0$. Thus,

$$
\begin{aligned}
\dot{V}_p(X(t)) &< e^{\top}(t)(-\mu_3)v_1^{(p)} + \theta^{\top}(t)(-\mu_3)v_2^{(p)} + \varepsilon^{\top}(t)(-\mu_3)v_3^{(p)} + \overline{\eta}^{\top}(t)(-\mu_3)v_4^{(p)} \\
&< -\mu_3 V_p(X(t)) + \varpi.
\end{aligned} \tag{45}
$$

By combining (40)–(45) and (32s), we can deduce that

$$
\begin{aligned}
V_{\sigma(t)}(X(t)) &\leqslant e^{(N_0 + \frac{t-t_{k_0}}{\tau_i})\sum_{i=1}^{n} \ln \lambda_i} e^{-\mu_3(t-t_{k_0})} V_{\sigma(t_{k_0})}(X(t_{k_0})) + \varpi \int_{t_0}^{t} e^{(-\mu_3)(t-\tau)}\, d\tau \\
&\leqslant \rho e^{-\phi(t-t_0)} V_{\sigma(t_0)}(X(t_0)) + \varpi \int_{t_0}^{t} e^{(-\mu_3)(t-\tau)}\, d\tau,
\end{aligned} \tag{46}
$$

where $\rho = e^{N_0 \sum_{i=1}^{n} \ln \lambda_i}$ and $\phi = \mu_3 - \frac{\sum_{i=1}^{n} \ln \lambda_i}{\tau_i}$. From Definitions 3 and 4, we can easily prove that system (36) reaches consensus.

Remark 8. Theorems 2 and 3 involve many conditions, all of which are derived based on LP and CLFs. For Theorem 2, conditions (26a)–(26c) are used to guarantee the positivity of leader system (22). Conditions (26d) and

(26e) are intended to reach the consensus of system (22). For Theorem 3, conditions (12a)–(12o) and (32b)–(32f) aim to guarantee the positivity of closed-loop system (36). Conditions (32e)–(32p) prove that system (36) reaches a practical consensus. Since there are only M leaders, it is necessary to discuss system (36) considering two parts: the first M agents and the last $N-M$ agents.

Remark 9. The security of MASs has attracted much attention [50,51]. This study focuses on complete consensus in different situations. On the one hand, it is suitable for scenarios with rigorous requirements. On the other hand, practical consensus refers to efficiency and performance in real-world applications. Thus, a balance between consensus and performance is achieved, making it suitable for high-throughput and low-delay scenarios. Compared to traditional consensus, practical consensus has two advantages. First, it is practical in real-world applications such as search and rescue and salvage operations. It allows agents to be distributed over an area for broader coverage rather than in a single path. Thus, the success rate of tasks can be increased. Second, the system security can be enhanced by practical consensus. Since each agent can follow a different trajectory, it is difficult for an attacker to damage the system even if the paths of some agents are blocked.

Remark 10. This system differs from a single-leader system in mainly two aspects. (i) The design of the controller with multiple leaders is more complex than that with a single leader. Leader systems with multiple leaders should consider the communication between different agents and have high demands on controllers [22,29]. However, there is no need to consider controller design in leader systems with only a single leader [11,24,30]. (ii) When some leaders in a system with multiple leaders fail, or the external environment alters, other leaders can still communicate with each other and lead their followers. Thus, this type of system is more robust and adaptable.

Remark 11. As stated in the introduction, CLFs and LP are suitable for handling the issues of positive systems. First, a quadratic Lyapunov function is unnecessary, and CLFs are more suitable for positive systems. Owing to the nonnegativity of the state, the positive definite property of the CLF $V(t)=x(t)^{\top}v$ can be guaranteed. Second, LP is more effective than LMI in handling the conditions involved. All positivity and stability conditions in (11), (12), (26), and (32) can be transformed into LP. Therefore, it is more reasonable to adopt LP for solving the conditions under study. Moreover, it is difficult to employ LMI to describe the positivity conditions. Based on the form of these conditions, vector variables are chosen. When using the LMI, a matrix variable is chosen. Vector-based conditions are easier to tackle than matrix-based conditions.

