

• Supplementary File •

# Robust stabilization of random impulsive logical dynamical systems via a hybrid-index model

Xinrong YANG<sup>1\*</sup> & Xiaodong LU<sup>2,3\*</sup>

<sup>1</sup>Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China;

<sup>2</sup>School of Automation and Electrical Engineering, University of Science and Technology Beijing, Beijing 100083, China;

<sup>3</sup>Key Laboratory of Knowledge Automation for Industrial Processes of Ministry of Education, Beijing 100083, China

## Appendix A Notations used in the body of this paper

- $\mathbb{N}$ ,  $\mathbb{Z}_+$  and  $\mathbb{R}$  represent the sets of nonnegative integers, positive integers and real numbers, respectively.
- Given  $a, b \in \mathbb{N}$  with  $a < b$ , denote  $\mathbb{N}_a^b := \{i \in \mathbb{N} : a \leq i \leq b\}$  and  $\mathbb{R}_a^b := \{i \in \mathbb{R} : a \leq i \leq b\}$ .
- “ $\times$ ” represents the semi-tensor product of matrices. In this article, we treat “ $\times$ ” as the default matrix product, and often omit it.
- “ $*$ ” represents the Khatri-Rao product of matrix.
- $I_n$  represents the  $n$ -dimensional identity matrix.
- $Col_i(A)$  denotes the  $i$ th column of  $A$ , and  $Col(A)$  denotes the set of column of  $A$ .
- $\Delta_n := Col(I_n)$ ,  $\delta_n^i := Col_i(I_n)$ , and  $\delta_n[i_1 \ i_2 \ \cdots \ i_m] := [\delta_n^{i_1} \ \delta_n^{i_2} \ \cdots \ \delta_n^{i_m}]$ .
- $\mathcal{L}_{n \times m}$ ,  $\mathbb{R}_{n \times m}$  and  $\Upsilon_{n \times m}$  represent the sets of  $n \times m$ -dimensional logical matrices, real matrices and column random matrices, respectively.
- $\mathbf{0}_{n \times m}$  is the  $n \times m$ -dimensional matrix with all elements being 0.
- $\mathbf{1}_n$  and  $\mathbf{0}_n$  represent the  $n$ -dimensional column vectors with all elements being 1 and 0, respectively.

## Appendix B Proofs in the body of the letter

### Appendix B.1 Proof of Proposition 1

We prove this proposition by induction. We obtain from the construction of  $\Phi_J$  that the proposition holds when  $d = 1$ . Assume that

$$\mathbb{P}\left\{x(t, j + d) = \delta_{k^n}^\beta, \prod_{s=0}^{d-1} \theta(t, j + s) = 1 | x(t, j) = \delta_{k^n}^\alpha\right\} = ((\Phi_J)^d)_{\beta, \alpha}$$

holds for certain integer  $d = d'$  with  $d' \geq 1$ . Consider the case of  $d = d' + 1$ . Since  $\prod_{s=0}^{d'} \theta(t, j + s) = 1$  is satisfied if and only if  $\theta(t, j + s) = 1, \forall s \in \mathbb{N}_0^{d'}$ , we have

$$\begin{aligned} & \mathbb{P}\left\{x(t, j + d' + 1) = \delta_{k^n}^\beta, \prod_{s=0}^{d'} \theta(t, j + s) = 1 | x(t, j) = \delta_{k^n}^\alpha\right\} \\ &= \sum_{i'=1}^{k^n} \mathbb{P}\left\{x(t, j + d' + 1) = \delta_{k^n}^\beta, \prod_{s=0}^{d'} \theta(t, j + s) = 1, x(t, j + 1) = \delta_{k^n}^{i'} | x(t, j) = \delta_{k^n}^\alpha\right\} \\ &= \sum_{i'=1}^{k^n} \mathbb{P}\left\{x(t, j + d' + 1) = \delta_{k^n}^\beta, \prod_{s=1}^{d'} \theta(t, j + s) = 1 | x(t, j + 1) = \delta_{k^n}^{i'}\right\} \times \mathbb{P}\{x(t, j + 1) = \delta_{k^n}^{i'}, \theta(t, j) = 1 | x(t, j) = \delta_{k^n}^\alpha\} \\ &= \sum_{i'=1}^{k^n} ((\Phi_J)^{d'})_{\beta, i'} \times (\Phi_J)_{i', \alpha} \\ &= ((\Phi_J)^{d'+1})_{\beta, \alpha}. \end{aligned}$$

To sum up, the conclusion follows.

