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Special Topic: Logical System Control

Optimal control of smart grids based on weighted network congestion games with time delay via state-flipped control

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Nowadays, a novel matrix product called the semi-tensor product (STP) of matrices has been proposed [1] for the representation and analysis of Boolean networks (BNs) and Boolean control networks (BCNs). This methodology has enabled breakthroughs in the theory of BNs/BCNs. Recently, Ref. [2] investigated the demand-side management and control of smart grids based on weighted network congestion games via the theory of potential games. In fact, time delays are a very common phenomenon, occurring in processes such as drug therapy, chemical reactions, long transmission lines in pneumatic systems, and evolutionary games. Naturally, an idea arises that we can consider optimal control of smart grids based on weighted network congestion games with time delay. This work will enrich not only the optimal control theory but also the theory of potential games.

Note that extending the existing results to systems involving time delay is by no means an easy task. In this study, we explore this problem by the method of block logical dynamical systems [3] and the theory of invariant subsets for Boolean control networks. All preliminaries are in Appendix A.

Mathematical model description. For the case of smart grids with two-layer power companies [2] as shown in Figure A1, the potential function has not been provided, and no further research has been conducted.

Firstly, smart grids with two-layer power companies can be modeled as a weighted network congestion game. Let E_0, E_1 and E_2 denote the sets of edges from the power station to the first layer companies, from the first layer companies to the second layer companies and from the second layer companies to the communities, respectively. Thus, the strategy of any community $i \in N$ can be represented as $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$ and $e_{0a} \in E_0, e_{aa'} \in E_1, e_{a'i} \in E_2$.

According to Definition A7, the payoff function is

$$c_i(s) = p_{e_{0a}}(w(s, e_{0a})) + p_{e_{aa'}}(w(s, e_{aa'})) + p_{e_{a'i}}(w(s, e_{a'i})), \quad (1)$$

where $s = \{s_1, s_2, \ldots, s_n\} \in S$ is a profile of the game, $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$ is the strategy of the community i, $e_{0a} \in E_0, e_{aa'} \in E_1, e_{a'i} \in E_2$ and $i \in N$.

Potential function. We present the definitions of external intersecting for two players i and j and internal intersecting in [2]. Furthermore, sufficient conditions for the weighted network game to be a potential game are established in Proposition A2. Based on that, we propose a feasible potential function as follows.

Theorem 1. When the weighted network congestion game is a potential game, its potential function can be formulated as

$$\begin{aligned} P(s) = &\sum_{e_{0a}\in E_0} \sum_{j\in \mathcal{M}_{s,e_{0a}}} \sum_{\lambda=1}^{|\mathcal{M}_{s,e_{0a}}|} [p_{e_{0a}}(\lambda w_j)/|\mathcal{M}_{s,e_{0a}}|] \\ &+ \sum_{e_{aa'}\in E_1} \sum_{j\in \mathcal{M}_{s,e_{aa'}}} \sum_{\lambda=1}^{|\mathcal{M}_{s,e_{aa'}}|} [p_{e_{aa'}}(\lambda w_j)/|\mathcal{M}_{s,e_{aa'}}|] \\ &+ \sum_{e_{a'i}\in s_i\cap E_2} p_{e_{a'i}}(w_i), \quad i \in N. \end{aligned} \quad (2)$$

The proof is in Appendix B.

Evolutionary game. Consider the potential function $P(s)$ as the objective function. Each community decides in a dynamic environment with available information about their neighbors. However, it takes time for information receiving and statistics, so it is inevitable that there is time delay in the strategy update rule of communities.

Assuming that $\tau \in Z^+$ is the constant time delay and the strategy update rule adopted by each community is the parallel myopic best response adjustment with τ delay (τ-MBRA) as follows:

$$s_i(t+1) = \begin{cases} s_i(t-\tau), & s_i(t-\tau) \in O_i(t-\tau), \\ \min O_i(t-\tau), & s_i(t-\tau) \notin O_i(t-\tau), \end{cases} \quad (3)$$

where $O_i(t-\tau) = \{s_i \in S_i | \operatorname{argmin}_{s_i\in S_i} c_i(s_i, s_{-i}(t-\tau))\}$ is the best response strategy set, $\min O_i(t-\tau)$ refers to a predefined ordering of the set S_i by strategy index and $s_{-i} \in S_{-i} := \prod_{j\neq i} S_j$ is a profile composed of the strategies of all other players except player i and $i \in N$.