Remark 12. The switching topology refers to the dynamic adjustment of communications among agents. In contrast, the switching of MASs focuses on the dynamic switching of different subsystems of agents. These two types of switching involve distinct aspects of a system and require separate consideration. To handle the switching topology, this study designed state estimation coupling terms in controllers (4) and (25) to effectively manage the dynamic behaviors under different topologies. For MASs with switching characteristics, the ADT switching law and CLFs are utilized to guarantee the positivity and stability of each agent.

Remark 13. The control gains in (4) are based on the matrix decomposition approach. This design strategy has two advantages. (i) A vector is chosen as the variable, thereby reducing the computation complexity of the conditions. (ii) Under the proposed control design framework, a simple Lyapunov function can be constructed to analyze the consensus of the systems. In Theorem 1, controller (4) comprises state estimation, disturbance estimation, output, a leader state, and a coupling term with state observation of followers. Owing to the introduction of multiple leaders (22), Theorem 2 integrates the leader-following error term $d_{k,\sigma_i(t)}(\widehat{x}_k(t)-\eta_k(t))$ and a constant term x^* into the controller (25). Proposed controllers (4) and (25) have the following advantages. (i) The controller design ensures system positivity, which is a fundamental requirement for a positive system. (ii) State and disturbance estimations are added to the controller, allowing the controller to precisely regulate the system behavior. (iii) The term $d_{k,\sigma_i(t)}(\widehat{x}_k(t)-\eta_k(t))$ and x^* in (25) enable the system to achieve the leader-following practical consensus, thereby allowing it to meet the practical engineering requirements. (iv) Under the control framework, LP can be adopted to devise a tractable computation method.

4 Illustrative example

In recent years, the MAS dynamics equation has been commonly used to formulate unmanned aerial vehicle (UAV) formations. A previous study [55] discussed the cooperative control of UAVs based on the MAS dynamics equation and proposed a multi-UAV control scheme. The designed MAS model and control scheme provide new perspectives and contributions in the field of UAVs. In [56], a smooth distributed consensus control method for MASs in UAVs was proposed, and the problem of controller chattering in traditional discontinuous control was solved. In [57], the authors comprehensively reviewed the modeling of dynamic equations and distributed control applications for UAVs.

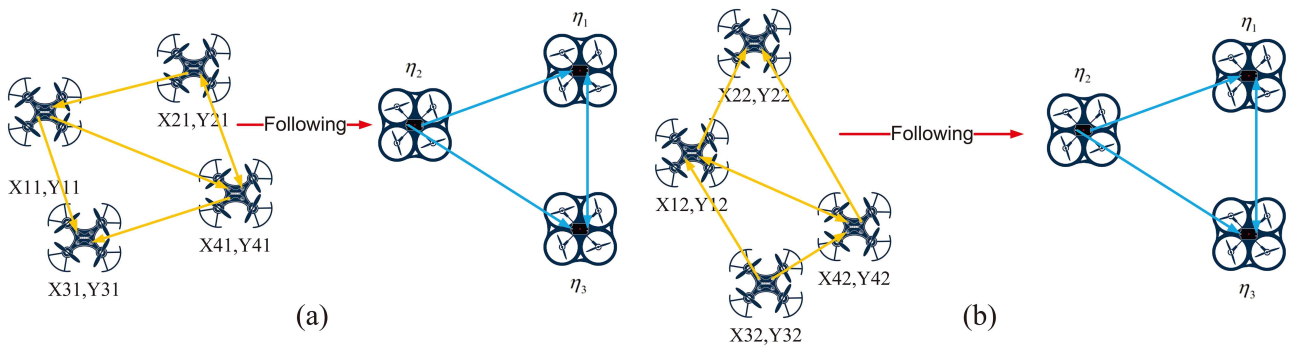

Figure 2 (Color online) Illustration of the formation of UAVs. (a) Switching subsystem 1; (b) switching subsystems 2.

