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• Supplementary File •

Probabilistic Bounds and Ramp Control in Linear Threshold Dynamics

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Appendix A Supermodularity

Definition 1 (Supermodularity)**.** [1]A function $f : \mathbb{R}^n \to \mathbb{R}$ is called supermodular if for $\forall x, y \in \mathbb{R}^n$,

$$f(x \vee y) + f(x \wedge y) \geqslant f(x) + f(y),$$

where $(x \vee y)_i = \max(x_i, y_i)$ and $(x \wedge y)_i = \min(x_i, y_i)$. Here, the operator $\vee$ denotes the element-wise maximum, and $\wedge$ denotes the element-wise minimum.

The asynchronous linear threshold dynamic system considered in this paper is in accordance with the supermodularity.

Appendix B Proof of Theorem 1

Proof. The assumption of half-cycle symmetry is justified by the symmetric structure of the control limits and the network weights. Path independence holds due to the Markov property of the system, where the process resets after each consensus is reached. Due to the symmetry of the cycle, we consider the half-cycle under the action of the constant external field h^+. Basd on Supermodularity and (h^-, h^+)-indecomposable, we can obtain that there exists an improved path of length L_+, from which it is possible to reach the $+\mathbf{1}$ consensus starting from any initial state. By the additivity of the Poisson distribution, each Poisson clock with a rate of 1 at each node is combined into a Poisson clock with a rate of n. Then, within time interval $[0, t]$, the probability of exactly L_+ events is,

$$P(N(t) = L_+) = e^{-nt} \frac{(nt)^{L_+}}{L_+!}. \tag{B1}$$

Considering the combined probability of path length L_+ and the total number of nodes n, conditional on exactly L_+ events occurring, the probability that these events strictly follow the improvement path sequence is at least $1/n^{L_+}$. Thus,

$$P(\mathcal{A} \mid N(t) = L_+) \geqslant \frac{1}{n^{L_+}}, \tag{B2}$$

where $\mathcal{A}$ denotes the event that the improvement path is activated. Therefore, $\alpha_+(t) \geqslant e^{-nt} \frac{(nt)^{L_+}}{L_+!} \cdot \frac{1}{n^{L_+}} = e^{-nt} \frac{t^{L_+}}{L_+!}$. Due to the symmetry of the cycle, it can be similarly proven that $\alpha_-(t) \geqslant e^{-nt} \frac{t^{L_-}}{L_-!}$.

A complete cycle consists of two independent half-cycles, so

$$\alpha(t_c) = \alpha_+(t_c)\alpha_-(t_c) \geqslant e^{-2nt_c} \frac{t_c^{L_+ + L_-}}{L_+! L_-!}. \tag{B3}$$

Let $N(T)$ be the number of successfully completed oscillations within time T. Since cycles are independent:

$$N(T) \succeq \mathrm{B}(M, \alpha(t_c)), \tag{B4}$$

where $\succeq$ denotes stochastic dominance. Therefore,

$$P(N(T) \geqslant K) \geqslant P(\mathrm{B}(M, \alpha(t_c)) \geqslant K). \tag{B5}$$

This completes the proof. ■

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Appendix C Simulation

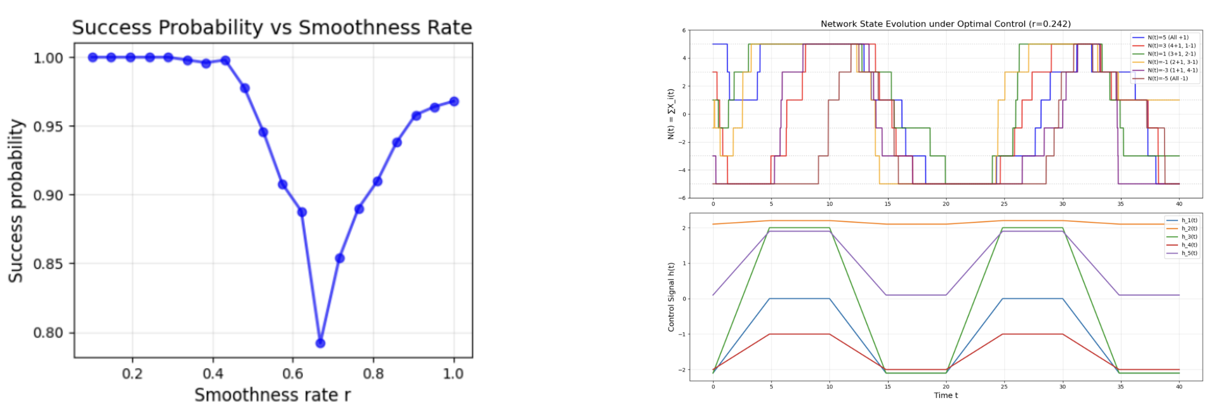
Example 1. Considering the linear threshold dynamics of 5-node, it is easy to verify that it satisfies Assumption 1 and 2.

$$W = \begin{bmatrix} 0&1&0&0&0 \\ 1&0&1&1&1 \\ 0&1&0&0&0 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix}, \quad h^- = \begin{bmatrix} -2.1 \\ 2.1 \\ -2.1 \\ -2 \\ 0.1 \end{bmatrix}, \quad h^+ = \begin{bmatrix} 0 \\ 2.2 \\ 2 \\ -1 \\ 1.9 \end{bmatrix}.$$

By substituting the specific values of the parameters into the optimization model, the following results can be obtained:

$$\begin{aligned} &\max_{r \in (0,1]} \quad [f_1(r),\ f_2(r)] \\ &f_1(r) = \mathbb{P}\left(N(100) \geqslant 4 \mid r\right), \\ &f_2(r) = -\int_0^{100} \left\| \frac{dh(t)}{dt} \right\|_2 dt, \end{aligned}$$

The simulation results are illustrated in Figure C1. Figure C1(a) depicts the Pareto frontier, illustrating the trade-off between the smoothness rate r and the success probability. Figure C1(b) displays the time evolution of the network state $N(t)$ and the control signal $h(t)$ under the optimal parameter $r = 0.242$. It also demonstrates that when $r = 0.242$, the ramp control was able to achieve oscillations between $-\mathbf{1}$ consensus and $+\mathbf{1}$ consensus for any initial state, $N(t) = \sum_{i=1}^{5} X_i(t)$. Unlike the Bang-Bang control, ramp control has a more gradual slope. It is worth noting that the Pareto frontier shows that the degree of smoothness and the probability of success exhibit a trend of first decreasing and then increasing. The first half of the text indicates that achieving the goal with less control energy comes at the cost of sacrificing the success probability. The latter part indicates that when r reaches a certain level, each node has sufficient time to spontaneously influence the state of its neighboring nodes, thereby enabling the system to successfully achieve oscillation. This is a conclusion that cannot be reached merely by relying on a single optimal solution.



(a) (b)

Figure C1 (a) The Pareto frontier between the smoothness rate r and the success probability P. (b) Take $r = 0.242$, the time-varying control signal $h(t)$ shows its curve that changes over time, and the state evolution diagram of any initial state $N(0)$ under the influence of $h(t)$.

References

1 Topkis D M. Supermodularity and Complementarity[M]. Princeton University Press, 1998.