Appendix A Preliminaries

Notation.

- Z^+ denotes the set of positive integers;
- $\Delta_n := \{\delta_n^i \,|\, i = 1, 2, \cdots, n\}$, where δ_n^i is the i-th column of the n-dimensional identity matrix I_n;
- An $n \times t$ matrix M is called a logical matrix, if $M = [\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$ and denote M briefly by $M = \delta_n[i_1\ i_2\ \cdots\ i_t]$;
- $\mathbb{R}_{m\times n}$, $\mathcal{L}_{m\times n}$, $\mathcal{B}_{m\times n}$ denote the set of $m \times n$ real matrices, logical matrices and Boolean matrices, respectively;
- $\mathcal{R}(\delta_\kappa^\theta) := \{x \in \Delta_\kappa | Lx = \delta_\kappa^\theta\}$ denote the one-step reachable set of δ_κ^θ, where $L \in \mathcal{L}_{\kappa\times\kappa}$ is the transition matrix of the logical system;
- $\mathbf{1}_{m\times n}$ and $\mathbf{0}_{m\times n}$ denote the $m \times n$-dimensional matrix with all entries being 1 and 0, respectively;
- $\delta_k^{i_1,i_2,\cdots,i_p} := [(\delta_k^{i_1})^\top (\delta_k^{i_2})^\top \cdots (\delta_k^{i_p})^\top]^\top \in \mathcal{N}_{p,k}$, where "$^\top$" is the transpose operator;
- $Col_i(L)$ ($Row_i(L)$) is the i-th column (row) of matrix L; $Col(L)$ ($Row(L)$) is the set formed by all columns (rows) of matrix L;
- $(L)_{ij}$ is the (i, j) -th entry in the matrix L;
- $[L]_{\mathcal{B}}$ represents converting the matrix L into a Boolean matrix, which implies that $([L]_{\mathcal{B}})_{ij} = 1$ iff $(L)_{ij} \geqslant 1$, otherwise $([L]_{\mathcal{B}})_{ij} = 0$;
- $sgn(\cdot)$ is the sign function, that is, $sgn(t) = 1$ iff $t > 0$; $sgn(t) = 0$ iff $t = 0$; $sgn(t) = -1$ iff $t < 0$;
- Two matrices $A, B \in \mathbb{R}_{m\times n}$, $A \sqsubseteq B$ means $A_{ij} \leqslant B_{ij}$ holds, for any $i = 1, 2, \cdots m$ and $j = 1, 2, \cdots n$;
- 2^A and $|A|$ represent the power set of the set A and its cardinality, respectively;
- "$\ltimes$" denotes the semi-tensor product of matrices, which can be omitted without confusion in this paper;
- e_{ab} denotes the directed edge from node a to node b.

Definition 1. [3] [Block logical matrix] Matrix

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \ldots & M_{n,n} \end{bmatrix} \in \mathcal{B}_{mn\times mn}$$

is called a block logical matrix, if for any $i \in \{1, 2, \cdots, n\}$, there exists a unique integer $i_* \in \{1, 2, \cdots, n\}$ such that $M_{i,i_*} \in \mathcal{L}_{m\times m}$ and $M_{i,j} = \mathbf{0}_{m\times m}$, for any $j \neq i_*$.

Note that if M and N are block logical matrices, then MN is also a block logical matrix. Especially, M^t is a block logical matrix for any $t \in Z^+$. We thus obtain the following representation for the block logical matrix M^t can also be the form as follows:

$$M^t = \begin{bmatrix} M_{1,1}^{(t)} & M_{1,2}^{(t)} & \cdots & M_{1,n}^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1}^{(t)} & M_{n,2}^{(t)} & \ldots & M_{n,n}^{(t)} \end{bmatrix} \in \mathcal{B}_{mn\times mn}.$$

Definition 2. [3] [Block logical dynamical system] System

$$z(t + 1) = Mz(t) \tag{A1}$$

is called a BLDS, if $z(t) = [z_1(t)^\top z_2(t)^\top \cdots z_n(t)^\top]^\top \in \mathcal{N}_{n,m},\ z_i(t) \in \Delta_m$ and $M \in \mathcal{B}_{mn\times mn}$ is a block logical matrix.

Next, we introduce the state-flipped control on k-value logical dynamic system in [2]. It is different from both state-flipped control on Boolean networks [3,4] and state-circulated control k-value logical system [6]. Actually, it is an extension of the two control methods mentioned above.

Definition 3. [1] An operator $\circlearrowleft$: $\Delta_\kappa \to \Delta_\kappa$, called the rotator, is defined as

$$\circlearrowleft s = \circlearrowleft \delta_\kappa^i = \delta_\kappa^{(i\ mod\ \kappa)+1},$$

where $s = \delta_\kappa^i \in \Delta_\kappa$.

Obviously, the state-circulated control in [6] can be achieved through rotator operator. In addition, let $l = 2$, $\circlearrowleft^l \delta_\kappa^i = \circlearrowleft (\circlearrowleft \delta_\kappa^i) = \circlearrowleft \delta_\kappa^{i'} = \delta_\kappa^{(i'\ mod\ \kappa)+1}$ holds, where $i' = (i\ mod\ \kappa)+1$. Note that for a fixed $s = \delta_\kappa^j \in \Delta_\kappa$ and any $\delta_\kappa^{j'} \in \Delta_\kappa$, there always exists $1 \leqslant l \leqslant \kappa-1$ such that $\circlearrowleft^l \delta_\kappa^j = \delta_\kappa^{j'}$, where l is the number of rotations. Thus, any two states in the finite set Δ_κ can be mutually flipped onto each other.

