

• Supplementary File •

Finite-time stability of stochastic block logical dynamical systems and its application

Xiangshan KONG^{1*}, Jinfeng PAN¹, Haitao CHE¹ & Kaikai CAO¹

¹*School of Mathematics and Statistics, Weifang University, Weifang 261061, China*

Appendix A Notations used in the body of this paper

- \mathbb{N} represents the set of nonnegative integers.
- $\mathbf{0}_{n \times m}$ denotes the $n \times m$ -dimensional matrix with all entries being 0. $\mathbf{1}_n := \underbrace{(1, \dots, 1)}_n^\top$.
- $\Delta_n := \{\delta_n^i : i = 1, \dots, n\}$, where δ_n^i is the i -th column of the n -dimensional identity matrix I_n . For example, $\delta_3^1 := [1 \ 0 \ 0]^\top$, $\delta_3^2 := [0 \ 1 \ 0]^\top$, $\delta_3^3 := [0 \ 0 \ 1]^\top$, $\Delta_3 := \{\delta_3^j : j = 1, 2, 3\}$.
- An $n \times m$ -dimensional matrix A is called a logical matrix, if all the columns of A belong to Δ_n . Particularly, an $n \times m$ -dimensional logical matrix $[\delta_n^{i_1} \ \dots \ \delta_n^{i_m}]$ is expressed as $\delta_n[i_1 \ \dots \ i_m]$. $\mathcal{L}_{n \times m}$ denotes the set of $n \times m$ -dimensional logical matrices. $\mathcal{B}^{n \times m}$ denotes the set of $n \times m$ -dimensional Boolean matrices.
- $\delta_n^{i_1, \dots, i_m} := [(\delta_n^{i_1})^\top \ \dots \ (\delta_n^{i_m})^\top]^\top$. $\Lambda_{m, n} := \{\delta_n^{i_1, \dots, i_m} : i_1, \dots, i_m \in \{1, \dots, n\}\}$.
- $(E)_{n, m}$ and $\text{Row}_n(E)$ denote the (n, m) -th element and the n -th row of matrix E , respectively.
- $\mathcal{D}_k := \{0, \dots, k-1\}$, $\mathcal{D}_k^m := \underbrace{\mathcal{D}_k \times \dots \times \mathcal{D}_k}_m$, $k \geq 2$.

Appendix B Proofs in the body of the letter

Appendix B.1 Proof of Theorem 1

(Necessity) Suppose that SBLDS (1) is stable at $v_d = \delta_n^{d_1, \dots, d_m}$.

Firstly, we prove that condition (i) holds. By Definition 1, there exists an integer $S > 0$ such that

$$P\{v(s) = \delta_n^{d_1, \dots, d_m} \mid v(0) = \delta_n^{e_1, \dots, e_m}\} = 1, \quad \forall s \geq S \quad (\text{B1})$$

holds for any $v_0 = \delta_n^{e_1, \dots, e_m}$.

For any v_0 , we have

$$\begin{aligned} & P\{v(s+1) = \delta_n^{d_1, \dots, d_m} \mid v(0) = v_0\} \\ &= \sum_{j_1, \dots, j_m=1}^n P\{v(s) = \delta_n^{j_1, \dots, j_m} \mid v(0) = v_0\} P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{j_1, \dots, j_m}\} \\ &= P\{v(s) = \delta_n^{d_1, \dots, d_m} \mid v(0) = v_0\} P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{d_1, \dots, d_m}\} \\ &\quad + \sum_{j_1 \neq d_1} P\{v(s) = \delta_n^{j_1, \dots, j_m} \mid v(0) = v_0\} P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{j_1, \dots, j_m}\} \\ &\quad + \dots \\ &\quad + \sum_{j_m \neq d_m} P\{v(s) = \delta_n^{j_1, \dots, j_m} \mid v(0) = v_0\} P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{j_1, \dots, j_m}\} \\ &= P\{v(s) = \delta_n^{d_1, \dots, d_m} \mid v(0) = v_0\} P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{d_1, \dots, d_m}\} \\ &= P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{d_1, \dots, d_m}\} \\ &= 1, \end{aligned} \quad (\text{B2})$$

that is,

$$P\{v(1) = \delta_n^{d_1, \dots, d_m} \mid \delta_n^{d_1, \dots, d_m}\} = 1. \quad (\text{B3})$$