The tracking problem is an important issue in UAV formation. In [58], the author investigated UAV formation control based on the consensus of a MAS and proposed a multiple UAV formation system based on the leader-following consensus. Another study [59] proposed an event-triggered strategy based on leader-following consensus and validated this strategy on a real UAV platform. The formation needs to ensure that all following UAVs can accurately track the leader UAV.

In the model, states $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ represent four UAVs, and states $\eta_1(t)$, $\eta_2(t)$, and $\eta_3(t)$ represent the UAVs that other UAVs are required to follow. Disturbance $w_i(t)$ refers to the external factors interfering with the UAVs; $y_i(t)$ is the output of the UAVs, and $u_i(t)$ is the control signal that manages the flight trajectories of UAVs. Matrices $A_{i,\sigma_i(t)}$, $B_{i,\sigma_i(t)}$, and $C_{i,\sigma_i(t)}$ denote the system state matrices of the UAVs, where $\sigma_i(t)$ denotes a switching signal. This means that when a signal changes, the position of the UAV is also altered. In addition, consider that communications between UAVs can change in different environments; thus, there are two communication topologies: $\mathcal{G}_1$ and $\mathcal{G}_2$. Therefore, based on the aforementioned results, the example considers the leader-following consensus problem with a leader and three followers under switching topologies. The specific principle is shown in Figure 2.

Consider system (1) with three agents, where each agent has two subsystems and three ADT switching signals:

$$A_{1,\sigma_1(t)} = \begin{pmatrix} -2.97 & 1.62 & 1.63 \\ 1.87 & -3.73 & 1.94 \\ 1.81 & 1.74 & -3.62 \end{pmatrix}, A_{2,\sigma_1(t)} = \begin{pmatrix} -2.93 & 1.82 & 1.39 \\ 1.85 & -2.65 & 1.82 \\ 1.62 & 1.76 & -3.57 \end{pmatrix},$$

$$A_{3,\sigma_1(t)} = \begin{pmatrix} -2.99 & 1.62 & 1.63 \\ 1.87 & -3.73 & 1.94 \\ 1.81 & 1.74 & -3.62 \end{pmatrix}, A_{1,\sigma_2(t)} = \begin{pmatrix} -2.96 & 1.61 & 1.62 \\ 1.85 & -3.71 & 1.76 \\ 1.82 & 1.76 & -3.61 \end{pmatrix},$$

$$A_{2,\sigma_2(t)} = \begin{pmatrix} -2.91 & 1.80 & 1.38 \\ 1.83 & -2.63 & 1.81 \\ 1.59 & 1.74 & -3.55 \end{pmatrix}, A_{3,\sigma_2(t)} = \begin{pmatrix} -2.98 & 1.61 & 1.62 \\ 1.86 & -3.72 & 1.94 \\ 1.80 & 1.74 & -3.61 \end{pmatrix},$$

$$B_{1,\sigma_1(t)} = \begin{pmatrix} 0.58 & 0.56 \\ 0.55 & 0.59 \\ 0.56 & 0.55 \end{pmatrix}, B_{2,\sigma_1(t)} = \begin{pmatrix} 0.78 & 0.76 \\ 0.75 & 0.79 \\ 0.76 & 0.75 \end{pmatrix}, B_{3,\sigma_1(t)} = \begin{pmatrix} 0.69 & 0.67 \\ 0.65 & 0.66 \\ 0.68 & 0.65 \end{pmatrix}, B_{1,\sigma_2(t)} = \begin{pmatrix} 0.57 & 0.54 \\ 0.58 & 0.57 \\ 0.54 & 0.52 \end{pmatrix},$$

$$B_{2,\sigma_2(t)} = \begin{pmatrix} 0.77 & 0.74 \\ 0.78 & 0.77 \\ 0.74 & 0.72 \end{pmatrix}, B_{3,\sigma_2(t)} = \begin{pmatrix} 0.68 & 0.69 \\ 0.64 & 0.65 \\ 0.68 & 0.66 \end{pmatrix}, E_{1,\sigma_1(t)} = \begin{pmatrix} 1.12 & 1.11 \\ 1.15 & 1.12 \\ 1.13 & 1.15 \end{pmatrix}, E_{2,\sigma_1(t)} = \begin{pmatrix} 1.28 & 1.28 \\ 1.29 & 1.20 \\ 1.26 & 1.25 \end{pmatrix},$$

