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Special Topic: Logical System Control

Probabilistic bounds and ramp control in linear threshold dynamics

Ziyu XUAN[1], Nan HUANG[1], Bangxin JIANG[2*] & Yang LIU[1,3]

[1]*School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China*
[2]*School of Computer Science and Technology, Zhejiang Normal University, Jinhua 321004, China*
[3]*Hangzhou School of Automation, Zhejiang Normal University, Hangzhou 311231, China*

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Linear threshold dynamics was first proposed in [1] as a mathematical model to explain the dynamics of collective behavior, providing an analytical framework for systems with network effects and herd behavior (such as financial panic [2] and information diffusion [3]). Each individual in this network model is affected heterogeneously, reflecting its unique threshold and structural position. Each of them has an activation threshold that reflects the structural status of the individual in the collective dynamics, as well as the acceptance and dependence on external influence in the process of behavior evolution. When the proportion (or absolute number) of activated neighbors exceeds the threshold of the individual, the individual will be activated.

In [4], it is mentioned that the 'robust yet fragile' nature of network evolution is the core issue of complex networks. In the directed weighted network, the problem of robust stability and control of asynchronous linear threshold dynamics affected by time-varying external fields is introduced and analyzed in [5]. Based on game theory and graph theory, the necessary and sufficient conditions for the system to robustly converge to consensus are established, and the ability of external control to regulate the system behavior is revealed. The work in [5] applies Bang-Bang control, which switches abruptly between extremes. While effective theoretically, such control often leads to high actuator stress and energy spikes in practice. In contrast, this study introduces ramp control, which restricts the rate of change of the control signal, thereby smoothing the transition and reducing the instantaneous control cost. Moreover, for the non-convergent controllable system, it is verified that the consensus oscillation can be induced by applying external control, but the optimal control strategy is not given. Identifying realistic and feasible control strategies is of paramount importance.

In this study, we first present the lower bound of the probability of achieving predefined-number consensus oscillations within a finite time, verifying the controllability within a finite time. To balance the control effect with the control cost, we consider a multi-objective optimization problem with the control success probability and control smoothness as the optimization objectives, solve the Pareto frontier, and obtain a set of optimal trade-off solutions rather than a single optimal solution, which can more effectively meet the needs of practical applications.

Notations. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the set of real numbers, integers, and positive integers, respectively. $||\mathbf{x}||_2$ is the Euclidean norm. $\mathbf{x} \in \mathbb{R}^Q$ is a vector in the space defined over a finite set Q. $\mathbf{x} \leqslant \mathbf{y}$ if and only if $\mathbf{x}_j \leqslant \mathbf{y}_j, \forall j \in Q$. $|Q|$ denotes the number of elements. $a \bmod b$ is the remainder of the division of a by b, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. $\mathbf{1}(\mathbf{-1})$ represents an n-dimensional column vector, where all elements are $1(-1)$.

Main result. We consider asynchronous linear threshold dynamics on a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{V} = \{1, \ldots, n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $W \in \mathbb{R}_+^{n\times n}$ is the weight matrix with $W_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$ and $W_{ii} = 0$. Each agent $i \in \mathcal{V}$ has a binary state $X_i(t) \in \{-1, +1\}$ evolving in continuous time $t \geqslant 0$, driven by an exogenous time-varying external field $h(t) \in \mathcal{H} \subseteq \mathbb{R}^n$, where $\mathcal{H}$ is the set of admissible control signals. Each X_i updates its state according to an independent rate-1 Poisson process such that when agent i's clock triggers at time t, the update function is as follows:

$$X_i(t^+) = \begin{cases} +1, & \text{if } \sum_{j=1}^n W_{ij}X_j(t) + h_i(t) > 0, \\ -1, & \text{if } \sum_{j=1}^n W_{ij}X_j(t) + h_i(t) < 0, \\ X_i(t), & \text{otherwise.} \end{cases} \tag{1}$$

For a subset of nodes $\mathcal{S} \subseteq \mathcal{V}$, the $\mathcal{S}$-restricted out-degree of node $i \in \mathcal{V}$ is defined as $w_i^{\mathcal{S}} = \sum_{j\in\mathcal{S}} W_{ij}$. When $\mathcal{S} = \mathcal{V}$, let $w_i = w_i^{\mathcal{V}}$ denote the out-degree of node i, and let $w = W\mathbf{1}$ be the vector of out-degrees. For a configuration $x \in \mathcal{X}$, define $w_i^-(x) = \sum_{j:x_j=-1} W_{ij}$ and $w_i^+(x) = \sum_{j:x_j=+1} W_{ij}$ as the aggregate weights from agent i to agents in state -1 and $+1$, respectively.

Assumption 1. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is (h^-, h^+)-indecomposable. That is, given $h^- \leqslant h^+ \in \mathbb{R}^{\mathcal{V}}$, for any 2-partition $\{V_-, V_+\}$ of the node set $\mathcal{V}$ satisfying

$$V = V_+ \cup V_-, V_+ \cap V_- = \emptyset, V_+ \neq \emptyset, V_- \neq \emptyset,$$

there exist $k \in \{-,+\}$ and a node $i \in V_k$ such that

$$w_i^k + kh_i^k < w_i^{-k}. \tag{2}$$

* Corresponding author (email: jiangbangxin@zjnu.edu.cn)

Intuitively, (h^-, h^+)-indecomposability implies that under the given external field bounds, no subset of nodes can permanently maintain a locally opposing state; the external influence is strong enough to eventually propagate through the entire network.