---

\* Corresponding author (email: xinrongyang2019@163.com, luxiaodong@ustb.edu.cn)

## Appendix B.2 Proof of Proposition 2

Since  $Col_\alpha((\Phi_J)^{k^n}) = (\Phi_J)^{k^n-t} Col_\alpha((\Phi_J)^t)$ ,  $\forall t \in \mathbb{N}_0^{k^n}$ , one obtains from  $Col_\alpha((\Phi_J)^{k^n}) \neq \mathbf{0}_{k^n}$  that  $Col_\alpha((\Phi_J)^t) \neq \mathbf{0}_{k^n}$  holds for any integer  $\forall t \in \mathbb{N}_0^{k^n}$ .

Now, we prove  $Col_\alpha((\Phi_J)^t) \neq \mathbf{0}_{k^n}$ ,  $\forall t > k^n$ . Since  $Col_\alpha((\Phi_J)^{k^n}) \neq \mathbf{0}_{k^n}$ , there exists  $\beta \in \mathbb{N}_1^{k^n}$  such that  $((\Phi_J)^{k^n})_{\beta,\alpha} \neq 0$ . Then, there exist  $\gamma_2, \dots, \gamma_{k^n}$  satisfying

$$(\Phi_J)_{\gamma_{k^n+1}, \gamma_{k^n}} \times \dots \times (\Phi_J)_{\gamma_2, \gamma_1} \neq 0, \quad (B1)$$

where  $\gamma_1 = \alpha$  and  $\gamma_{k^n+1} = \beta$ . According to  $G \in \mathcal{L}_{k^n \times k^n}$  and (4), we conclude that there exists  $i \in \mathbb{N}_1^{k^n}$  such that  $\gamma_{k^n+1} = \gamma_i$ . Denote

$$i^* := \max\{i \in \mathbb{N}_1^{k^n} : \gamma_i = \gamma_{k^n+1}\}.$$

Then, by (B1), for any integer  $t > k^n$  there exists  $\beta' \in \{\gamma_{i^*}, \dots, \gamma_{k^n+1}\}$  satisfying  $((\Phi_J)^t)_{\beta', \alpha} \neq 0$ . Hence, the conclusion follows.

## Appendix B.3 Proof of Theorem 1

(Necessity) We prove the necessity by absurdity. Assume that there exists  $\alpha \in \mathbb{N}_1^{k^n}$  such that  $Col_\alpha((\Phi_J)^{k^n}) \neq \mathbf{0}_{k^n}$ . Then, based on Proposition 2, it holds that  $Col_\alpha((\Phi_J)^t) \neq \mathbf{0}_{k^n}$ ,  $\forall t \in \mathbb{N}$ . Then, we obtain from (5) that

$$\mathbb{P}\{t_k = 0 | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} \neq 0$$

holds for any  $k \in \mathbb{Z}_+$ , any  $\bar{\mathbf{u}} \subseteq \Delta_{k^m}$  and any  $\xi \subseteq \Delta_{k^l}$ , which contradicts with the forward completeness of system (1).

(Sufficiency) We only need to prove that for any state  $\delta_{k^n}^\alpha \in \Delta_{k^n}$ , any state-feedback controller (2), any disturbance sequence  $\xi$  and any integer  $T \geq 0$ , the following condition holds:

$$\mathbb{P}\{t_d > T | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} = 1, \quad \forall d \geq k^n(T+1). \quad (B2)$$

When  $T = 0$ , one sees from  $(\Phi_J)^{k^n} = \mathbf{0}_{k^n \times k^n}$  and (5) that

$$\mathbb{P}\{t_{k^n} = 0 | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} = 0$$

is satisfied for any  $\delta_{k^n}^\alpha \in \Delta_{k^n}$ , any  $\bar{\mathbf{u}} \subseteq \Delta_{k^m}$ , and any  $\xi \subseteq \Delta_{k^l}$ . Therefore, (B2) follows when  $T = 0$ . Now, assume that (B2) holds for integer  $T = T' \geq 0$ . Consider the case of  $T = T' + 1$ . Since  $(\Phi_J)^{k^n} = \mathbf{0}_{k^n \times k^n}$ , we obtain