$$E_{3,\sigma_1(t)} = \begin{pmatrix} 1.39 & 1.35 \\ 1.37 & 1.32 \\ 1.34 & 1.37 \end{pmatrix}, E_{1,\sigma_2(t)} = \begin{pmatrix} 1.10 & 1.12 \\ 1.13 & 1.13 \\ 1.11 & 1.13 \end{pmatrix}, E_{2,\sigma_2(t)} = \begin{pmatrix} 1.26 & 1.26 \\ 1.27 & 1.26 \\ 1.25 & 1.23 \end{pmatrix}, E_{3,\sigma_2(t)} = \begin{pmatrix} 1.38 & 1.34 \\ 1.36 & 1.31 \\ 1.33 & 1.35 \end{pmatrix},$$

$$C_{1,\sigma_1(t)} = \begin{pmatrix} 1.63 & 1.68 & 1.65 \\ 1.63 & 1.65 & 1.66 \end{pmatrix}, C_{2,\sigma_1(t)} = \begin{pmatrix} 1.43 & 1.48 & 1.45 \\ 1.43 & 1.45 & 1.46 \end{pmatrix}, C_{3,\sigma_1(t)} = \begin{pmatrix} 1.77 & 1.75 & 1.74 \\ 1.72 & 1.77 & 1.73 \end{pmatrix},$$

$$C_{1,\sigma_2(t)} = \begin{pmatrix} 1.63 & 1.68 & 1.65 \\ 1.63 & 1.65 & 1.66 \end{pmatrix}, C_{2,\sigma_2(t)} = \begin{pmatrix} 1.43 & 1.48 & 1.45 \\ 1.43 & 1.45 & 1.46 \end{pmatrix}, C_{3,\sigma_2(t)} = \begin{pmatrix} 1.75 & 1.74 & 1.76 \\ 1.71 & 1.76 & 1.72 \end{pmatrix},$$

$$D_{1,\sigma_1(t)} = \begin{pmatrix} 1.28 & 1.24 \\ 0 & 0 \end{pmatrix}, D_{2,\sigma_1(t)} = \begin{pmatrix} 1.24 & 1.27 \\ 0 & 0 \end{pmatrix}, D_{3,\sigma_1(t)} = \begin{pmatrix} 1.59 & 1.59 \\ 0 & 0 \end{pmatrix},$$

$$D_{1,\sigma_2(t)} = \begin{pmatrix} 1.35 & 1.35 \\ 0 & 0 \end{pmatrix}, D_{2,\sigma_2(t)} = \begin{pmatrix} 1.23 & 1.26 \\ 0 & 0 \end{pmatrix}, D_{3,\sigma_2(t)} = \begin{pmatrix} 1.56 & 1.56 \\ 0 & 0 \end{pmatrix}.$$

Additionally, the matrices of systems (2) and (3) and the corresponding Laplacian matrices are as follows:

$$\varGamma_{1,\sigma_1(t)} = \begin{pmatrix} 1.10 & 0.20 \\ 0.10 & 0.12 \end{pmatrix}, \varGamma_{2,\sigma_1(t)} = \begin{pmatrix} 1.12 & 0.10 \\ 0.10 & 0.12 \end{pmatrix}, \varGamma_{3,\sigma_1(t)} = \begin{pmatrix} 1.13 & 0.13 \\ 0.12 & 0.11 \end{pmatrix},$$

$$\varGamma_{1,\sigma_2(t)} = \begin{pmatrix} 1.08 & 0.10 \\ 0.10 & 0.12 \end{pmatrix}, \varGamma_{2,\sigma_2(t)} = \begin{pmatrix} 1.10 & 0.10 \\ 0.10 & 0.12 \end{pmatrix}, \varGamma_{3,\sigma_2(t)} = \begin{pmatrix} 1.12 & 0.11 \\ 0.12 & 0.10 \end{pmatrix},$$

$$S = \begin{pmatrix} -2.69 & 0.76 & 0.77 \\ 0.78 & -2.59 & 0.71 \\ 0.75 & 0.76 & -2.69 \end{pmatrix}, \mathcal{L}_1 = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \mathcal{L}_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

