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Special Topic: Logical System Control

# Finite-time stability of stochastic block logical dynamical systems and its application

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Recently, a new type of discrete-time nonlinear systems, called block logical dynamical systems (BLDSs), was proposed in [1]. Based on the method of studying the stability of logical networks, the stability of BLDSs was analyzed. In addition, the stability of BLDSs was used to study the stability of logical networks with time delay. In fact, the equivalent algebraic systems of logical networks can be regarded as a special type of BLDSs. The state transition matrix of BLDSs is a Boolean matrix. In the real world, stochasticity is very common. Extending BLDSs to stochastic BLDSs (SBLDSs) is reasonable. Prior to this, logical networks were well extended to probabilistic logical networks (PLNs). Based on the semi-tensor product (STP) of matrices [2, 3], many theoretic results related to PLNs have been developed [4]. However, the methods used to study PLNs mentioned above are difficult to apply to the study of SBLDSs. Therefore, it is very meaningful and challenging to find research methods suitable for SBLDSs. It is worth noting that PLNs with time delay seem more suitable for describing real-world biological systems [5]. Given that the instability of many systems is caused by time delay, the impact of time delay on the asymptotic stability of PLNs naturally becomes an attractive and intriguing research topic. Therefore, establishing appropriate methods to study the stability of SBLDSs and applying them to PLNs affected by factors such as time delay is a very meaningful task.

In this study, we analyze the stability of SBLDSs and apply it to delayed PLNs. A new type of stochastic nonlinear systems, called SBLDSs, is proposed. Combining with the STP and the law of total expectation, a new method for constructing an equivalent form of SBLDSs is established. Based on the equivalent form, new criteria are proposed for the stability of SBLDSs. The stability criterion for SBLDSs is applied to verify the stability of PLNs with time delay. Compared with the general method for delayed PLNs, the proposed SBLDSs formulation reduces the stability test to a substantially lower-dimensional problem, yielding efficiency and scalability (see Appendix D). In future work, we will further investigate the robust stability of BLDSs with stochastic function perturbations, providing more theoretical tools for practical problems such as the stability of delayed logical networks with stochastic function perturbations.

*Transition probability matrix of SBLDS.* Consider the following SBLDS:

$$v(s+1)=\begin{bmatrix}\Psi_{1,1}^{\sigma(s)} & \cdots & \Psi_{1,m}^{\sigma(s)}\\ \vdots & \ddots & \vdots\\ \Psi_{m,1}^{\sigma(s)} & \cdots & \Psi_{m,m}^{\sigma(s)}\end{bmatrix}v(s):=\Psi^{\sigma(s)}v(s),\qquad (1)$$

where $\sigma:\mathbb{N}\mapsto L:=\{1,\cdots,l\}$ denotes the switching signal, $v(s)\in\Lambda_{m,n}$ denotes the state, and $\Psi^{\sigma(s)}\in\mathcal{B}^{mn\times mn}$ is a block logical matrix, which means that for any $i\in\{1,\cdots,m\}$, there exists a unique integer $j_i\in\{1,\cdots,m\}$ such that $\Psi_{i,j_i}^{\sigma(s)}\in\mathcal{L}_{n\times n}$ and $\Psi_{i,j}^{\sigma(s)}=\mathbf{0}_{n\times n}, j\neq j_i$. In addition, the switching signal $\sigma$ is an i.i.d. random sequence with the probability distribution $P\{\sigma(s)=i\}:=p_i>0,\ i\in L$. Obviously, $\sum_{i=1}^{l}p_i=1$.

Let $P\{v(s)=\delta_n^{i_1,\cdots,i_m}\mid v_0\}$ denote the probability of $v(s)=\delta_n^{i_1,\cdots,i_m}$ starting from $v(0)=v_0$. Then, it is obvious that $P\{v(s)=\delta_n^{i_1,\cdots,i_m}\mid v_0\}\geqslant 0,\ i_1,\cdots,i_m\in\{1,\cdots,n\}$, $\sum_{i_1,\cdots,i_m=1}^{n}P\{v(s)=\delta_n^{i_1,\cdots,i_m}\mid v_0\}=1$. In addition, from the distribution of $v(s)$ starting from $v(0)=v_0$, one can obtain the expectation of $v(s)$ as

$$E\{v(s)\mid v_0\}=\sum_{i_1,\cdots,i_m=1}^{n}\delta_n^{i_1,\cdots,i_m}P\{v(s)=\delta_n^{i_1,\cdots,i_m}\mid v_0\}.$$