$$\begin{aligned} & \mathbb{P}\{t_{k^n(T'+2)} = T' + 1 | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} \\ &= \sum_{\beta=1}^{k^n} \mathbb{P}\{t_{k^n(T'+2)} = T' + 1, x(t_{k^n(T'+1)}), j_{k^n(T'+1)} = \delta_{k^n}^\beta, t_{k^n(T'+1)} > T' | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} \\ &= \sum_{\beta=1}^{k^n} \mathbb{P}\{t_{k^n(T'+2)} = T' + 1, x(t_{k^n(T'+1)}), j_{k^n(T'+1)} = \delta_{k^n}^\beta, t_{k^n(T'+1)} = T' + 1 | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} \\ &= \sum_{\beta=1}^{k^n} \mathbb{P}\{t_{k^n(T'+2)} = T' + 1 | x(t_{k^n(T'+1)}), j_{k^n(T'+1)} = \delta_{k^n}^\beta, t_{k^n(T'+1)} = T' + 1, \bar{\mathbf{u}}, \xi\} \\ & \quad \times \mathbb{P}\{x(t_{k^n(T'+1)}), j_{k^n(T'+1)} = \delta_{k^n}^\beta, t_{k^n(T'+1)} = T' + 1 | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} \\ &\leq \sum_{\beta=1}^{k^n} \mathbb{P}\{t_{k^n(T'+2)} = t_{k^n(T'+1)} | x(t_{k^n(T'+1)}), j_{k^n(T'+1)} = \delta_{k^n}^\beta, \bar{\mathbf{u}}, \xi\} \\ &= \sum_{\beta=1}^{k^n} \sum_{i=1}^{k^n} ((\Phi_J)^{k^n})_{i,\beta} = 0, \end{aligned}$$

which together with  $\mathbb{P}\{t_{k^n(T'+2)} > T' | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} = 1$  implies that

$$\mathbb{P}\{t_{k^n(T'+2)} > T' + 1 | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \xi\} = 1.$$

Thus, (B2) holds for  $T = T' + 1$ .

To sum up, the conclusion follows.

## Appendix B.4 Proof of Remark 1

Based (1) and (7), it holds that

$$\begin{aligned} & \mathbb{P}\{z(s+1) = \delta_{k^n}^\beta | z(s) = \delta_{k^n}^\alpha, u_\phi(s) = \delta_{k^m}^\nu, \xi_\phi(s) = \delta_{k^l}^\epsilon\} \\ &= \mathbb{P}\{z(s+1) = \delta_{k^n}^\beta | \theta(s) = 0, z(s) = \delta_{k^n}^\alpha, u_\phi(s) = \delta_{k^m}^\nu, \xi_\phi(s) = \delta_{k^l}^\epsilon\} \times \mathbb{P}\{\theta(s) = 0 | z(s) = \delta_{k^n}^\alpha\} \\ & \quad + \mathbb{P}\{z(s+1) = \delta_{k^n}^\beta | \theta(s) = 1, z(s) = \delta_{k^n}^\alpha, u_\phi(s) = \delta_{k^m}^\nu, \xi_\phi(s) = \delta_{k^l}^\epsilon\} \times \mathbb{P}\{\theta(s) = 1 | z(s) = \delta_{k^n}^\alpha\} \\ &= \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta | \theta(t_s, j_s) = 0, x(t_s, j_s) = \delta_{k^n}^\alpha, u(t_s) = \delta_{k^m}^\nu, \xi(t_s) = \delta_{k^l}^\epsilon\} \times \mathbb{P}\{\theta(t_s, j_s) = 0 | x(t_s, j_s) = \delta_{k^n}^\alpha\} \\ & \quad + \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta | \theta(t_s, j_s) = 1, x(t_s, j_s) = \delta_{k^n}^\alpha\} \times \mathbb{P}\{\theta(t_s, j_s) = 1 | x(t_s, j_s) = \delta_{k^n}^\alpha\} \\ &= \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta, \theta(t_s, j_s) = 0 | x(t_s, j_s) = \delta_{k^n}^\alpha, u(t_s) = \delta_{k^m}^\nu, \xi(t_s) = \delta_{k^l}^\epsilon\} \\ & \quad + \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta, \theta(t_s, j_s) = 1 | x(t_s, j_s) = \delta_{k^n}^\alpha\} \\ &= \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta | x(t_s, j_s) = \delta_{k^n}^\alpha, u(t_s) = \delta_{k^m}^\nu, \xi(t_s) = \delta_{k^l}^\epsilon\}, \quad \forall s \in \mathbb{N}. \end{aligned}$$