We choose the constants as follows: $\mu_1 = 0.03$, $\alpha_1 = -4.1$, $\alpha_2 = -4.2$, $\lambda_1 = 3$, $\overline{\gamma}_1 = 81.1$, $\underline{\gamma}_1 = 81.0$, $\overline{\gamma}_2 = 81.2$, $\underline{\gamma}_2 = 81.1$, $\overline{\gamma}_3 = 2.05$, $\underline{\gamma}_3 = 2.02$. From Theorem 1, we obtain the following gain matrices:

$$K_{1,1,\sigma_1(t)} = \begin{pmatrix} -0.0326 & 0.4141 & -0.0326 \\ 0.1685 & -0.0324 & 0.3767 \end{pmatrix}, K_{1,1,\sigma_2(t)} = \begin{pmatrix} 0.0980 & 0.0980 & 0.0980 \\ -0.0336 & -0.0336 & -0.0336 \end{pmatrix},$$

$$K_{1,2,\sigma_1(t)} = \begin{pmatrix} 0.2248 & -0.0329 & 0.1042 \\ -0.0327 & 0.1020 & -0.0327 \end{pmatrix}, K_{1,2,\sigma_2(t)} = \begin{pmatrix} 0.1636 & 0.1636 & 0.1636 \\ -0.0333 & -0.0333 & -0.0333 \end{pmatrix},$$

$$K_{1,3,\sigma_1(t)} = \begin{pmatrix} 0.1010 & 0.1010 & 0.1010 \\ -0.0333 & -0.0333 & -0.0333 \end{pmatrix}, K_{1,3,\sigma_2(t)} = \begin{pmatrix} -0.0330 & -0.0330 & -0.0330 \\ 0.1030 & 0.1030 & 0.1030 \end{pmatrix},$$

$$K_{3,1,\sigma_1(t)} = \begin{pmatrix} 0 & 0 \\ -1.2799 & 0 \end{pmatrix}, K_{3,1,\sigma_2(t)} = \begin{pmatrix} 0 & 0 \\ 0 & -0.4656 \end{pmatrix}, K_{3,2,\sigma_1(t)} = \begin{pmatrix} 0 & 0 \\ -0.3687 & 0 \end{pmatrix},$$

$$K_{3,2,\sigma_2(t)} = \begin{pmatrix} 0 & 0 \\ -0.3841 & 0 \end{pmatrix}, K_{3,3,\sigma_1(t)} = \begin{pmatrix} 0 & 0 \\ -0.4168 & 0 \end{pmatrix}, K_{3,3,\sigma_2(t)} = \begin{pmatrix} 0 & 0 \\ 0 & -0.8940 \end{pmatrix},$$

$$K_{4,1,\sigma_1(t)} = \begin{pmatrix} -0.0326 & -0.0326 & -0.0326 \\ -0.0326 & -0.0326 & -0.0326 \end{pmatrix}, K_{4,1,\sigma_2(t)} = \begin{pmatrix} -0.0324 & -0.0324 & -0.0324 \\ -0.0324 & -0.0324 & -0.0324 \end{pmatrix},$$

$$K_{4,2,\sigma_1(t)} = \begin{pmatrix} -0.0329 & -0.0329 & -0.0329 \\ -0.0329 & -0.0329 & -0.0329 \end{pmatrix}, K_{4,2,\sigma_2(t)} = \begin{pmatrix} -0.0327 & -0.0327 & -0.0327 \\ -0.0327 & -0.0327 & -0.0327 \end{pmatrix},$$

$$K_{4,3,\sigma_1(t)} = \begin{pmatrix} -0.0327 & -0.0327 & -0.0327 \\ -0.0327 & -0.0327 & -0.0327 \end{pmatrix}, K_{4,3,\sigma_2(t)} = \begin{pmatrix} -0.0330 & -0.0330 & -0.0330 \\ -0.0330 & -0.0330 & -0.0330 \end{pmatrix},$$