According to the properties of the STP [2], one can obtain $(\delta_{mn}^{i_1})^\top E\{v(s)\mid v_0\}=\sum_{j_2,\cdots,j_m=1}^{n}P\{v(s)=\delta_n^{i_1,j_2,\cdots,j_m}\mid v_0\}:=P(1,i_1,s,v_0)$. Similarly, we have

$$\begin{aligned}&(\delta_{mn}^{(k-1)n+i_k})^\top E\{v(s)\mid v_0\}\\ &=\sum_{j_1,\cdots,j_{k-1},j_{k+1},\cdots,j_m=1}^{n}P\{v(s)=\delta_n^{j_1,\cdots,i_k,\cdots,j_m}\mid v_0\}\\ &:=P(k,i_k,s,v_0),\ k=2,\cdots,m.\end{aligned}\qquad (2)$$

In addition, since $\bigcap_{k=1}^{m}\{\delta_n^{j_1,\cdots,j_{k-1},i_k,j_{k+1},\cdots,j_m}: j_1,\cdots,j_{k-1},j_{k+1},\cdots,j_m=1,\cdots,n\}=\{\delta_n^{i_1,\cdots,i_m}\}$, $j_0=i_1$, we have the following proposition.

**Proposition 1.** $P\{v(s)=\delta_n^{i_1,i_2,\cdots,i_m}\mid v_0\}=1$, if and only if $P(k,i_k,s,v_0)=1,\ k=1,\cdots,m$.

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On the other hand, from the law of total expectation, one can obtain $E\{v(s) \mid v_0\} = \sum_{i=1}^{l} E\{v(s) \mid v_0, \sigma(s) = i\}P\{\sigma(s) = i\}$. Let $w_i(s) = E\{v(s) \mid v_0, \sigma(s) = i\}P\{\sigma(s) = i\}$. Then, we have $E\{v(s) \mid v_0\} = \sum_{i=1}^{l} w_i(s)$. In addition, for any initial state $v_0 = \delta_n^{\alpha_1,\cdots,\alpha_m}$, it holds that $w_i(0) = \delta_n^{\alpha_1,\cdots,\alpha_m} p_i,\ i = 1,\cdots l$. Thus, by the law of total expectation, we have

$$\begin{aligned} w_i(s+1) &= E\{v(s+1) \mid v_0, \sigma(s+1) = i\}P\{\sigma(s+1) = i\} \\ &= \sum_{j=1}^{l} \Psi^j E\{v(s) \mid v_0, \sigma(s) = j\}P\{\sigma(s) = j\}p_i \\ &= \sum_{j=1}^{l} p_i \Psi^j w_j(s). \end{aligned} \tag{3}$$

Letting $w(s) = [(w_1(s))^\top \ \cdots \ (w_l(s))^\top]^\top$, one can obtain

$$\begin{aligned} w(s+1) &= \left[ \left( \sum_{j=1}^{l} p_1 \Psi^j w_j(s) \right)^\top \cdots \left( \sum_{j=1}^{l} p_l \Psi^j w_j(s) \right)^\top \right]^\top \\ &= \begin{bmatrix} p_1\Psi^1 & \cdots & p_1\Psi^l \\ \vdots & \ddots & \vdots \\ p_l\Psi^1 & \cdots & p_l\Psi^l \end{bmatrix} w(s) := Mw(s), \end{aligned} \tag{4}$$

where $M \in \mathcal{R}^{mnl \times mnl}$ is the transition probability matrix. In addition, for any initial state $v_0 = \delta_n^{\alpha_1,\cdots,\alpha_m}$, it holds that $w(0) = [(\delta_n^{\alpha_1,\cdots,\alpha_m} p_1)^\top \ \cdots \ (\delta_n^{\alpha_1,\cdots,\alpha_m} p_l)^\top]^\top$.

Then, we have the following result of SBLDS (1):

$$\begin{aligned} P(k, i_k, s, v_0) &= (\delta_{mn}^{(k-1)n+i_k})^\top \sum_{j=1}^{l} w_j(s) \\ &= (\delta_{mn}^{(k-1)n+i_k})^\top (\mathbf{1}_l)^\top w(s). \end{aligned} \tag{5}$$

*Stability of SBLDS*. Based on the above analysis, we provide the definition of stability for SBLDS (1).