Definition 4. [2] [State-flipped function] Let $F := \{i_1, i_2, \cdots, i_{|F|}\} \subseteq N$ be the flipped set and $l := \{l_1, l_2, \cdots, l_{|F|}\}$ be the number of rotations corresponding to each node in F . The state-flipped function with respect to F and l, denoted by $g_F^{\circlearrowleft^l}$, is defined as

$$g_F^{\circlearrowleft^l}(s) = s_F^{\circlearrowleft^l} = (s_1, \cdots, \circlearrowleft^{l_1} s_{i_1}, \cdots, \circlearrowleft^{l_{|F|}} s_{i_{|F|}}, \cdots, s_n),$$

where $\circlearrowleft^{l_i} s_{i_j}$ denotes the state of s_{i_j} after l_j rotations, $1 \leqslant l_j \leqslant k_i - 1$ and $s_F^{\circlearrowleft^l}$ can be achieved by flipping the i_1-th, i_2-th, $\cdots$, i_F-th components of s with l times, which is denoted by $s \xrightarrow{g_F^{\circlearrowleft^l}} s_F^{\circlearrowleft^l}$.

Definition 5. [2] [State-flipped matrix] For a flipped set $F = \{i_1, i_2, \cdots, i_{|F|}\} \subseteq N$ and the number of rotations $l := \{l_1, l_2, \cdots, l_{|F|}\}$. The state-flipped matrix, denoted by $G_F^{\circlearrowleft^l} \in \mathcal{L}_{\kappa\times\kappa}$, is defined as

$$Col_j(G_F^{\circlearrowleft^l}) = \delta_\kappa^i, \quad if\ \ \delta_\kappa^j \xrightarrow{g_F^{\circlearrowleft^l}} \delta_\kappa^i,$$

where $j = 1, 2, \cdots, \kappa$.

Definition 6. [2] [Combinatorial flipped matrix] Let $V = \{i_1, i_2, \cdots, i_v\} \subseteq \mathcal{N}$. The combinatorial flipped matrix of V, denoted by $\mathcal{G}_V \in \mathcal{B}_{\kappa\times\kappa}$, is defined as

$$(\mathcal{G}_V)_{ij} = \begin{cases} 1, & if\ \exists\, F \in 2^V,\ l = \{l_1, l_2, \cdots, l_{|F|}\}, \\ & s.t.\ (G_F^{\circlearrowleft^l})_{ij} = 1\ ; \\ 0, & otherwise. \end{cases}$$

Consequently, $\mathcal{G}_V$ is a Boolean matrix and can be calculated by $\mathcal{G}_V = \sum_{F\in 2^V} \sum_{l=\{l_1,l_2,\cdots,l_{|F|}\},\ 1\leqslant l_j\leqslant k_i-1} G_F^{\circlearrowleft^l}$. Moreover, the j-th column of $\mathcal{G}_V$ contains all states reachable from $s = \delta_\kappa^j$ by applying state-flipped control corresponding to set V.

The demand-side management of smart grids refers to optimizing the allocation of resources by changing the choice of the community to minimize the total cost of electricity consumption. The smart grids have been modeled as weighted network congestion game to consider demand-side management problem [2].

Definition 7. [2] [Weighted network congestion game] A weighted network congestion game is a triple $G = \{N, (O, E), S, W, C\}$, where

- $N = \{1, 2, \cdots, n\}$ are n players (communities).
- (O, E) is a directed graph, where O and E are the set of nodes and edges, respectively.
- $S = \prod_{i=1}^n S_i$, where $S_i = \{s_1^i, s_2^i, \cdots, s_{k_i}^i\}$ is the strategy set of play i and s_j^i is a directed path from the power station to the community i.
- $W = \{w_1, w_2, \cdots, w_n\}$ are weights of n communities.
- $C = \{c_1, c_2, \cdots, c_n\}$ is the set of payoff functions, where $c_i : S \to \mathbb{R}$ can be defined by

$$c_i(s) = \sum_{e\in s_i} p_e(w(s, e)), \quad s \in S\ and\ e \in E, \tag{A2}$$

where $w(s, e) = \sum_{j\in\mathcal{M}_{s,e}} w_j$ represents the sum of weights of players choosing edge e under the profile s, $\mathcal{M}_{s,e} = \{j \in N | e \in s_j\}$ is the set of players who select the edge e under the profile s and p_e is the price function at different weights on edge e.

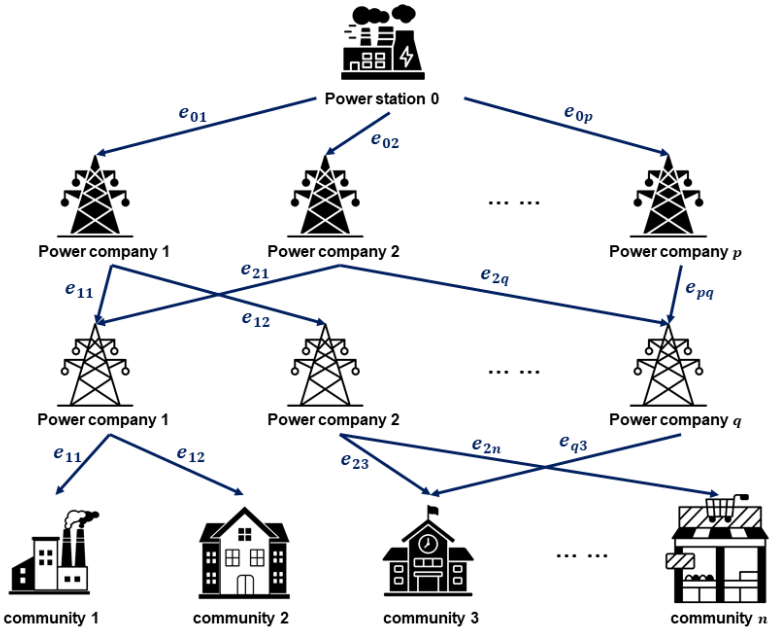

Figure A1 Smart grids with two-layer power companies.