* Corresponding author (email: kxs_stp@163.com)

According to Proposition 1 and (6), we have

$$\begin{aligned}
 P(j, d_j, 1, \delta_n^{d_1, \dots, d_m}) &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top w(1) \\
 &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M w(0) \\
 &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M ((\delta_n^{d_1, \dots, d_m} p_1)^\top, \dots, (\delta_n^{d_1, \dots, d_m} p_l)^\top)^\top \\
 &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M \left(\sum_{i=1}^l p_i \delta_i^i \delta_n^{d_1, \dots, d_m} \right) \\
 &= \sum_{i=1}^l p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M \delta_i^i \delta_n^{d_1, \dots, d_m} \\
 &= \sum_{i=1}^l p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M \delta_i^i \sum_{k=1}^m \delta_{mn}^{(k-1)n+d_k} \\
 &= \sum_{i=1}^l \sum_{k=1}^m p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M \delta_i^i \delta_{mn}^{(k-1)n+d_k} \\
 &= 1, \quad j = 1, \dots, m.
 \end{aligned} \tag{B4}$$

On account of $\sum_{i=1}^l p_i = 1$ and $0 \leq (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M \delta_i^i \sum_{k=1}^m \delta_{mn}^{(k-1)n+d_k} \leq 1$, by (B4), we have

$$\sum_{k=1}^m (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M \delta_i^i \delta_{mn}^{(k-1)n+d_k} = 1, \quad i = 1, \dots, l, \quad j = 1, \dots, m, \tag{B5}$$

that is, condition (i) holds.

Secondly, we prove that there exists an integer $S > 0$ such that

$$(\delta_m^j)^\top \mathbf{1}_l^\top M^s \delta_l^i \mathbf{1}_m = \delta_n \underbrace{[d_j \dots d_j]}_n, \quad i = 1, \dots, l, \quad j = 1, \dots, m \tag{B6}$$

holds for any $s \geq S$.

Similar to (B4), according to Proposition 1 and (6), we have

$$\begin{aligned}
 P(j, d_j, s, \delta_n^{e_1, \dots, e_m}) &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top w(s) \\
 &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s w(0) \\
 &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s ((\delta_n^{e_1, \dots, e_m} p_1)^\top, \dots, (\delta_n^{e_1, \dots, e_m} p_l)^\top)^\top \\
 &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \left(\sum_{i=1}^l p_i \delta_i^i \delta_n^{e_1, \dots, e_m} \right) \\
 &= \sum_{i=1}^l p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \delta_i^i \delta_n^{e_1, \dots, e_m} \\
 &= \sum_{i=1}^l p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \delta_i^i \sum_{k=1}^m \delta_{mn}^{(k-1)n+e_k} \\
 &= \sum_{i=1}^l \sum_{k=1}^m p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \delta_i^i \delta_{mn}^{(k-1)n+e_k} \\
 &= 1, \quad j = 1, \dots, m.
 \end{aligned} \tag{B7}$$

On account of $\sum_{i=1}^l p_i = 1$ and $0 \leq (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \delta_i^i \sum_{k=1}^m \delta_{mn}^{(k-1)n+e_k} \leq 1$, by (B7), we have

$$\sum_{k=1}^m (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \delta_i^i \delta_{mn}^{(k-1)n+e_k} = 1, \quad i = 1, \dots, l, \quad j = 1, \dots, m, \tag{B8}$$

that is, (B6) holds for any $s \geq S$.