$$P_{1,\sigma_1(t)} = \begin{pmatrix} 2.4110 & -0.0326 & 3.0528 \\ -0.0324 & 3.2745 & -0.0324 \end{pmatrix}, P_{1,\sigma_2(t)} = \begin{pmatrix} 1.0694 & -0.0324 & 1.8558 \\ -0.0336 & 2.1262 & -0.0336 \end{pmatrix},$$

$$P_{2,\sigma_1(t)} = \begin{pmatrix} 0.6385 & -0.0329 & 1.3481 \\ -0.0327 & 0.6317 & -0.0327 \end{pmatrix}, P_{2,\sigma_2(t)} = \begin{pmatrix} 0.7485 & -0.0324 & 1.3212 \\ -0.0333 & 0.6471 & -0.0333 \end{pmatrix},$$

$$P_{3,\sigma_1(t)} = \begin{pmatrix} 1.0337 & -0.0326 & 1.6342 \\ -0.0333 & 1.9542 & -0.0333 \end{pmatrix}, P_{3,\sigma_2(t)} = \begin{pmatrix} 1.8961 & -0.0330 & 2.4482 \\ -0.0330 & 2.7464 & -0.0330 \end{pmatrix}.$$

Assume that there exists a leader with three followers, as shown in Figure 3. The topology between the followers can be switched. Furthermore, each agent in the system is subject to a solitary switching signal. In Figure 4, the switching process occurs at different moments. Since the state and external disturbance of the system are

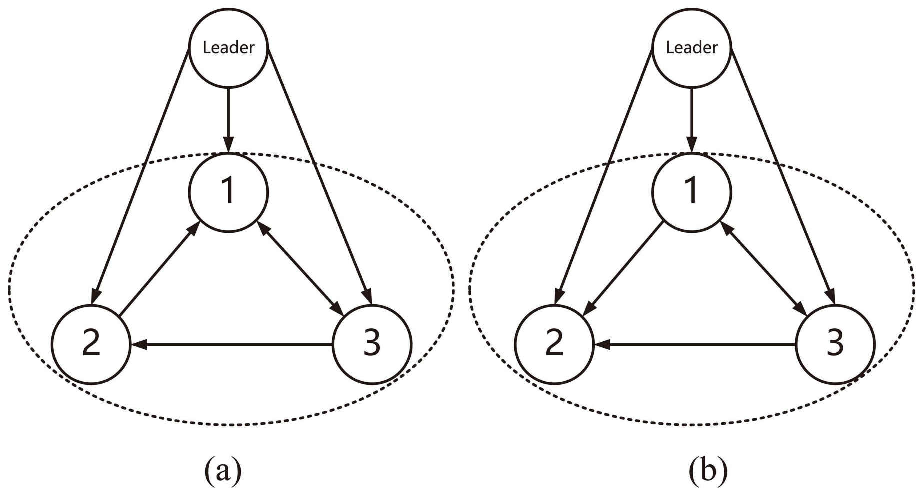

Figure 3 Communication topology. (a) Topology $\mathcal{G}_1$; (b) topology $\mathcal{G}_2$.