## Appendix B.5 Proof of Proposition 3

It is easy to see from Remark 1 that

$$\begin{aligned}
 & \mathbb{P}\{z(s+1) = \delta_{k^n}^\beta | z(s) = \delta_{k^n}^\alpha, u_\phi(s) = \delta_{k^m}^\nu, \xi_\phi(s) = \delta_{k^l}^\varepsilon\} \\
 &= \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta, \theta(t_s, j_s) = 0 | x(t_s, j_s) = \delta_{k^n}^\alpha, u(t_s) = \delta_{k^m}^\nu, \xi(t_s) = \delta_{k^l}^\varepsilon\} \\
 & \quad + \mathbb{P}\{x(t_{s+1}, j_{s+1}) = \delta_{k^n}^\beta, \theta(t_s, j_s) = 1 | x(t_s, j_s) = \delta_{k^n}^\alpha\} \\
 &= (L_\nu^\varepsilon * \mathbf{p})_{\beta, \alpha} + (\Phi_J)_{\beta, \alpha} \\
 &= (\Phi_\nu^\varepsilon)_{\beta, \alpha}.
 \end{aligned}$$

## Appendix B.6 Proof of Theorem 2

Before we show the proof of Theorem 2, we first prove the following fact.

For any initial state  $z(0) := x(0, 0) = \delta_{k^n}^\alpha$  and integer  $s \in \mathbb{N}$ , the following conditions are equivalent:

(a)  $\mathbb{P}\{z(s) = x_e | z(0), \bar{\mathbf{u}}_\phi, \xi_\phi\} = 1$  holds for any  $\xi_\phi := \{\xi_\phi(s)\}_{s \in \mathbb{N}} \subseteq \Delta_{k^l}$ , where  $\bar{\mathbf{u}}_\phi$  represents the control sequence generated by

$$u_\phi(s) = Hz(s); \quad (\text{B3})$$

(b)  $\mathbb{P}\{x(t_s, j_s) = x_e | x(0, 0), \bar{\mathbf{u}}, \xi\} = 1$  holds for any  $\xi \subseteq \Delta_{k^l}$ , where  $\bar{\mathbf{u}}$  represents the control sequence generated by (2).

When  $s = 1$ , similar to the proof of Remark 1, item (a) is equivalent to item (b). Suppose that item (a) is equivalent to item (b) when  $s = d$ . Assume that item (a) holds for  $s = d + 1$ , that is,

$$\mathbb{P}\{z(d+1) = x_e | z(0), \bar{\mathbf{u}}_\phi, \xi_\phi\} = 1, \quad \forall \xi_\phi \subseteq \Delta_{k^l}. \quad (\text{B4})$$

For any  $\xi := \{\xi(0), \xi_1\} \subseteq \Delta_{k^l}$ , we have

$$\mathbb{P}\{x(t_{d+1}, j_{d+1}) = x_e | x(0, 0), \bar{\mathbf{u}}, \xi\} = \sum_{\beta \in \Gamma} \mathbb{P}\{x(t_{d+1}, j_{d+1}) = x_e | x(t_1, j_1) = \delta_{k^n}^\beta, \bar{\mathbf{u}}, \xi\} \times \mathbb{P}\{x(t_1, j_1) = \delta_{k^n}^\beta | x(0, 0), \bar{\mathbf{u}}, \xi(0)\}, \quad (\text{B5})$$

where

$$\Gamma := \{\beta \in \mathbb{N}_1^{k^n} : \mathbb{P}\{x(t_1, j_1) = \delta_{k^n}^\beta | x(0, 0), \bar{\mathbf{u}}, \xi(0)\} > 0\}.$$

We see from Remark 1 and (B3) that  $\mathbb{P}\{x(t_1, j_1) = \delta_{k^n}^\beta | x(0, 0), \bar{\mathbf{u}}, \xi(0)\} = \mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0), \bar{\mathbf{u}}_\phi, \xi_\phi(0)\}$ , which implies

$$\sum_{\beta \in \Gamma} \mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0), \bar{\mathbf{u}}_\phi, \xi_\phi(0)\} = 1$$

and  $\mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0), \bar{\mathbf{u}}_\phi, \xi_\phi(0)\} > 0, \forall \beta \in \Gamma$ . Based on (B4), one concludes  $\mathbb{P}\{z(d+1) = x_e | z(1) = \delta_{k^n}^\beta, \bar{\mathbf{u}}_\phi, \xi_\phi^1\} = 1, \forall \xi_\phi^1 := \{\xi_\phi(1), \dots, \xi_\phi(d)\} \subseteq \Delta_{k^l}, \forall \beta \in \Gamma$ . Then, according to (B5), we have

$$\mathbb{P}\{x(t_{d+1}, j_{d+1}) = x_e | x(0, 0), \bar{\mathbf{u}}, \xi\} = \sum_{\beta \in \Gamma} \mathbb{P}\{x(t_1, j_1) = \delta_{k^n}^\beta | x(0, 0), \bar{\mathbf{u}}, \xi(0)\} = 1.$$

From the arbitrary of  $\xi \subseteq \Delta_{k^l}$ , item (b) holds for  $s = d + 1$ . Similarly, under the case of  $s = d + 1$ , one can derive item (a) when item (b) holds. Therefore, item (a) is equivalent to item (b) for  $s = d + 1$ . To sum up, the conclusion follows by induction.