Finally, we prove $S \leq n^m - 1$ by reduction to absurdity. Assume that $S > n^m - 1$ holds. Based on the above discussion and $S > n^m - 1$, there exist $e_\beta \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, l\}$ such that

$$\left((\delta_m^j)^\top \mathbf{1}_l^\top M^{n^m-1} \delta_l^i \mathbf{1}_m \right)_{d_j, e_\beta} < 1. \tag{B9}$$

Thus, there is $\eta \neq d_j$ that satisfies $\left((\delta_m^j)^\top \mathbf{1}_l^\top M^{n^m-1} \delta_l^i \mathbf{1}_m \right)_{\eta, e_\beta} > 0$. Because $v_d = \delta_n^{d_1, \dots, d_m}$ is the equilibrium point, there exist $\eta_1 \notin \{d_1, \dots, d_m\}, \dots, \eta_{n^m-2} \notin \{d_1, \dots, d_m\}, j_1, \dots, j_{n^m-1} \in \{1, \dots, m\}$ and $i_1, \dots, i_{n^m-1} \in \{1, \dots, l\}$ that satisfy

$$\left((\delta_m^{j_{n^m-1}})^\top \mathbf{1}_l^\top M \delta_{l_{n^m-1}}^{i_{n^m-1}} \mathbf{1}_m \right)_{\eta_{n^m-1}, \eta_{n^m-2}} \dots \left((\delta_m^{j_1})^\top \mathbf{1}_l^\top M \delta_{l_1}^{i_1} \mathbf{1}_m \right)_{\eta_1, \eta_0} > 0, \tag{B10}$$

where $\eta_0 = e_\beta, \eta_{n^m-1} = \eta$. Because $\eta_i \in \{1, \dots, n\}$, there exist $0 \leq k_1 < k_2 \leq n^m - 1$, $j' \in \{1, \dots, m\}$ and $i' \in \{1, \dots, l\}$ such that $\eta_{k_1} = \eta_{k_2} := \eta'$ and $((\delta_m^{j'})^\top \mathbf{1}_l^\top M^{k_2-k_1} \delta_l^{i'} \mathbf{1}_m)_{\eta', \eta'} > 0$. According to the definition of matrix product, we have

$$\begin{aligned} & ((\delta_m^{j'})^\top \mathbf{1}_l^\top M^{t(k_2-k_1)} \delta_l^{i'} \mathbf{1}_m)_{\eta', \eta'} \\ & \geq \underbrace{((\delta_m^{j'})^\top \mathbf{1}_l^\top M^{k_2-k_1} \delta_l^{i'} \mathbf{1}_m)_{\eta', \eta'} \cdots ((\delta_m^{j'})^\top \mathbf{1}_l^\top M^{k_2-k_1} \delta_l^{i'} \mathbf{1}_m)_{\eta', \eta'}}_t > 0. \end{aligned} \quad (\text{B11})$$

Therefore, for any positive integer S , there is always a positive integer t that satisfies $t(k_2 - k_1) > S$ and

$$((\delta_m^{j'})^\top \mathbf{1}_l^\top M^{t(k_2-k_1)} \delta_l^{i'} \mathbf{1}_m)_{d_j, \eta'} < 1, \quad (\text{B12})$$

which contradicts the fact that SBLDS (1) is stable at v_e . Hence, $S \leq n^m - 1$.

(Sufficiency) Assuming condition (i) and condition (ii) hold, we have

$$((\delta_m^j)^\top \mathbf{1}_l^\top M^s \delta_l^i \mathbf{1}_m)_{d_j, e_\beta} = 1 \quad (\text{B13})$$

for any $e_\beta \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, $i \in \{1, \dots, l\}$ and $s > n^m - 1$. Therefore, by (6), we have

$$\begin{aligned} P(j, d_j, s, \delta_n^{e_1, \dots, e_m}) &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top w(s) \\ &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s w(0) \\ &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s ((\delta_n^{e_1, \dots, e_m} p_1)^\top, \dots, (\delta_n^{e_1, \dots, e_m} p_l)^\top)^\top \\ &= (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \left(\sum_{i=1}^l p_i \delta_l^i \delta_n^{e_1, \dots, e_m} \right) \\ &= \sum_{i=1}^l p_i (\delta_{mn}^{(j-1)n+d_j})^\top \mathbf{1}_l^\top M^s \delta_l^i \delta_n^{e_1, \dots, e_m} \end{aligned} \quad (\text{B14})$$

$$\begin{aligned} &= \sum_{i=1}^l p_i ((\delta_m^j)^\top \mathbf{1}_l^\top M^s \delta_l^i \mathbf{1}_m)_{d_j, e_\beta} \\ &= 1 \end{aligned} \quad (\text{B15})$$

for any $v(0) = \delta_n^{e_1, \dots, e_m}$, $j \in \{1, \dots, m\}$ and $s > n^m - 1$.