Definition 8. [5] [Potential game] A finite game is a potential game if there exists a function $P : S \to \mathbb{R}$, called the potential function, such that

$$c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}), \quad i \in N,$$

for any two distinct profiles (s_i, s_{-i}) , $(s_i', s_{-i}) \in S$.

Proposition A1. [5] A game is a potential game if and only if for every $i, j \in N$, for every $s_{-\{i,j\}} \in S_{-\{i,j\}} = \prod_{k\neq i,j} S_k$ and for every $s_i,\ s_i' \in S_i$ and $s_j,\ s_j' \in S_j$, the equation $c_i(B) - c_i(A) + c_j(C) - c_j(B) + c_i(D) - c_i(C) + c_j(A) - c_j(D) = 0$ holds, where $A = \{s_i, s_j, s_{-\{i,j\}}\}$, $B = \{s_i', s_j, s_{-\{i,j\}}\}$, $C = \{s_i', s_j', s_{-\{i,j\}}\}$ and $D = \{s_i, s_j', s_{-\{i,j\}}\}$ hold.

The sufficient conditions of the weighted network congestion game with single-layer and multi-layer power companies being a potential game are presented, respectively. For the weighted network congestion game with single-layer power companies modeled as a potential game, the corresponding potential function has been established in [2]. They further investigated how state-flipped control can be applied to drive the system toward the optimal Nash equilibrium.

Definition 9. [2] [External intersecting] Player i and j are external intersecting, if there exist two strategies $s_i \in S_i$, $s_j \in S_j$ and an edge e such that $e \in s_i \cap s_j$ holds. The edge e is called the external intersecting edge.

Definition 10. [2] [Internal intersecting] Player i is called internal intersecting, if there exists an edge e such that $e \in \cap_{j=1}^{k_i} s_j^i$ holds, where k_i denotes the total number of strategies for player i and s_j^i is the j-th strategy among them. The edge e is an internal intersecting edge.

In the following, the sufficient conditions for the weighted network congestion game to be a potential game are presented.

Proposition A2. The weighted network congestion game G with two-layer companies is a potential game if one of the following conditions holds:

(1) All players are not external intersecting;

(2) When there exist two players i and j are external intersecting, their external intersecting edges are also the internal intersecting edges of player i and j;

(3) When there exists an external intersecting edge e_{ab} of player i and j that is not the internal intersecting edge of player i and j, then the weights of i and j are equal, i.e. $w_i = w_j := w_{e_{ab}}$.

Proof.

Case (1): All players are not external intersecting. For any players i and j, choose any $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$, $s_i' = \{e_{0b}, e_{bb'}, e_{b'i}\} \in S_i$,$s_j = \{e_{0c}, e_{cc'}, e_{c'i}\}$, $s_j' = \{e_{0d}, e_{dd'}, e_{d'i}\} \in S_j$ and a fixed profile $s_{-\{i,j\}}$. For simplicity, denote that $A = \{s_i, s_j, s_{-\{i,j\}}\}$, $B = \{s_i', s_j, s_{-\{i,j\}}\}$, $C = \{s_i', s_j', s_{-\{i,j\}}\}$ and $D = \{s_i, s_j', s_{-\{i,j\}}\}$.

By (A2), it is easy to calculate that $c_i(B) - c_i(A) + c_j(C) - c_j(B) + c_i(D) - c_i(C) + c_j(A) - c_j(D) = 0$ holds. Based on Proposition A1, G is a potential game under Case (1).

Case (2): Suppose players i and j satisfy Case (2). For any $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$, $s_i' = \{e_{0a}, e_{ab'}, e_{b'i}\} \in S_i$,$s_j = \{e_{0a}, e_{ac'}, e_{c'i}\}$, $s_j' = \{e_{0a}, e_{ad'}, e_{d'i}\} \in S_j$ and a fixed profile $s_{-\{i,j\}}$, we have players i and j are external intersecting, their external intersecting edge e_{0a} is also the internal intersecting edges of player i and j.

Let w_{-1}, w_{-2}, w_{-3}, w_{-4} and w_{-5} be the total weights of edges $e_{0a} \in s_{-\{i,j\}}$, $e_{aa'} \in s_{-i}$, $e_{ab'} \in s_{-i}$, $e_{ac'} \in s_{-j}$ and $e_{ad'} \in s_{-j}$, respectively.

According to (A2), we have

$$
\begin{aligned}
c_i(B) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ab'}}(w_{-3} + w_i) + p_{e_{b'i}}(w_i),\\
c_i(A) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{aa'}}(w_{-2} + w_i) + p_{e_{a'i}}(w_i),\\
c_j(C) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ad'}}(w_{-5} + w_j) + p_{e_{d'i}}(w_j),\\
c_j(B) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ac'}}(w_{-4} + w_j) + p_{e_{c'i}}(w_j),\\
c_i(D) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{aa'}}(w_{-2} + w_i) + p_{e_{a'i}}(w_i),\\
c_i(C) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ab'}}(w_{-3} + w_i) + p_{e_{b'i}}(w_i),\\
c_j(A) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ac'}}(w_{-4} + w_j) + p_{e_{c'i}}(w_j),\\
c_j(D) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ad'}}(w_{-5} + w_j) + p_{e_{d'i}}(w_j).
\end{aligned}
$$

By simple calculation, we obtain that $c_i(B) - c_i(A) + c_j(C) - c_j(B) + c_i(D) - c_i(C) + c_j(A) - c_j(D) = 0$ holds. Based on Proposition A1, G is a potential game under Case (2).