This work discusses in detail the existence and form of linear threshold dynamic absorption into consensus structure for different external domains $[h^-, h^+]$ [5]. The first three cases show that the system can achieve good robust consensus convergence. For the non-convergent controllable system under the influence of the fourth external domain, it is of great research significance to find a control that enables the system to realize the preset number of consensus oscillations in a limited time. Therefore, this work mainly studies this kind of system and introduces Assumption 2.

Assumption 2. Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, external domain $\mathcal{H} = [h^-, h^+]$. Suppose the conditions $w \ngeqslant -h^-$ and $w \ngeqslant h^+$ are satisfied.

We present the oscillation probability bound theorem. It proves that there exists a ramp control r such that the probability of achieving K oscillations within a finite time is greater than zero. Because the optimization problem is inherently non-differentiable, the theorem provides a rational basis for the approximate solution.

Theorem 1 (Oscillation probability bound)**.** Let system (1) satisfy Assumptions 1 and 2. Applying ramp control with period 2τ,

$$h(t) = \begin{cases} h^- + \frac{t_{mod}}{r}(h^+ - h^-), & 0 \leqslant t_{mod} < r, \\ h^+, & r \leqslant t_{mod} < \tau, \\ h^+ - \frac{t_{mod}-\tau}{r}(h^+ - h^-), & \tau \leqslant t_{mod} < \tau + r, \\ h^-, & \tau + r \leqslant t_{mod} < 2\tau, \end{cases} \tag{3}$$

where $t_{mod} = t \mod (2\tau)$. Then, the probability of achieving at least K complete consensus oscillations within a finite time T satisfies

$$P(N(T) \geqslant K) \geqslant P(\mathrm{B}(M, \alpha(t_c)) \geqslant K), \tag{4}$$

where $\tau = \frac{T}{2K}$, $t_c = \tau - r$, $\alpha(t_c) = \alpha_+(t_c)\alpha_-(t_c) \geqslant e^{-2nt_c}\frac{t_c^{L_+ + L_-}}{L_+!L_-!}$. $\alpha_+(t_c) = e^{-nt_c}\frac{t_c^{L_+}}{L_+!}, \alpha_-(t_c) = e^{-nt_c}\frac{t_c^{L_-}}{L_-!}$. $\mathrm{B}(M, \alpha(t_c))$ represents a Binomial distribution. $L_+(L_-)$ is the minimum number of activation events required to reach $+\mathbf{1}(-\mathbf{1})$ consensus from $\forall X(0)$.

Theorem 1 guarantees the existence of ramp control and provides a strict lower bound on probability. Furthermore, the more important issue in reality is how to obtain the appropriate value of r and the probability of achieving a certain number of oscillations K within a limited time. Therefore, based on the analysis, we establish a generalized mathematical model with the smoothing rate r and the probability of successful oscillation as optimization objectives. Sometimes, if the probability of success requirement is not very high, the control cost can be reduced by using a larger r. Hence, to systematically characterize this trade-off, we formulate the following multi-objective optimization problem and obtain the Pareto frontier:

$$\max_{r} \quad [f_1(r),\ f_2(r)],$$
$$f_1(r) = \mathbb{P}(N(T) \geqslant K \mid r),$$
$$f_2(r) = -\int_0^T \left\| \frac{dh(t)}{dt} \right\|_2 dt,$$

where the system follows linear threshold dynamics (1) with ramp control $h_i(t)$ defined piecewise linearly in terms of r (3), and fixed network parameters W, h^-, h^+. The objective function $f_2(r)$, defined as the negative integral of the control signal's derivative, serves as a metric for control smoothness. Minimizing the total variation of the control signal reduces the demand for rapid actuator response, which is directly correlated with the operational cost and stability of physical systems.

However, the direct optimization of the formulated multi-objective problem faces a fundamental challenge: while Theorem 1 provides a strict theoretical lower bound for $f_1(r)$, this bound is derived under worst-case assumptions and is often overly conservative for practical optimization purposes. The analytical expression for the true success probability $f_1(r)$ is mathematically intractable due to the high-dimensional, stochastic, and time-inhomogeneous nature of the asynchronous threshold dynamics.

To address this, we employ a Monte Carlo simulation to estimate the objective functions. This approach is justified by the following considerations. On the one hand, the analytical lower bound, while providing a theoretical guarantee, is too loose to facilitate meaningful trade-off analysis. It fails to capture the actual performance variations across different r values, rendering it unsuitable for guiding practical optimization. On the other hand, Monte Carlo estimation directly simulates the stochastic system dynamics, thereby capturing the true average performance. The strong law of large numbers ensures that the sample average converges almost surely to the expected performance.

Conclusion. In linear threshold dynamics, we have derived the lower bound of the probability of predefined-number consensus oscillations within a finite time. The theorem verifies the finite-time controllability of consensus oscillations and provides a theoretical basis for setting the optimization objective of finding the optimal control strategy. We consider using ramp control to replace Bang-Bang control, transforming the control strategy search problem into a multi-objective optimization problem of the control success probability and control smoothness. Finally, a simulation example is given in Appendix C to illustrate the theoretical result. The results can provide a reference basis for designing feasible control.

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Supporting information Appendixes A–C. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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