From Proposition 1 and Definition 1, SBLDS (1) is stable at v_d .

Appendix C Examples

Appendix C.1 An example used to illustrate how transition probability matrix of SBLDS (1) works

Consider the following SBLDS:

$$v(s+1) = \Psi^{\sigma(s)} v(s), \quad (\text{C1})$$

where $\Psi^1 = \begin{bmatrix} \Psi_{1,1}^1 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \Psi_{2,2}^1 \end{bmatrix}$, $\Psi^2 = \begin{bmatrix} \Psi_{1,1}^2 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \Psi_{2,1}^2 \end{bmatrix}$, $\Psi_{1,1}^1 = \delta_4[2 \ 2 \ 4 \ 2]$, $\Psi_{2,2}^1 = \delta_4[4 \ 2 \ 2 \ 2]$, $\Psi_{1,1}^2 = \delta_4[2 \ 2 \ 1 \ 2]$, $\Psi_{2,1}^2 = \delta_4[3 \ 2 \ 2 \ 2]$, $P\{\sigma(s) = 1\} = 0.3$, and $P\{\sigma(s) = 2\} = 0.7$.

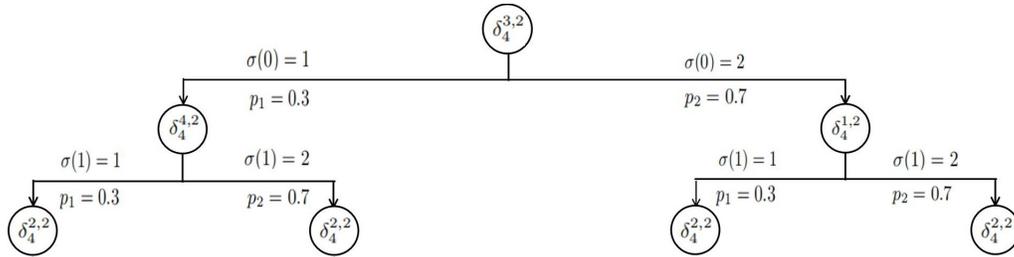


Figure 1: State trajectory for $v(s)$ with the initial state $v_0 = \delta_4^{3,2}$.

According to (5), we have the transition probability matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.3 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0.3 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0.7 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.7 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0.7 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (C2)$$

Given initial state $v_0 = \delta_4^{3,2}$, by (3), (5) and (6), we have

$$\begin{aligned} w(0) &= [0 \ 0 \ 0.3 \ 0 \ 0 \ 0.3 \ 0 \ 0 \ 0 \ 0 \ 0.7 \ 0 \ 0 \ 0.7 \ 0 \ 0]^\top, \\ w(1) &= Mw(0) = [0.21 \ 0 \ 0 \ 0.09 \ 0 \ 0.3 \ 0 \ 0 \ 0.49 \ 0 \ 0 \ 0.21 \ 0 \ 0.7 \ 0 \ 0]^\top, \\ E\{v(1) | v_0\} &= [0.7 \ 0 \ 0 \ 0.3 \ 0 \ 1 \ 0 \ 0]^\top, \\ P(1, 2, 1, v_0) &= \sum_{j_2=1}^4 P\{v(1) = \delta_4^{2,j_2} | v_0 = \delta_4^{3,2}\} = (\delta_8^2)^\top E\{v(1) | v_0\} = 0, \\ P(2, 2, 1, v_0) &= \sum_{j_1=1}^4 P\{v(1) = \delta_4^{j_1,2} | v_0 = \delta_4^{3,2}\} = (\delta_8^6)^\top E\{v(1) | v_0\} = 1, \\ w(2) &= Mw(1) = [0 \ 0.3 \ 0 \ 0 \ 0 \ 0.3 \ 0 \ 0 \ 0 \ 0.7 \ 0 \ 0 \ 0 \ 0.7 \ 0 \ 0]^\top, \\ E\{v(2) | v_0\} &= [0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0]^\top, \\ P(1, 2, 2, v_0) &= \sum_{j_2=1}^4 P\{v(3) = \delta_4^{2,j_2} | v_0 = \delta_4^{3,2}\} = (\delta_8^2)^\top E\{v(3) | v_0\} = 1, \\ P(2, 2, 2, v_0) &= \sum_{j_1=1}^4 P\{v(3) = \delta_4^{j_1,2} | v_0 = \delta_4^{3,2}\} = (\delta_8^6)^\top E\{v(3) | v_0\} = 1. \end{aligned}$$