Case (3): Assume players i and j satisfy Case (3). For any $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$, $s_i' = \{e_{0a}, e_{ab'}, e_{b'i}\} \in S_i$, $s_j = \{e_{0a}, e_{aa'}, e_{a'i}\}$, $s_j' = \{e_{0d}, e_{dd'}, e_{d'i}\} \in S_j$ and a fixed profile $s_{-\{i,j\}}$, we have players i and j are external intersecting, but their external intersecting edges e_{0a} and $e_{aa'}$ are not the internal intersecting edges of j.

Let w_{-1}, w_{-2}, w_{-3}, w_{-4} and w_{-5} be the total weights of edges $e_{0a} \in s_{-\{i,j\}}$, $e_{aa'} \in s_{-\{i,j\}}$, $e_{ab'} \in s_{-i}$, $e_{0d} \in s_{-j}$ and $e_{dd'} \in s_{-j}$, respectively.

Based on the payoff function in (A2), we have

$$
\begin{aligned}
c_i(B) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{ab'}}(w_{-3} + w_i) + p_{e_{b'i}}(w_i),\\
c_i(A) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{aa'}}(w_{-2} + w_i + w_j) + p_{e_{a'i}}(w_i),\\
c_j(C) &= p_{e_{0d}}(w_{-4} + w_j) + p_{e_{dd'}}(w_{-5} + w_j) + p_{e_{d'i}}(w_j),\\
c_j(B) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{aa'}}(w_{-2} + w_j) + p_{e_{a'i}}(w_j),\\
c_i(D) &= p_{e_{0a}}(w_{-1} + w_i) + p_{e_{aa'}}(w_{-2} + w_i) + p_{e_{a'i}}(w_i),\\
c_i(C) &= p_{e_{0a}}(w_{-1} + w_i) + p_{e_{ab'}}(w_{-3} + w_i) + p_{e_{b'i}}(w_i),\\
c_j(A) &= p_{e_{0a}}(w_{-1} + w_i + w_j) + p_{e_{aa'}}(w_{-2} + w_i + w_j) + p_{e_{a'i}}(w_j),\\
c_j(D) &= p_{e_{0d}}(w_{-4} + w_j) + p_{e_{dd'}}(w_{-5} + w_j) + p_{e_{d'i}}(w_j).
\end{aligned}
$$

By simple calculation, we obtain that $c_i(B) - c_i(A) + c_j(C) - c_j(B) + c_i(D) - c_i(C) + c_j(A) - c_j(D) = p_{aa'}(w_{-2} + w_i) - p_{aa'}(w_{-2} + w_i) = 0$, if $w_i = w_j$ holds. Based on Proposition A1, G is a potential game under Case (3).

This completes the proof for Proposition A2. □

Definition 11. [1] System (5) is said to be globally stable at $s^* \in \Delta_\kappa$, if there exists $T \in Z^+$, such that $s(t; s(-\tau), s(-\tau + 1), \cdots, s(0)) = s^*$ holds for any initial state trajectory $\{s(-\tau), s(-\tau + 1), \cdots, s(0)\}$ and any integer $t \geqslant T$.

Definition 12. [3] System (6) is said to be globally stable at $z^* \in \mathcal{N}_{\tau+1,\kappa}$, if there exists $T \in Z^+$ such that $z(t; z(0)) = z^*$ holds for any initial state $z(0)$ and any integer $t \geqslant T$.

Definition 13. Given a flipped set $V \subseteq N$, system (5) under state-flipped control is said to be globally stabilized to $s^* \in \Delta_\kappa$, if for arbitrary initial trajectory $\{s(-\tau), s(-\tau + 1), \cdots, s(0)\}$, there exist $T \in Z^+$, a flipped set sequence $\Gamma = \{F_0, F_1, \cdots, F_t\}$ and the number of rotations $\Lambda := \{l(0), l(1), \cdots, l(t)\}$ such that $s(t; s(-\tau), s(-\tau + 1), \cdots, s(0)) = s^*$ holds for any $t \geqslant T$, where $F_t \subseteq V$ and $l(t) = \{l_j(t)\}_{j=1}^{|F_t|}$, $l_j(t) \in Z^+$ are the flipped set and the number of rotations at time t, respectively.

Note that the one-step reachable set of state δ_κ^θ for system (5) is $\mathcal{R}(\delta_\kappa^\theta) := \{x \in \Delta_\kappa | Lx = \delta_\kappa^\theta\}$. Construct a state set for system (7), denoted by $S = \{\hat{z} = \delta_\kappa^{\theta,\theta_1,\cdots,\theta_\tau} \in \mathcal{N}_{\tau+1,\kappa} | \delta_\kappa^{\theta_i} \in \mathcal{R}(\delta_\kappa^\theta), i \in \{1,2,\cdots,\tau\}\}$. Next, based on the definition of set stabilization in [2], we present the definition of stabilization of system (7) to the given set S under state-flipped control, which is also referred to as S-stabilization.

Definition 14. [Set Stabilization] Given $S \subseteq \mathcal{N}_{\tau+1,\kappa}$ and a flipped set $V \subseteq N$, system (7) is said to be S-stabilizable, if for arbitrary initial state $z(0)$, there exist $T \in Z^+$, a flipped set sequence $\Gamma = \{F_j\}_{j=0}^t$ and times sequence $\Lambda = \{l(j)\}_{j=0}^t$ such that $\hat{z}(t; z(0)) \in S$ holds for any $t \geqslant T$, where $F_t \subseteq V$ and $l(t) = \{l_j(t)\}_{j=1}^{|F_t|}$, $l_j(t) \in Z^+$ are the flipped set and the number of rotations at time t, respectively.

Lemma A1. Given nonnegative matrices $\mathbf{0}_{m\times n} \sqsubseteq A, B \in \mathbb{R}_{m\times n}$ and $\mathbf{0}_{p\times q} \sqsubseteq C, D \in \mathbb{R}_{p\times q}$, if $A \sqsubseteq B$ and $C \sqsubseteq D$ hold, then we can get that $AC \sqsubseteq BD$ holds.