Now, we prove Theorem 2.

(Necessity) Based on Definition 2 and Theorem 1, we derive that item (i) holds.

Assume that under state-feedback controller (2),

$$\mathbb{P}\{x(t_s, j_s) = x_e | x(0, 0) = x_0, \bar{\mathbf{u}}, \xi\} = 1 \quad (\text{B6})$$

holds for any initial state  $x_0 \in \Delta_{k^n}$ , any disturbance sequence  $\xi \subseteq \Delta_{k^l}$ , and any positive integer  $s \geq d$ . Then, we derive that under state-feedback controller (B3),

$$\mathbb{P}\{z(s) = x_e | z(0) = x_0, \bar{\mathbf{u}}_\phi, \xi_\phi\} = 1 \quad (\text{B7})$$

holds for any initial state  $x_0 \in \Delta_{k^n}$ , any disturbance sequence  $\xi_\phi \subseteq \Delta_{k^l}$  and any positive integer  $s \geq d$ . Thus, from

$$\begin{aligned}
 & \mathbb{P}\{z(d+1) = x_e | z(0) = x_0, \bar{\mathbf{u}}_\phi, \xi_\phi\} \\
 &= \sum_{\alpha=1}^{k^n} \mathbb{P}\{z(d+1) = x_e | z(d) = \delta_{k^n}^\alpha, u_\phi(d), \xi_\phi(d)\} \times \mathbb{P}\{z(d) = \delta_{k^n}^\alpha | z(0) = x_0, \bar{\mathbf{u}}_\phi, \xi_\phi\} \\
 &= \mathbb{P}\{z(d+1) = x_e | z(d) = x_e, u_\phi(d), \xi_\phi(d)\}, \quad \forall x_0 \in \Delta_{k^n}, \forall \xi_\phi \subseteq \Delta_{k^l},
 \end{aligned}$$

we have  $\mathbb{P}\{z(d+1) = x_e | z(d) = x_e, u_\phi(d), \xi_\phi(d)\} = 1, \forall \xi_\phi(d) \in \Delta_{k^l}$ . Therefore,  $R_0(x_e) = \{\eta\}$  is satisfied via Proposition 3.

Now, based on induction, for any  $s \in \mathbb{Z}_+$ , we prove that if  $\mathbb{P}\{z(s) = x_e | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \xi_\phi\} = 1$  holds for any  $\xi_\phi \subseteq \Delta_{k^l}$ , then  $\alpha \in R_s(x_e)$ . If  $\mathbb{P}\{z(1) = x_e | z(0) = \delta_{k^n}^\alpha, u_\phi(0) = \delta_{k^m}^\mu, \xi_\phi(0) = \delta_{k^l}^\varepsilon\} = 1, \forall \varepsilon \in \mathbb{N}_1^{k^l}$ , then, we obtain from Proposition 3 that  $(\Phi_{\mu_\alpha}^\varepsilon)_{\eta, \alpha} = 1, \forall \varepsilon \in \mathbb{N}_1^{k^l}$ . Thus,  $\alpha \in R_1(x_e)$  is satisfied. Assume that  $\mathbb{P}\{z(d') = x_e | z(0) = x_0, \bar{\mathbf{u}}_\phi, \xi_\phi\} = 1, \forall \xi_\phi \subseteq \Delta_{k^l}$  implies  $x_0 \in R_{d'}(x_e)$ . When  $s = d' + 1$ , we derive from

$$\mathbb{P}\{z(d'+1) = x_e | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \xi_\phi\}$$

$$\begin{aligned}
 &= \sum_{\beta=1}^{k^n} \mathbb{P}\{z(d'+1) = x_e | z(1) = \delta_{k^n}^\beta, \bar{\mathbf{u}}_\phi, \boldsymbol{\xi}_\phi^1\} \times \mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \xi_\phi(0)\} \\
 &= 1, \quad \forall \boldsymbol{\xi}_\phi := \{\xi_\phi(0), \boldsymbol{\xi}_\phi^1\} \subseteq \Delta_{k^l}
 \end{aligned}$$

that  $\sum_{\beta \in R_{d'}(x_e)} \mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \xi_\phi(0)\} = 1, \forall \xi_\phi(0) \in \Delta_{k^l}$ . Therefore, based on Proposition 3,

$$\sum_{\beta \in R_{d'}(x_e)} (L\xi_\phi(0)\delta_{k^n}^{\mu_\alpha} * \mathbf{p} + \Phi_J)_{\beta, \alpha} = 1$$

is satisfied for any  $\xi_\phi(0) \in \Delta_{k^l}$ , that is,  $\alpha \in R_{d'+1}(x_e)$ . To sum up, the conclusion follows.