It is easy to see that the results $P(1, 2, s, v_0)$, $P(2, 2, s, v_0)$, $s = 1, 2$ match the state trajectory in Figure 1.

Appendix C.2 An example used to show how Theorem 1 works

Consider the SBLDS (C1). From (C2), one can obtain that

$$\left((\delta_2^1)^\top \mathbf{1}_2^\top M \delta_2^1 \mathbf{1}_2 \right)_{2,2} = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right)_{2,2} = 1, \quad \left((\delta_2^2)^\top \mathbf{1}_2^\top M \delta_2^1 \mathbf{1}_2 \right)_{2,2} = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right)_{2,2} = 1,$$

$$\left((\delta_2^1)^\top \mathbf{1}_2^\top M \delta_2^2 \mathbf{1}_2 \right)_{2,2} = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)_{2,2} = 1, \quad \left((\delta_2^2)^\top \mathbf{1}_2^\top M \delta_2^2 \mathbf{1}_2 \right)_{2,2} = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)_{2,2} = 1,$$

$$(\delta_2^1)^\top \mathbf{1}_2^\top M^{15} \delta_2^1 \mathbf{1}_2 = \delta_4 [2 \ 2 \ 2 \ 2], \quad (\delta_2^2)^\top \mathbf{1}_2^\top M^{15} \delta_2^1 \mathbf{1}_2 = \delta_4 [2 \ 2 \ 2 \ 2],$$

$$(\delta_2^1)^\top \mathbf{1}_2^\top M^{15} \delta_2^2 \mathbf{1}_2 = \delta_4 [2 \ 2 \ 2 \ 2], \quad (\delta_2^2)^\top \mathbf{1}_2^\top M^{15} \delta_2^2 \mathbf{1}_2 = \delta_4 [2 \ 2 \ 2 \ 2].$$

Then, by Theorem 1, the SBLDS (C1) is stable at $v_d = \delta_4^{2,2}$.

We propose a simulation to verify the effectiveness of our results (see Figure 2). Since $v_d = \delta_4^{2,2}$, it can be seen from Figure 2 that $P\{v(s) = \delta_4^{2,2} | v(0) = \delta_4^{3,2}\} = 1$ holds for any integer $s \geq 2$.

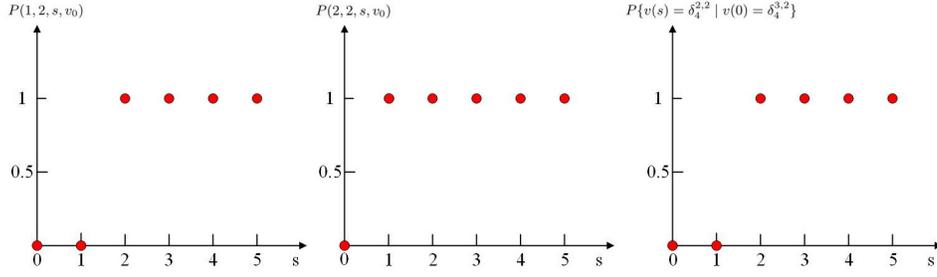


Figure 2: Probability trajectory for $P(1, 2, s, v_0)$, $P(2, 2, s, v_0)$, $P\{v(s) = \delta_4^{2,2} \mid v(0) = \delta_4^{3,2}\}$ with the initial state $\delta_4^{3,2}$.