Then the system evolves according to the following k-value logical dynamic network with time delay

$$s_i(t+1) = f_i(s(t-\tau)), \ i \in N, \quad (4)$$

where $s(t-\tau) = \{s_1(t-\tau), s_2(t-\tau), \ldots, s_n(t-\tau)\}$ is the profile at time $t-\tau$ and f_i is the strategy update rule in (3).

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Identify $s_i = s^i_j \in S_i$ as $\delta^j_{k_i} \in \Delta_{k_i}$, the algebraic form of system (4) can be obtained as [1]

$$s(t+1) = Ls(t-\tau), \tag{5}$$

where $s(t) = \ltimes_{i=1}^n s_i(t) \in \Delta_\kappa$, $\kappa := \prod_{i=1}^n k_i$ is the profile at time t and $L \in \mathcal{L}_{\kappa\times\kappa}$ is the profile transition matrix.

By leveraging the theory of block logical dynamical systems (BLDSs) in [3], we can derive an augmented system for system (5).

Let $z(t) = [s(t)^\top s(t-1)^\top \cdots s(t-\tau)^\top]^\top \in \mathcal{N}_{\tau+1,\kappa}$. Then, system (5) can be converted into the following form:

$$\begin{aligned} z(t+1) &= [s(t+1)^\top s(t)^\top \cdots s(t-\tau+1)^\top]^\top \\ &= \begin{bmatrix} \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & L \\ I_\kappa & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \dots & I_\kappa & \mathbf{0}_{\kappa\times\kappa} \end{bmatrix} \begin{bmatrix} s(t) \\ s(t-1) \\ \vdots \\ s(t-\tau) \end{bmatrix} \\ &:= Mz(t), \end{aligned} \tag{6}$$

where $M \in \mathcal{L}_{(\tau+1)\kappa\times(\tau+1)\kappa}$ is a block logical matrix and states $z(t), t \geqslant -\tau$ belong to the finite set $\mathcal{N}_{\tau+1,\kappa}$. Therefore, system (6) is called a BLDS.

Next, we introduce the state-flipped control to BLDS (6). In fact, at time t, we can only flip profile $s(t)$. The states $s(t-1)$, $s(t-2)$, …, $s(t-\tau)$ have been flipped and recorded as $\hat{s}(t-1)$, $\hat{s}(t-2)$, …, $\hat{s}(t-\tau)$. Especially, $\hat{s}(i) = s(i)$ holds, for any $i \in \{-\tau, -\tau+1, \dots, -1\}$. In addition, not all players can be controlled in practical problems. Thus, we give a subset $V \subseteq \mathcal{N}$ in advance, and the communities in it can be controlled.

Let $\hat{z}(t) = [s(t)^\top \hat{s}(t-1)^\top \cdots \hat{s}(t-\tau)^\top]^\top \in \mathcal{N}_{\tau+1,\kappa}$. Then the BLDS (6) under state-flipped control can be obtained

$$\begin{aligned} \hat{z}(t+1) &= \begin{bmatrix} \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & L \\ G_F^{\circlearrowleft^l} & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \dots & I_\kappa & \mathbf{0}_{\kappa\times\kappa} \end{bmatrix} \begin{bmatrix} s(t) \\ \hat{s}(t-1) \\ \vdots \\ \hat{s}(t-\tau) \end{bmatrix} \\ &:= M(G_F^{\circlearrowleft^l})\hat{z}(t), \end{aligned} \tag{7}$$

where $M(G_F^{\circlearrowleft^l}) \in \mathcal{L}_{\kappa(\tau+1)\times\kappa(\tau+1)}$ remains a block logical matrix and system (7) is a BLDS. But now $M(G_F^{\circlearrowleft^l})$ exhibits time-varying dependence on the state-flipped matrix $G_F^{\circlearrowleft^l}$ selection at time t, where $G_F^{\circlearrowleft^l}$ is defined in Definition A5, $F = \{i_1, i_2, \dots, i_{|F|}\} \subseteq N$ and $l := \{l_1, l_2, \dots, l_{|F|}\}$ are the flipped set and the number of rotations, respectively.