Appendix B Proof of Theorem 1

Proof. For Theorem 1, it suffices to prove that $c_i(s_i, s_{-i}) - c_i(s_i^{'}, s_{-i}) = P(s_i, s_{-i}) - P(s_i^{'}, s_{-i})$ holds for any $i \in N$ and every two distinct profiles (s_i, s_{-i}) , $(s_i^{'}, s_{-i}) \in S$.

For any player $i \in N$, let $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$, $s_i' = \{e_{0b}, e_{bb'}, e_{b'i}\} \in S_i$ be strategies of player i. Denote n_1, n_2, n_3 and n_4 as the number of players selecting the edges e_{0a}, e_{0b}, $e_{aa'}$ and $e_{bb'}$ in s_{-i}, respectively.

The proof can be divided into two cases.

Case 1: All players are not external intersecting. Hence, we have that $n_1 = n_2 = n_3 = n_4 = 0$ holds. According to Equation (A2) and Equation (2), we conclude

$$\begin{aligned}
&P(s_i, s_{-i}) - P(s_i', s_{-i})\\
=&p_{e_{0a}}(w_i) + p_{e_{aa'}}(w_i) + p_{e_{a'i}}(w_i) -\\
&[p_{e_{0b}}(w_i) + p_{e_{bb'}}(w_i) + p_{e_{b'i}}(w_i)]\\
=&c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}).
\end{aligned}$$

Case 2: At least one of n_1, n_2, n_3 or n_4 is nonzero, which implies that at least one other player is externally intersected with player i. Assume that $n_1 > 0$ and $n_2 = n_3 = n_4 = 0$ hold. Therefore, we have e_{0a} is the external intersecting edge. Then, there are two subcases as follows.

Subcase (1): The edge e_{0a} is also the internal intersecting edge of all players choosing the same edge e_{0a}. That is $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$ and $s_i' = \{e_{0a}, e_{ab'}, e_{b'i}\} \in S_i$ hold. According to Equation (A2) and Equation (2), we conclude

$$\begin{aligned}
&P(s_i, s_{-i}) - P(s_i', s_{-i})\\
=&p_{e_{aa'}}(w_i) + p_{e_{a'i}}(w_i) - [p_{e_{ab'}}(w_i) + p_{e_{b'i}}(w_i)]\\
=&c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}).
\end{aligned}$$

Subcase (2): The edge e_{0a} is not the internal intersecting edge of all players choosing the same edge e_{0a}. That is $s_i = \{e_{0a}, e_{aa'}, e_{a'i}\}$ and $s_i' = \{e_{0b}, e_{bb'}, e_{b'i}\} \in S_i$ hold. According to the Proposition A2, the weights of all players choosing the same edge e_{0a} should be equal, denoted by $w_{e_{0a}}$. By Equation (A2) and Equation (2), we conclude

$$\begin{aligned}
&P(s_i, s_{-i}) - P(s_i', s_{-i})\\
=&p_{e_{0a}}((n_1+1)w_{e_{0a}}) + p_{e_{aa'}}(w_{e_{0a}}) + p_{e_{a'i}}(w_{e_{0a}}) -\\
&[p_{e_{0b}}(w_{e_{0a}}) + p_{e_{bb'}}(w_{e_{0a}}) + p_{e_{b'i}}(w_{e_{0a}})]\\
=&c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}).
\end{aligned}$$

This completes the proof for Theorem 1. □

Appendix C Proof of Theorem 2

Proof. [Necessity] Based on Definition 13, system (5) under state-flipped control is globally stabilized to $s^* = \delta_\kappa^\theta$, that is, for any initial trajectory $\{s(i)\}_{i=-\tau}^0$, there exist $T \in Z^+$ and $T \geqslant \tau$, a flipped set sequence $\Gamma = \{F_j\}_{j=0}^t$ and times sequence $\Lambda = \{l(j)\}_{j=0}^t$ such that $s(t; \{s(i)\}_{i=-\tau}^0) = s^*$ holds for any $t \geqslant T$. Let $\lambda \geqslant 0$ holds, we have that

$$\hat{z}(T+\lambda+1) = M(G_{F_{T+\lambda}}^{\circlearrowleft l(T+\lambda)})\hat{z}(T+\lambda) = \begin{bmatrix} L\hat{s}(T+\lambda-\tau)\\ \hat{s}(T+\lambda)\\ \vdots\\ \hat{s}(T+\lambda+1-\tau)\end{bmatrix}.$$

Thus, $L\hat{s}(T+\lambda-\tau) = \delta_\kappa^\theta$ holds, for any $\lambda \geqslant 0$.

To sum up, We can obtain that for any $t \geqslant T$ and any $z(0) \in \mathcal{N}_{\tau+1,\kappa}$, $\hat{z}(t; z(0)) = \delta_\kappa^{\theta,\theta_1,\cdots,\theta_\tau}$ holds, where $\delta_\kappa^{\theta_i} \in \mathcal{R}(\delta_\kappa^\theta)$ and $i \in \{1,2,\cdots,\tau\}$, that is, $\hat{z}(t; z(0)) \in S$ By Definition 14, system (7) is S-stabilizable.