**Definition 1.** For an equilibrium point $v_d = \delta_n^{d_1,\cdots,d_m} \in \Lambda_{m,n}$, SBLDS (1) is considered to be finite-time stable at $v_d$, if there exists a positive integer $S$ such that for any $v_0 \in \Lambda_{m,n}$, it holds that $P\{v(s) = v_d \mid v(0) = v_0\} = 1,\ \forall\ s \geqslant S$.

Then, based on the transition probability matrix $M$ in (4), we obtain the following main result.

**Theorem 1.** For an equilibrium point $v_d = \delta_n^{d_1,\cdots,d_m} \in \Lambda_{m,n}$, SBLDS (1) is finite-time stable at $v_d$, if and only if

(i) $\left( (\delta_m^j)^\top \mathbf{1}_l^\top M \delta_l^i \mathbf{1}_m \right)_{d_j,d_k} = 1$;

(ii) $(\delta_m^j)^\top \mathbf{1}_l^\top M^{n^m-1} \delta_l^i \mathbf{1}_m = \delta_n[\underbrace{d_j \ \cdots \ d_j}_{n}]$,

where $M$ is given by (4), $i = 1,\cdots,l,\ j,k = 1,\cdots,m$ (see Appendix B.1 for the proof).

*Application in PLNs with state delays*. Based on the stability of SBLDS, we study the stability of PLNs with state delays.

Consider the following PLN with state delays [6]:

$$x_i(s+1) = g_i(X(s-\tau(s))),\ i = 1,\cdots,n, \tag{6}$$

where $\tau : \mathbb{N} \mapsto \Upsilon := \{0,\cdots,\gamma-1\}$ denotes the state delay, $X(s) := (x_1(s),\cdots,x_n(s)) \in \mathcal{D}_k^n$ denotes the state, and $g_i : \mathcal{D}_k^n \mapsto \mathcal{D}_k,\ i = 1,\cdots,n$ are logical functions. In addition, the state delay $\tau$ is an i.i.d. random sequence with the probability distribution $P\{\tau(s) = i\} = p_i > 0,\ i \in \Upsilon$.

Letting $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{k^n}$, one can convert system (6) into the following equivalent algebraic form:

$$x(s+1) = Gx(s-\tau(s)), \tag{7}$$

where $G \in \mathcal{L}_{k^n \times k^n}$ is said to be state transition matrix.

Let $v(s) = [(x(s))^\top \ \cdots \ (x(s-\gamma+1))^\top]^\top \in \Lambda_{\gamma,k^n}$. Similar to the construction in [1], system (7) with the state delay $\tau$ can be equivalently converted into the following form:

$$v(s+1) = \Psi^{\tau(s)} v(s), \tag{8}$$

where

$$\Psi^0 = \begin{bmatrix} G & \cdots & \mathbf{0} & \mathbf{0} \\ I_{k^n} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & I_{k^n} & \mathbf{0} \end{bmatrix}, \cdots, \Psi^{\gamma-1} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & G \\ I_{k^n} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & I_{k^n} & \mathbf{0} \end{bmatrix},$$

and $\mathbf{0}_{k^n \times k^n} := \mathbf{0}$.

As mentioned in [1], we have the following result.

**Proposition 2.** System (7) is finite-time stable at $x_d = \delta_{k^n}^{\theta}$, if and only if SBLDS (8) is finite-time stable at $v_d = \delta_{k^n}^{\theta,\cdots,\theta}$.

By Theorem 1 and Proposition 2, we propose the following new criterion to verify the stability of system (6).

**Corollary 1.** System (6) is finite-time stable at $x_d = \delta_{k^n}^{\theta}$, if and only if

(i) $\left( (\delta_\gamma^j)^\top \mathbf{1}_\gamma^\top M \delta_\gamma^i \mathbf{1}_\gamma \right)_{\theta,\theta} = 1$;

(ii) $(\delta_\gamma^j)^\top \mathbf{1}_\gamma^\top M^{k^{mn}-1} \delta_\gamma^i \mathbf{1}_\gamma = \delta_{k^n}[\underbrace{\theta \ \cdots \ \theta}_{k^n}]$,

where $M = \begin{bmatrix} p_0\Psi^0 & \cdots & p_0\Psi^{\gamma-1} \\ \vdots & \ddots & \vdots \\ p_{\gamma-1}\Psi^0 & \cdots & p_{\gamma-1}\Psi^{\gamma-1} \end{bmatrix}$, $i,j = 1,\cdots,\gamma$.

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**Supporting information** Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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