According to the combinatorial flipped matrix $\mathcal{G}_V \in \mathcal{B}_{\kappa\times\kappa}$ defined in Definition A6, we can obtain a new representation of system (6) under state-flipped control as follows:

$$\begin{aligned} A(t+1) &= \begin{bmatrix} \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & L \\ \mathcal{G}_V & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \dots & I_\kappa & \mathbf{0}_{\kappa\times\kappa} \end{bmatrix} \begin{bmatrix} a(t) \\ \hat{a}(t-1) \\ \vdots \\ \hat{a}(t-\tau) \end{bmatrix} \\ &:= M(\mathcal{G}_V)A(t), \end{aligned} \tag{8}$$

where transition matrix $M(\mathcal{G}_V) \in \mathcal{B}_{\kappa(\tau+1)\times\kappa(\tau+1)}$, state $A(t) = [a(t)^\top \hat{a}(t-1)^\top \cdots \hat{a}(t-\tau)^\top]^\top$, $\hat{a}(t) = \mathcal{G}_V a(t) \in \mathbb{R}_{\kappa\times 1}$ and $a(i) = x(i), i \in \{-\tau, -\tau+1, \dots, 0\}$.

Stability and stabilization. Next, equivalent conditions for global stabilization of the system (5) under state-flipped control will be given.

Note that for any $F \subseteq V$, $G_F^{\circlearrowleft^l} \sqsubseteq \mathcal{G}_V$ and $[M(G_F^{\circlearrowleft l})]^t \sqsubseteq [M(\mathcal{G}_V)]^t$ hold, where $M(G_F^{\circlearrowleft l})$ and $M(\mathcal{G}_V)$ are defined in equation (7), $t \in Z^+$ and the relation "$\sqsubseteq$" is defined in Appendix A.

Theorem 2. System (5) under state-flipped control is globally stabilized to $s^* = \delta_\kappa^\theta \in \Delta_\kappa$ if and only if system (7) is S-stabilizable, where set $S = \{\hat{z} = \delta_\kappa^{\theta,\theta_1,\dots,\theta_\tau} \in \mathcal{N}_{\tau+1,\kappa} | \, \delta_\kappa^{\theta_i} \in \mathcal{R}(\delta_\kappa^\theta), i \in \{1,2,\dots,\tau\}\}$.

The proof is in Appendix C.

Theorem 3. Given target state $\delta_\kappa^\theta \in \Delta_\kappa$, the state set $S = \{\hat{z} = \delta_\kappa^{\theta,\theta_1,\dots,\theta_\tau} \in \mathcal{N}_{\tau+1,\kappa} | \, \delta_\kappa^{\theta_i} \in \mathcal{R}(\delta_\kappa^\theta), i \in \{1,2,\dots,\tau\}\}$ can be derived from (5). System (7) is S-stabilizable if and only if

(1) The (θ,θ)-th entry of matrix $L\mathcal{G}_V$ is positive, that is, $(L\mathcal{G}_V)_{\theta\theta} > 0$ holds;

(2) $\mathrm{Row}_\theta[(L\mathcal{G}_V)^\kappa] > \mathbf{0}_{1\times\kappa}$,

where $L \in \mathcal{L}_{\kappa\times\kappa}$ is the transition matrix of system (5) and $\mathcal{G}_V$ is the combinatorial flipped matrix of set V.

The proof is in Appendix D.

Remark 1. The computational complexity measures in this article are all $O(\kappa)$.

Example. Consider a smart grid with two-layer power companies in Figure E1. First, it can be modeled as a weighted network congestion game, denoted by $G = \{N, (O,E), S, W, C\}$. The strategy sets are $S_1 = \{\{e_{01}, e_{11}, e_{11}\}, \{e_{02}, e_{22}, e_{21}\}, \{e_{02}, e_{23}, e_{11}\}\}$, $S_2 = \{\{e_{01}, e_{13}, e_{32}\}, \{e_{02}, e_{21}, e_{12}\}\}$, $S_3 = \{\{e_{01}, e_{11}, e_{13}\}, \{e_{01}, e_{12}, e_{23}\}\}$, $S_4 = \{\{e_{01}, e_{13}, e_{34}\}, \{e_{02}, e_{21}, e_{14}\}\}$, $S_5 = \{\{e_{01}, e_{11}, e_{35}\}\}$, respectively. Based on Proposition A2, if the weights of the players satisfy the following equations: $w_1 = w_2 = w_3 = w_4 = w_5$, then the game G is a potential game. Without loss of generality, suppose that $w_1 = w_2 = w_3 = w_4 = w_5 = 1$ holds. Then, the price function p_e related to different weights on the edge e in Figure E1 is listed. More details can be found in Appendix E.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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