[Sufficiency] Since system (7) is S-stabilizable, we have that for any initial state $z(0) \in \mathcal{N}_{\tau+1,\kappa}$, there exist $T \in Z^+$, $T \geqslant \tau$, a flipped set sequence $\Gamma = \{F_j\}_{j=0}^t$ and times sequence $\Lambda = \{l(j)\}_{j=0}^t$ such that $\hat{z}(t; z(0)) = [s(t)^\top \hat{s}(t-1)^\top \cdots \hat{s}(t-\tau)^\top]^\top = \delta_\kappa^{\theta,\theta_1,\cdots,\theta_\tau}$ holds for any $t \geqslant T$. Thus, we can obtained that $s(t; \{s(i)\}_{i=-\tau}^0) = \delta_\kappa^\theta$ holds for any $t \geqslant T$ and any initial trajectory $\{s(i)\}_{i=-\tau}^0$, where $s(i) \in \Delta_\kappa$. By Definition 13, system (5) is globally stabilized to $s^* = \delta_\kappa^\theta$ under state-flipped control. □

Appendix D Proof of Theorem 3

Proof. [Necessity] Since system (7) is S-stabilizable, by Definition 14, we have for any initial state $z(0) \in \mathcal{N}_{\tau+1,\kappa}$, there exist $T \in Z^+$, a flipped set sequence $\Gamma = \{F_j\}_{j=0}^t$ and times sequence $\Lambda = \{l(j)\}_{j=0}^t$ such that $\hat{z}(t; z(0)) \in S$ holds for any $t \geqslant T$.

On the one hand, for the given T and τ, it can be known that $\hat{z}(T+\tau+1; z(0)) = [\prod_{t=T+\tau}^{T} M(G_{F_t}^{\circlearrowleft l(t)})]\hat{z}(T)$ holds. According to Lemma A1 and some simple calculations, we can obtain that $[\prod_{t=T+\tau}^{T} M(G_{F_t}^{\circlearrowleft l(t)})] \sqsubseteq [M(\mathcal{G}_V)]^{\tau+1}$ and

$$[M(\mathcal{G}_V)]^{\tau+1} = \begin{bmatrix} L\mathcal{G}_V & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \mathbf{0}_{\kappa\times\kappa} & \mathcal{G}_V L & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathcal{G}_V L \end{bmatrix}.$$

Since $\hat{z}(T+\tau+1; z(0)) = \hat{z}(T; z(0)) = z^*$ holds, it follows that $[\prod_{t=T+\tau}^{T} M(G_{F_t}^{\circlearrowleft l(t)})]_{\theta\theta} > 0$ and $[L\mathcal{G}_V]_{\theta\theta} > 0$ hold.

On the other hand, it can be calculated that

$$[M(\mathcal{G}_V)]^{\kappa(\tau+1)} = \begin{bmatrix} [L\mathcal{G}_V]^\kappa & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \mathbf{0}_{\kappa\times\kappa} & [\mathcal{G}_V L]^\kappa & \cdots & \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times\kappa} & \cdots & \mathbf{0}_{\kappa\times\kappa} & [\mathcal{G}_V L]^\kappa \end{bmatrix},$$

where matrix $L \in \mathcal{L}_{\kappa\times\kappa}$ holds.

Then, for any positive integer $p \geqslant \kappa$, it holds that $Col([(L\mathcal{G}_V)^\kappa]_{\mathcal{B}}) = Col([(L\mathcal{G}_V)^p]_{\mathcal{B}})$. Suppose that p is taken sufficiently large such that the inequality $p(\tau+1) \geqslant T$ holds. Let $t = p(\tau+1)+t_1$, where $0 \leqslant t_1 \leqslant \tau$, it can be calculated that the non-zero block matrix in the first row of matrix $[M(\mathcal{G}_V)]^t$ is $[M(\mathcal{G}_V)]_{1,1*}^{(t)} = [L\mathcal{G}_V]^p L^{sgn(t_1)}$ and $Row_\theta([M(\mathcal{G}_V)]_{1,1*}^{(t)}) > 0$ holds. Since $L \in \mathcal{L}_{\kappa\times\kappa}$ is a logic matrix, we have $Col([[M(\mathcal{G}_V)]_{1,1*}^{(t)}]_{\mathcal{B}}) \subseteq Col([(L\mathcal{G}_V)^p]_{\mathcal{B}}) = Col([(L\mathcal{G}_V)^\kappa]_{\mathcal{B}})$. In conclusion, we can obtain the result that $Row_\theta[(L\mathcal{G}_V)^\kappa] > 0$ holds.

The proof of necessity is complete.

[Sufficiency] First, since $(L\mathcal{G}_V)_{\theta\theta} > 0$ and $\mathcal{G}_V = \sum_{F\in 2^V} \sum_{l=\{l_1,l_2,\cdots,l_{|F|}\},\ 1\leqslant l_j\leqslant k_i-1} G_F^{\circlearrowleft l}$ hold, it follows that there exist at least one flipped set F and corresponding number of rotations l, such that $(LG_F^{\circlearrowleft l})_{\theta\theta} = 1 > 0$ holds.

Second, since $Row_\theta[(L\mathcal{G}_V)^\kappa] > 0$ and $\mathcal{G}_V = \sum_{F\in 2^V} \sum_{l=\{l_1,l_2,\cdots,l_{|F|}\},\ 1\leqslant l_j\leqslant k_i-1} G_F^{\circlearrowleft l}$ hold, it follows that for any $x(0) = \delta_\kappa^j \in \Delta_\kappa$, there exist a corresponding flipped set sequence $\Gamma^j = \{F_t^j\}_{t=0}^{\kappa-1}$ and times sequence $\Lambda^j = \{l^j(t)\}_{j=t}^{\kappa-1}$ such that $\prod_{t=\kappa-1}^{0}(LG_{F_t^j}^{\circlearrowleft l^j(t)})\delta_\kappa^j = \delta_\kappa^\theta$.