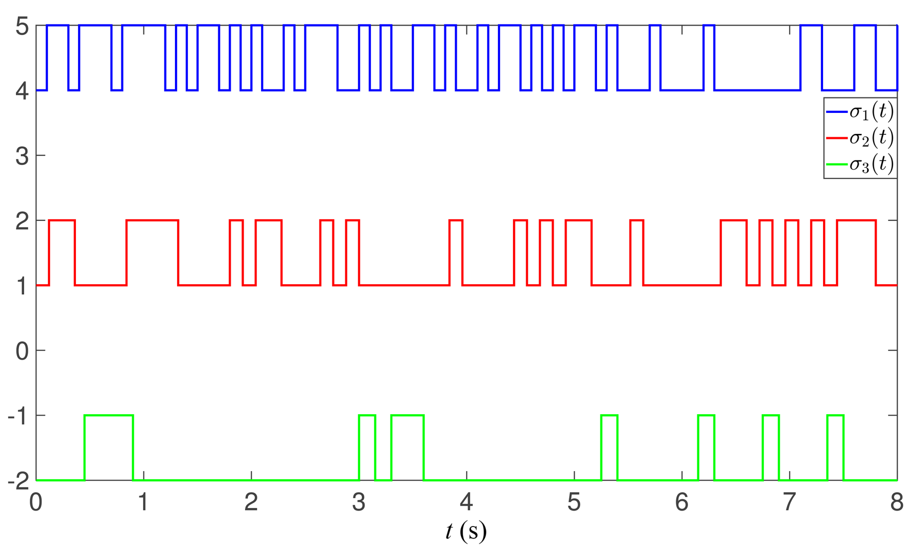

Figure 4 (Color online) ADT switching signals of three agents.

assumed to be unknown in this study, we designed a state and a disturbance observer. Figures 5 and 6 represent the trend plots of state error $e_i(t)$ and disturbance error $\theta_i(t)$, respectively. The observer errors are all zero after 8 s. This indicates that dual observers (5) and (6) designed for SPMASs are working. Figure 7 shows the tracking error. When error $\epsilon_i(t)$ is zero, it signifies that the followers can track the leader. From Figures 3 and 4, it can be found that each UAV has distinct coordinates at different points in time. From Figures 5–7, it is seen that the leader-following errors, state errors, and disturbance errors converge to zero. Then, the designed controller ensures the consensus of UAVs. However, it may be erroneous to conclude that UAVs are no longer working. In fact, the zero point represents an equilibrium point for the considered system. In the control phase, the equilibrium point enables the follower UAVs to track the leader UAVs.

In [17], the positivity of MASs was considered, and some conditions for positive consensus under a fixed topology were derived. However, the agents' switching properties were neglected. In [18], the consensus of PMASs with switching topologies was addressed using LMI-based conditions. None of the above literature considers the unmeasurable state and external disturbances. For a three-dimensional system, the simulation time length is 8 s, and the step size is 10, the computational complexity is: $O(8 \times 10 \times 3^3)$. Table 1 lists the complexity of Theorems 1 and 2 in [18]. As shown in Table 1, the complexity of LP and CLFs is lower when dealing with the consensus of SPMASs.

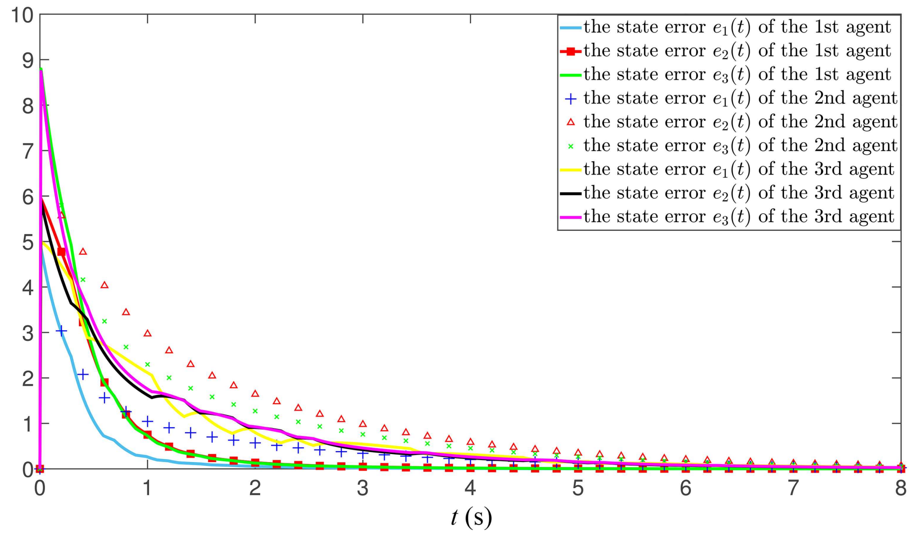

Figure 5 (Color online) State errors of three agents.