Therefore, from (B7), we have  $R_d(x_e) = \mathbb{N}_1^{k^n}$ , which together with  $R_0(x_e) = \{\eta\}$  implies  $R_{k^n}(x_e) = \mathbb{N}_1^{k^n}$ .

(Sufficiency) If item (i) holds, then we obtain from Theorem 1 that system (1) is forward complete.

Since item (ii) holds, denote

$$d^* := \min\{d \in \mathbb{N}_1^{k^n} : R_d(x_e) = \mathbb{N}_1^{k^n}\}. \quad (\text{B8})$$

Let

$$\Lambda_\eta = \left\{ \nu \in \mathbb{N}_1^{k^m} : \sum_{\varepsilon=1}^{k^l} (\Phi_\nu^\varepsilon)_{\eta, \eta} = k^l \right\}, \quad (\text{B9})$$

and for any  $\alpha \in R_d(x_e) \setminus R_{d-1}(x_e)$ ,  $d \in \mathbb{N}_1^{d^*}$ , let

$$\Lambda_\alpha = \left\{ \nu \in \mathbb{N}_1^{k^m} : \sum_{\varepsilon=1}^{k^l} \sum_{\beta \in R_{d-1}(x_e)} (\Phi_\nu^\varepsilon)_{\beta, \alpha} = k^l \right\}. \quad (\text{B10})$$

Then, we construct the state-feedback controller  $u_\phi(s) = Hz(s)$  with  $H := \delta_{k^m}[\mu_1 \cdots \mu_{k^n}]$ , where  $\mu_\eta \in \Lambda_\eta$  and  $\mu_\alpha \in \Lambda_\alpha$ ,  $\forall \alpha \in \mathbb{N}_1^{k^n} \setminus \{\eta\}$ . According to Proposition 3, we have  $\mathbb{P}\{z(1) \in R_{d-1}(x_e) | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \xi_\phi(0) = \delta_{k^l}^\varepsilon\} = 1, \forall \alpha \in R_d(x_e), \forall \delta_{k^l}^\varepsilon \in \Delta_{k^l}, d \in \mathbb{N}_1^{d^*}$ . Hence, one derives that

$$\mathbb{P}\{z(d) = x_e | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \boldsymbol{\xi}_\phi\} = 1, \quad (\text{B11})$$

holds for any  $\delta_{k^n}^\alpha \in R_d(x_e)$  and any  $\boldsymbol{\xi}_\phi \subseteq \Delta_{k^l}$ . Then, from  $R_0(x_e) = \{\eta\}$ , it holds that

$$\mathbb{P}\{z(d) = x_e | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \boldsymbol{\xi}_\phi\} = 1,$$

$\forall \delta_{k^n}^\alpha \in \Delta_{k^n}, \forall \boldsymbol{\xi}_\phi \subseteq \Delta_{k^l}, \forall k^n \leq d \in \mathbb{Z}_+$ . Therefore, system (1) is robust  $x_e$ -FFSPO.  $\square$

### Appendix B.7 Proof of Theorem 3

First of all, we briefly summarize the process of constructing the feedback gain matrix  $H$  based on Algorithm 1 into the following four steps: (i) Calculate  $R_d(x_e)$ ,  $d \in \mathbb{N}_0^{d^*}$  via (8); (ii) For  $\eta \in R_0(x_e)$ , we construct  $\Lambda_\eta = \left\{ \nu \in \mathbb{N}_1^{k^m} : \sum_{\varepsilon=1}^{k^l} (\Phi_\nu^\varepsilon)_{\eta, \eta} = k^l \right\}$ ; (iii) For  $\alpha \in R_d(x_e) \setminus R_{d-1}(x_e)$ , we construct  $\Lambda_\alpha = \left\{ \nu \in \mathbb{N}_1^{k^m} : \sum_{\varepsilon=1}^{k^l} \sum_{\beta \in R_{d-1}(x_e)} (\Phi_\nu^\varepsilon)_{\beta, \alpha} = k^l \right\}$ ,  $d \in \mathbb{N}_1^{d^*}$ ; (iv) Design the feedback gain matrix  $H$  with  $\text{Col}_\alpha(H) \in \Lambda_\alpha, \forall \alpha \in \mathbb{N}_1^{k^n}$ .