Let $T = \kappa(\tau+1)$ holds, then for any initial state of system (7) denoted by

$$z(0) = \begin{bmatrix} s(0) \\ s(-1) \\ \vdots \\ s(-\tau) \end{bmatrix} = \begin{bmatrix} \delta_\kappa^{j_0} \\ \delta_\kappa^{j-1} \\ \vdots \\ \delta_\kappa^{j-\tau} \end{bmatrix} \in \mathcal{N}_{\tau+1,\kappa}.$$

According to equation (5), it follows that $s(t+1) = Ls(t-\tau) = \delta_\kappa^{j_{t+1}}$, where $t \in \{0, 1, \cdots, \tau-1\}$.

Next, for the above states $\delta_\kappa^{j_i}$ and $i \in \{0, 1, \cdots, \tau\}$, we can construct the flipped set sequence $\Gamma = \{F_0^{j_0}, F_0^{j_1}, \cdots, F_0^{j_\tau}, \cdots, F_{\kappa-1}^{j_0}, F_{\kappa-1}^{j_1}, \cdots, F_{\kappa-1}^{j_\tau}\}$ and the times sequence $\Lambda = \{l^{j_0}(0), l^{j_1}(0), \cdots, l^{j_\tau}(0), \cdots, l^{j_0}(\kappa-1), l^{j_1}(\kappa-1), \cdots, l^{j_\tau}(\kappa-1)\}$ such that

$$\hat{z}(T; z(0)) = \begin{bmatrix} \prod_{t=\kappa-1}^{0}(LG_{F_t^{j_0}}^{\circlearrowleft l^{j_0}(t)})\delta_\kappa^{j_0} \\ \prod_{t=\kappa-1}^{0}(G_{F_t^{j_\tau}}^{\circlearrowleft l^{j_\tau}(t)} L)\delta_\kappa^{j-1} \\ \vdots \\ \prod_{t=\kappa-1}^{0}(G_{F_t^{j_1}}^{\circlearrowleft l^{j_1}(t)} L)\delta_\kappa^{j-\tau} \end{bmatrix} = \begin{bmatrix} \delta_\kappa^\theta \\ \delta_\kappa^{\theta_1} \\ \vdots \\ \delta_\kappa^{\theta_\tau} \end{bmatrix} \in S,$$

where $L\delta_\kappa^{\theta_i} = \delta_\kappa^\theta$ and $i \in \{1, 2, \cdots, \tau\}$ hold.

Third, suppose that when $t = \gamma > T$, there exist a flipped set sequence $\Gamma = \{F_j\}_{j=0}^{\gamma}$ and times sequence $\Lambda = \{l(j)\}_{j=0}^{\gamma}$ such that $\hat{z}(\gamma; z(0)) = \hat{z}^*$ holds. Next, when $t = \gamma+1$ holds, construct the flipped set sequence $\Gamma = \{F_j\}_{j=0}^{\gamma+1}$ and times sequence $\Lambda = \{l(j)\}_{j=0}^{\gamma+1}$, where $F_{\gamma+1} = F$, $l(\gamma+1) = l$ and $(LG_F^{\circlearrowleft l})_{\theta\theta} > 0$ hold. Thus, it follows that $G_F^{\circlearrowleft l}\delta_\kappa^\theta \in \mathcal{R}(\delta_\kappa^\theta)$ and

$$\hat{z}(\gamma+1; z(0)) = M(G_F^{\circlearrowleft l})\hat{z}(\gamma) = \begin{bmatrix} \delta_\kappa^\theta \\ G_F^{\circlearrowleft l}\delta_\kappa^\theta \\ \delta_\kappa^{\theta_1} \\ \vdots \\ \delta_\kappa^{\theta_{\tau-1}} \end{bmatrix} \in S.$$

In conclusion, by Definition 14, system (7) is S-stabilizable. □

Appendix E Results of example

Consider a smart grid with two-layer power companies in Figure E1. First, it can be modeled as a weighted network congestion game, denoted by $G = \{N, (O, E), S, W, C\}$.

The strategy sets are $S_1 = \{\{e_{01}, e_{11}, e_{11}\}, \{e_{02}, e_{22}, e_{21}\}, \{e_{02}, e_{23}, e_{11}\}\}$, $S_2 = \{\{e_{01}, e_{13}, e_{32}\}, \{e_{02}, e_{21}, e_{12}\}\}$, $S_3 = \{\{e_{01}, e_{11}, e_{13}\}, \{e_{01}, e_{12}, e_{23}\}\}$, $S_4 = \{\{e_{01}, e_{13}, e_{34}\}, \{e_{02}, e_{21}, e_{14}\}\}$, $S_5 = \{\{e_{01}, e_{11}, e_{35}\}\}$, respectively. Based on Proposition A2, if the weights of the players satisfy the following equations: $w_1 = w_2 = w_3 = w_4 = w_5$, then the game G is a potential game. Without loss of generality, suppose that $w_1 = w_2 = w_3 = w_4 = w_5 = 1$ holds. Then, the price function p_e related to different weights on the edge e in Figure E1 in Appendix can be listed as follows (see Table E1).