Appendix C.3 An example used to show how Corollary 1 works

Consider the following probabilistic logical network with coincident state delays [6]:

$$\begin{cases} x_1(s+1) = x_1(s - \tau(s)) \wedge \neg x_2(s - \tau(s)), \\ x_2(s+1) = x_1(s - \tau(s)), \end{cases} \quad (\text{C3})$$

where the state delay $\tau(s)$ takes values in $\{0, 1\}$ with the probability distribution $P\{\tau(s) = 0\} = 2/3$ and $P\{\tau(s) = 1\} = 1/3$.

Letting $x(t) = \times_{i=1}^2 x_i(t) \in \Delta_{2,2}$, one can convert system (C3) into the equivalent algebraic form

$$x(s+1) = Gx(s - \tau(s)), \quad (\text{C4})$$

where $G = \delta_4[3 \ 1 \ 4 \ 4]$.

Setting $v(s) = [(x(s))^\top (x(s-1))^\top]^\top$, we have

$$v(s+1) = \Psi^{\tau(s)} v(s), \quad (\text{C5})$$

where $\Psi^0 = \begin{bmatrix} G & \mathbf{0}_{2^2 \times 2^2} \\ I_{2^2} & \mathbf{0}_{2^2 \times 2^2} \end{bmatrix}$, $\Psi^1 = \begin{bmatrix} \mathbf{0}_{2^2 \times 2^2} & G \\ I_{2^2} & \mathbf{0}_{2^2 \times 2^2} \end{bmatrix}$.

According to (5), we have the transition probability matrix $M = \begin{bmatrix} 2/3\Psi^0 & 2/3\Psi^1 \\ 1/3\Psi^0 & 1/3\Psi^1 \end{bmatrix}$. Furthermore, one can obtain that

$$\left((\delta_2^1)^\top \mathbf{1}_2^\top M \delta_2^1 \mathbf{1}_2 \right)_{4,4} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right)_{4,4} = 1, \quad \left((\delta_2^2)^\top \mathbf{1}_2^\top M \delta_2^1 \mathbf{1}_2 \right)_{4,4} = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)_{4,4} = 1,$$

$$\left((\delta_2^1)^\top \mathbf{1}_2^\top M \delta_2^2 \mathbf{1}_2 \right)_{4,4} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right)_{4,4} = 1, \quad \left((\delta_2^2)^\top \mathbf{1}_2^\top M \delta_2^2 \mathbf{1}_2 \right)_{4,4} = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)_{4,4} = 1,$$

$$(\delta_2^1)^\top \mathbf{1}_2^\top M^{15} \delta_2^1 \mathbf{1}_2 = \delta_4[4 \ 4 \ 4 \ 4], \quad (\delta_2^2)^\top \mathbf{1}_2^\top M^{15} \delta_2^1 \mathbf{1}_2 = \delta_4[4 \ 4 \ 4 \ 4],$$

$$(\delta_2^1)^\top \mathbf{1}_2^\top M^{15} \delta_2^2 \mathbf{1}_2 = \delta_4[4 \ 4 \ 4 \ 4], \quad (\delta_2^2)^\top \mathbf{1}_2^\top M^{15} \delta_2^2 \mathbf{1}_2 = \delta_4[4 \ 4 \ 4 \ 4].$$

Then, by Corollary 1, system (C3) is stable at $x_d = \delta_4^4$.

Appendix D A remark

In Theorem 1 of [6], one needs to calculate the $(k^{n(\gamma+1)} + 1) \times (k^{n(\gamma+1)} + 1)$ -dimensional matrix L for the stability of system (6). Compared with [6], in Corollary 1, one only needs to calculate the $\gamma^2 k^n \times \gamma^2 k^n$ -dimensional matrix M for the stability of system (6). If γ and n are large enough, then $\gamma^2 k^n \ll k^{n(\gamma+1)+1}$ and Corollary 1 is more efficient.