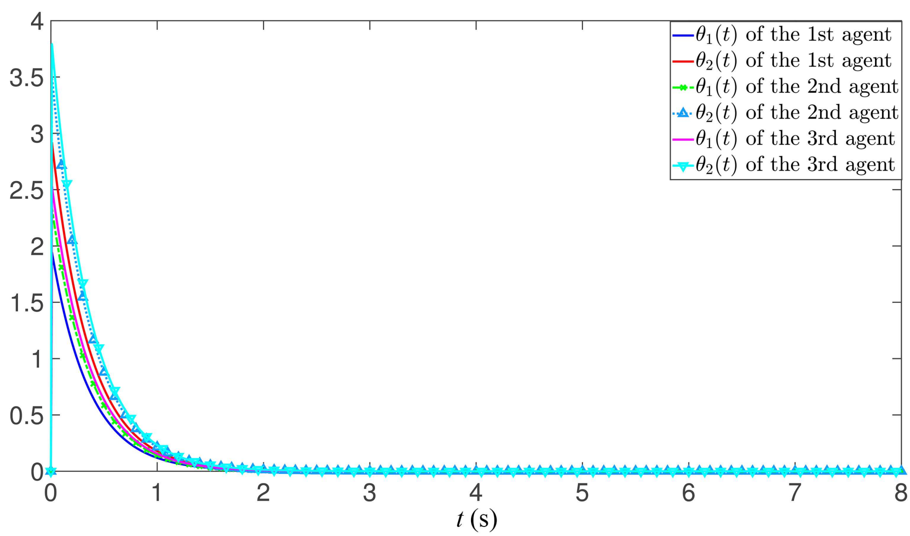

Figure 6 (Color online) Disturbance errors of three agents.

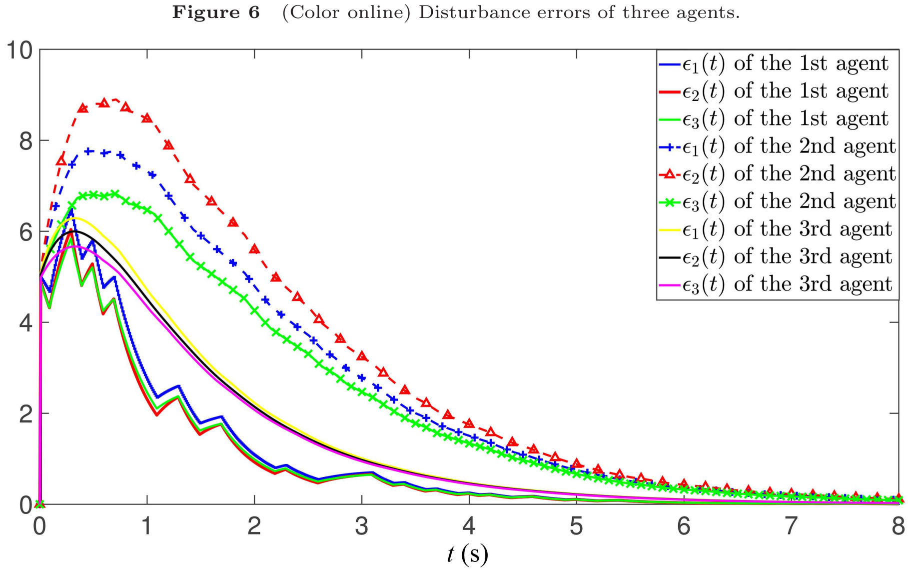

Figure 7 (Color online) Tracking errors of three agents.

Table 1 Comparison of computational complexity.

Theorem	Employing method	Computational complexity
Theorem 1 in [18]	LMI	24×24^3
Theorem 2 in [18]	LMI	24×8^3
Theorem 1 in this paper	LP	80×3^3

5 Conclusion

A double-observer framework based on states and disturbance was established for SPMASs. The tracking problem of SPMASs was solved for a leader. The practical consensus with multiple leaders was also discussed. By using multiple CLFs and LP, the consensus of SPMASs was achieved. In future work, more complete state-disturbance-based observer frameworks in SPMASs will be considered. Moreover, it is also an interesting topic for further research on the positive consensus of MASs with switching topologies.

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