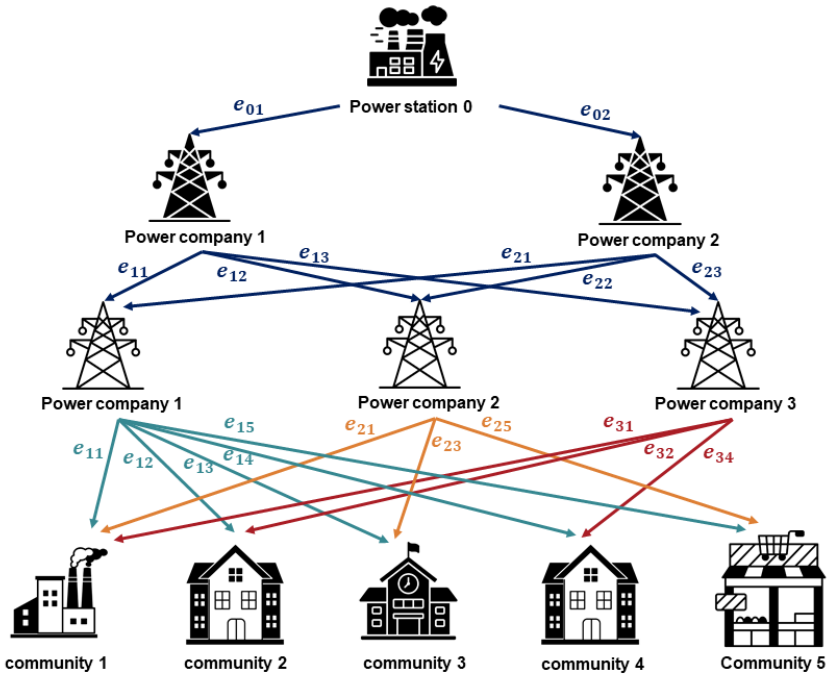

Figure E1 A smart grid with two-layer power companies in Example.

Table E1 Price Table of Example

Edge set	Price	Weights				
		1	**2**	**3**	**4**	**5**
E_0	$p_{e_{01}}$	1	1.5	2	2.5	3
	$p_{e_{02}}$	1.5	2	2.5	2.5	3.5
E_1	$p_{e_{11}}$	1	1.5	2	2.5	3
	$p_{e_{12}}$	1.5	2	2.5	3	4
	$p_{e_{13}}$	1	1.5	2	2.5	3
	$p_{e_{21}}$	1	2	2.5	3	3.5
	$p_{e_{22}}$	1	1.5	2	2.5	3
	$p_{e_{23}}$	1	1.5	2	2.5	3
E_2	$p_{e_{11}}(1) = 0.5$	$p_{e_{21}}(1) = 1$		$p_{e_{31}}(1) = 0.5$		
	$p_{e_{12}}(1) = 1$	$p_{e_{22}}(1) = 1$		$p_{e_{32}}(1) = 1$		
	$p_{e_{13}}(1) = 0.5$	$p_{e_{23}}(1) = 1$		$p_{e_{33}}(1) = 0.5$		
	$p_{e_{14}}(1) = 0.5$	$p_{e_{24}}(1) = 0.5$		$p_{e_{34}}(1) = 0.5$		
	$p_{e_{15}}(1) = 1$	$p_{e_{25}}(1) = 2$		$p_{e_{35}}(1) = 1$		

According to the 2-MBRA rule and matrix tool STP, the logical dynamic form of the above game can be denoted by

$$s(t+1) = Ls(t-2), \tag{E1}$$

where matrix $L = \delta_{24}[22, 18, 22, 18, 22, 18, 22, 1, 22, 18, 22, 18, 21, 17, 21, 1, 22, 18, 22, 18, 21, 17, 21, 1]$.

Let $z(t) = [s(t)^\top s(t-1)^\top s(t-2)^\top]^\top \in \mathcal{N}_{3,24}$, system (E1) can be converted into

$$z(t+1) = \begin{bmatrix} \mathbf{0}_{24\times 24} & \mathbf{0}_{24\times 24} & L \\ I_{24} & \mathbf{0}_{24\times 24} & \mathbf{0}_{24\times 24} \\ \mathbf{0}_{24\times 24} & I_{24} & \mathbf{0}_{24\times 24} \end{bmatrix} \begin{bmatrix} s(t) \\ s(t-1) \\ s(t-2) \end{bmatrix} := Mz(t), \tag{E2}$$

where $M \in \mathcal{L}_{72\times 72}$ is a block logical matrix and states $z(t), t \geqslant -2$ belong to the finite set $\mathcal{N}_{3,24}$.

From the state transition matrix, it can be obtained that δ_{24}^{18} and δ_{24}^{21} are two fixed points of system (E1). According to Theorem 1, we have $P(\delta_{24}^{18}) = 17$ and $P(\delta_{24}^{21}) = 17$ hold. Because the values of their potential function are equal, any one of them can be taken as the optimal Nash equilibrium profile.

Assume that the state δ_{24}^{18} is chosen as the target point and the flipped set $V = \{4\}$. Through some calculations, we have $\mathcal{G}_V = G_{\emptyset}^{\circlearrowleft 1} + G_{\{4\}}^{\circlearrowleft 1} = [\delta_{24}^{1,2}, \delta_{24}^{1,2}, \delta_{24}^{3,4}, \delta_{24}^{3,4}, \delta_{24}^{5,6}, \delta_{24}^{5,6}, \delta_{24}^{7,8}, \delta_{24}^{7,8}, \delta_{24}^{9,10}, \delta_{24}^{9,10}, \delta_{24}^{11,12}, \delta_{24}^{11,12}, \delta_{24}^{13,14}, \delta_{24}^{13,14}, \delta_{24}^{15,16}, \delta_{24}^{15,16}, \delta_{24}^{17,18}, \delta_{24}^{17,18}, \delta_{24}^{19,20}, \delta_{24}^{19,20}, \delta_{24}^{21,22}, \delta_{24}^{21,22}, \delta_{24}^{23,24}, \delta_{24}^{23,24}]$. Furthermore, through some matrix calculations, we obtain that $(L\mathcal{G}_V)_{18,18} = 1 > 0$ and $Row_{18}[(L\mathcal{G}_V)^{24}] = 41943041_{1\times 24} > \mathbf{0}_{1\times 24}$ hold. According to Theorem 2 and 3 , system (E1) can realize global stabilization to the profile $\{\{e_{02}, e_{23}, e_{31}\}, \{e_{01}, e_{13}, e_{32}\}, \{e_{01}, e_{11}, e_{13}\}, \{e_{02}, e_{21}, e_{14}\}, \{e_{01}, e_{11}, e_{15}\}\}$ under state-flipped control.

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