Next, we prove that the state-feedback controller (2) is time-optimal state-feedback controller, if and only if  $\text{Col}_\eta(H) \in \Lambda_\eta$  and  $\text{Col}_\alpha(H) \in \Lambda_\alpha, \forall \alpha \in \mathbb{N}_1^{k^n} \setminus \{\eta\}$ , where  $\Lambda_\eta$  and  $\Lambda_\alpha, \forall \alpha \in \mathbb{N}_1^{k^n} \setminus \{\eta\}$  are shown in Algorithm 1.

(Necessity) Based (B11), for any state-feedback controller  $\mathbf{u}$  designed by Algorithm 1,  $T^*(x_0, \mathbf{u}) \leq d$  holds for any  $x_0 \in R_d(x_e)$ . Suppose that the state-feedback controller (2) is a time-optimal state-feedback controller. Based on Definition 2,  $\text{Col}_\eta(H) \in \Lambda_\eta$  obviously holds. In the sequel, we prove  $\text{Col}_\alpha(H) \in \Lambda_\alpha, \forall \alpha \in \mathbb{N}_1^{k^n} \setminus \{\eta\}$  by absurdity. Assume that there exist  $d \in \mathbb{N}_1^{d^*}$  and  $\alpha \in R_d(x_e) \setminus R_{d-1}(x_e)$  satisfying  $\text{Col}_\alpha(H) \notin \Lambda_\alpha$ . According to  $\text{Col}_\alpha(H) \notin \Lambda_\alpha$  and the construction of  $\Lambda_\alpha$ , there exist  $\varepsilon' \in \mathbb{N}_1^{k^l}$  and  $\beta \notin R_{d-1}(x_e)$  such that

$$\mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \delta_{k^l}^{\varepsilon'}\} > 0.$$

Then, one concludes from  $\beta \notin R_{d-1}(x_e)$  that there exists disturbance sequence  $\boldsymbol{\xi}'_\phi$  satisfying

$$\mathbb{P}\{z(d-1) \neq x_e | z(0) = \delta_{k^n}^\beta, \bar{\mathbf{u}}_\phi, \boldsymbol{\xi}'_\phi\} > 0.$$

Hence, under the disturbance sequence  $\boldsymbol{\xi}''_\phi := \{\delta_{k^l}^{\varepsilon'}, \boldsymbol{\xi}'_\phi\}$ , we have

$$\begin{aligned}
 &\mathbb{P}\{z(d) \neq x_e | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \boldsymbol{\xi}''_\phi\} \\
 &\geq \mathbb{P}\{z(d-1) \neq x_e | z(0) = \delta_{k^n}^\beta, \bar{\mathbf{u}}_\phi, \boldsymbol{\xi}'_\phi\} \times \mathbb{P}\{z(1) = \delta_{k^n}^\beta | z(0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}_\phi, \delta_{k^l}^{\varepsilon'}\} \\
 &> 0,
 \end{aligned}$$

which contradicts with the optimality of the state feedback controller (2). Therefore,  $\text{Col}_\alpha(H) \in \Lambda_\alpha$  holds for any  $\alpha \in \mathbb{N}_1^{k^n} \setminus \{\eta\}$ .

(Sufficiency) Based on (8), we derive  $T^*(\delta_{k^n}^\alpha, \bar{\mathbf{u}}) = d, \forall \alpha \in R_d(x_e) \setminus R_{d-1}(x_e)$ , where feedback gain matrix  $H$  is obtained via Algorithm 1. In order to prove the sufficiency, we only need to prove

$$T^*(\delta_{k^n}^\alpha) = d, \quad \forall \alpha \in R_d(x_e) \setminus R_{d-1}(x_e), \quad \forall d \in \mathbb{N}_1^{d^*}. \quad (\text{B12})$$

We prove (B12) by absurdity. Assume that there exist integers  $s < d \in \mathbb{N}_1^{d^*}$  and  $\alpha \in R_d(x_e) \setminus R_{d-1}(x_e)$  such that  $T^*(\delta_{k^n}^\alpha) = s$ . Accordingly, there exists a state feedback control  $u(t) = H'x(t, j_t^+) \in \mathbf{u}^*$  such that  $\mathbb{P}\{x(t_s, j_s) = x_e | x(0, 0) = \delta_{k^n}^\alpha, \bar{\mathbf{u}}, \boldsymbol{\xi}\} = 1, \forall \boldsymbol{\xi} \subseteq \Delta_{k^l}$ , where  $\bar{\mathbf{u}}$  is the control sequence generated by  $u(t) = H'x(t, j_t^+)$ . Based on (B4) and the proof of necessity in Theorem 2, we derive  $\alpha \in R_s(x_e)$ , which contradicts with  $\alpha \in R_d(x_e) \setminus R_{d-1}(x_e)$ . Therefore, the conclusion follows.

## Appendix C Example

Consider system (1) with 3 state variables, 1 input variable, 1 disturbance variable and  $k = 2$ . Let

$$L = \delta_8[1 \ 1 \ 5 \ 2 \ 3 \ 3 \ 4 \ 4 \ 2 \ 1 \ 5 \ 6 \ 3 \ 1 \ 8 \ 6 \ 1 \ 6 \ 8 \ 6 \ 7 \ 7 \ 2 \ 3 \ 1 \ 1 \ 3 \ 3 \ 1 \ 1 \ 5 \ 5],$$

$G = \delta_8[4 \ 6 \ 2 \ 3 \ 1 \ 1 \ 2 \ 2]$ ,  $\mathbf{p} = [1 \ 0.8 \ 0 \ 1 \ 0 \ 0.6 \ 1 \ 1]$  and  $\mathbf{q} = [0 \ 0.2 \ 1 \ 0 \ 1 \ 0.4 \ 0 \ 0]$ . Denote  $x_e = \delta_8^1$ .

Based on (4), we obtain

$$\Phi_J = G * \mathbf{q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0.4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}_{8 \times 8}.$$

Based on Theorem 1, the considered system is forward complete since  $(\Phi_J)^4 = \mathbf{0}_{8 \times 8}$  holds.

Splitting  $L$  into 4 blocks, we denote  $L := [L_1^1 \ L_2^1 \ L_1^2 \ L_2^2]$ . Then, we have

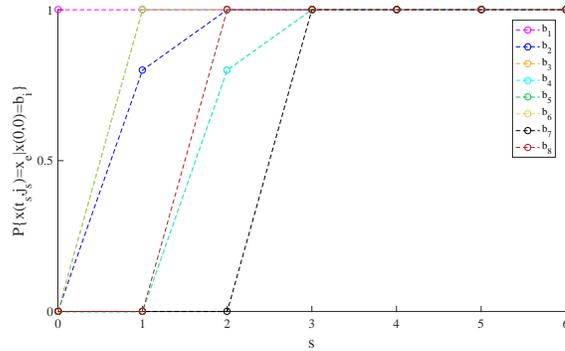
$$\Phi = [L_1^1 * \mathbf{p} + \Phi_J \ L_2^1 * \mathbf{p} + \Phi_J \ L_1^2 * \mathbf{p} + \Phi_J \ L_2^2 * \mathbf{p} + \Phi_J].$$

We omit the details of  $\Phi$  for saving space. According to (8) and  $(L_1^1 * \mathbf{p} + L_2^1 * \mathbf{p})_{1,1} = 2$ , we have  $R_0(x_e) = \{1\}$ . Furthermore, one can calculate  $R_1(x_e) = \{1, 5, 6\}$ ,  $R_2(x_e) = \{1, 2, 5, 6, 8\}$  and  $R_3(x_e) = \mathbb{N}_1^8$ . Then, we derive from Theorem 2 that the considered system is robust  $x_e$ -FFSPO.

Furthermore, according to Algorithm 1, the time-optimal state-feedback controller  $u(t) = Hx(t, j_t^+)$  can be obtained, whose feedback gain matrix is shown below:

$$H = \delta_2[1 \ \diamond \ \diamond \ 1 \ \diamond \ 2 \ 2 \ 2] \in \mathcal{L}_{2 \times 8},$$

where the symbol “ $\diamond$ ” is arbitrarily chosen from  $\{1, 2\}$ . Taking feedback gain matrix  $H = \delta_2[1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 2]$  as an example and assuming disturbance variable  $\xi(t) \equiv \delta_2^1$ ,  $\forall t \in \mathbb{N}$ , the probability trajectories of  $x_e$  for the considered system are shown in Fig. 1. Notably, starting from any initial state  $x(0, 0) \in \Delta_8$ , the state trajectory of the considered system reaches the control-fixed point  $x_e$  with probability one after three steps, which verifies the effectiveness of the state-feedback controller.



**Fig 1:** Probability trajectory of target state  $x_e$  for the considered system, where  $b_i := \delta_8^i$ ,  $i \in \mathbb{N}_1^8$ .