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• **Supplementary File** •

A Boolean algebra approach for the disturbance decoupling problem of large-scale Boolean control networks

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Appendix A Proof of Theorem 1

Proof. (*Sufficiency*) Assume that for any $k \in [1, q]$, every $y_j(t), j \in [1, p]$, satisfies at least one of the two conditions.

If Condition 1 is satisfied, i.e., for $k \in [1, q]$, the restricted expressions $y_j(t)|_{\xi_k(t-1)=1}$ and $y_j(t)|_{\xi_k(t-1)=0}$ are identical, then ξ_k is redundant for y_j, implying that ξ_k has no influence on y_j, $j \in [1, p]$.

If Condition 2 holds, we show that there exists a control input u such that the two restricted expressions $y_j(t)|_{\xi_k(t-1)=1}$ and $y_j(t)|_{\xi_k(t-1)=0}$ become equal. Without loss of generality, suppose $y_j(t)|_{\xi_k(t-1)=1} = y_j(t)|_{\xi_k(t-1)=0}$ can be transformed into $E \vee u = E$. Then, based on the properties of logical operations, by setting $u = 0$, we obtain $E \vee 0 = E$, which holds regardless of E. Hence, the two restricted expressions coincide under this control. Similar reasoning applies to the other five forms listed in Theorem 1.

Therefore, in either case, $\xi_k(t-1)$ has no effect on $y_j(t)$. Since k and j are arbitrary, the system is disturbance decoupled.

Necessity. Suppose BCN (1) is disturbance decoupled. Then, for every ξ_k, $k \in [1, q]$, and every y_j, $j \in [1, p]$, the disturbance variable ξ_k must be redundant for y_j under some appropriate control. This leads to two possibilities:

- **Case 1:** The restricted expressions $y_j(t)|_{\xi_k=1}$ and $y_j(t)|_{\xi_k=0}$ are identical without any control, i.e., Condition 1 is satisfied.
- **Case 2:** The two expressions are not initially identical, but there exists a control u that makes them equal.

In Case 2, the equation $y_j(t)|_{\xi_k=1} = y_j(t)|_{\xi_k=0}$ must be solvable for u. In general, such a Boolean equation can be rewritten in one of the following eight forms:

$$E \vee u = E, \quad E \wedge u = E, \quad E \vee u = 1, \quad E \wedge u = 0,$$
$$E \wedge u = u, \quad E \vee u = u, \quad E \vee u = 0, \quad E \wedge u = 1.$$

However, the last two forms, $E \vee u = 0$ and $E \wedge u = 1$, may not be solvable for u:

- If $E = 1$, then $E \vee u = 0$ has no solution.
- If $E = 0$, then $E \wedge u = 1$ has no solution.

Therefore, only the first six forms, which are exactly those listed in Condition 2, guarantee the existence of a control u that equalizes the two restricted expressions. This completes the proof of necessity.

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Appendix B Algorithm 1

Algorithm B1 Solvability check for DDP of BCN (1)

Input: Dynamics of BCN (1)
Output: List of restricted expressions' pairs (if any)

for $i = 1$ **to** q **do**
  if $y_j(t)$ satisfies the conditions of Theorem 1 **then**
    if $y_j(t)$ satisfies condition 2 in Theorem 1 **then**
      Add this restricted expression pair to the list
    else
      Omit the expression for the controller design
    end if
  else
    DDP is unsolvable; discard the list (if any) and exit
  end if
end for

Appendix C Algorithm 2

Algorithm C1 Controller design for DDP of BCN (1)

Step 1. For each pair of restricted expressions obtained from Algorithm 1, convert it into one of the following forms:

$$E \vee u = E, \quad E \wedge u = E, \quad E \vee u = 1,$$
$$E \wedge u = 0, \quad E \wedge u = u, \quad E \vee u = u,$$

where E is a Boolean expression.
Step 2. Following the rules in Table 1, construct a simplified K-map using the state variables present in E along with one additional state variable.
Step 3. Design a state feedback controller that satisfies the constraints represented in the constructed K-map.

Appendix D A remark

Remark 1. For the DDP in BCNs, traditional approaches often employ the STP-based state space method [2,3]. These methods require constructing the system's structural matrix, whose dimension grows exponentially with the number of system variables. As a result, verifying solvability conditions via traditional STP techniques incurs at least exponential time complexity (at least $O(2^n)$). In contrast, our approach evaluates the solvability of DDP by checking the equivalence of restricted expressions using Boolean algebra and its properties. The associated computational complexity is $O(pq)$, where p and q denote the numbers of outputs and disturbances, respectively.

The conventional design of disturbance decoupling controllers for BCNs relies on the STP method [2,3]. Since the solvability conditions derived via this method involve exponential complexity (at least $O(2^n)$), the controller design based on these conditions also entails exponential time complexity (at least $O(2^n)$). Another method, presented in [4], designs controllers using K-maps. However, a K-map for a Boolean expression with n variables contains 2^n cells. Hence, this approach still entails a computational complexity of at least $O(2^n)$, rendering it unsuitable for large-scale Boolean control networks. In this work, we propose a simplified K-map constructed using only a subset of state variables related to outputs and disturbances rather than all state variables. This leads to a constructive algorithm for designing state feedback controllers with a time complexity of $O(2^{n_1})$, where n_1 is generally much smaller than the total number of state variables n. This approach significantly reduces the computational burden compared to existing methods.

We demonstrate the effectiveness of this method through two examples of large-scale Boolean control networks (see Appendix E).

Appendix E Examples

Example 1. Consider a BCN with 40 state variables, 5 disturbance variables, 5 control inputs and 3 output variables:

$$x_1(t+1) = x_2(t) \wedge \neg x_3(t) \vee \xi_1(t) \wedge u_1(t),$$
$$x_2(t+1) = x_1(t) \vee x_4(t),$$

$$
\begin{aligned}
&x_3(t+1) = x_2(t) \wedge x_5(t) \wedge u_2(t) \vee \xi_2(t),\\
&x_4(t+1) = \neg x_3(t) \vee x_6(t),\\
&x_5(t+1) = x_4(t) \wedge \neg x_7(t),\\
&x_6(t+1) = x_5(t) \vee x_8(t) \vee u_3(t),\\
&x_7(t+1) = x_6(t) \wedge x_9(t),\\
&x_8(t+1) = \neg x_7(t) \vee x_{10}(t),\\
&x_9(t+1) = x_8(t) \wedge \neg x_{11}(t) \wedge u_4(t) \vee \xi_4(t),\\
&x_{10}(t+1) = x_9(t) \vee x_{12}(t),\\
&x_{11}(t+1) = x_{10}(t) \wedge x_{13}(t),\\
&x_{12}(t+1) = \neg x_{11}(t) \vee x_{14}(t) \vee \xi_5(t) \wedge u_5(t),\\
&x_{13}(t+1) = x_{12}(t) \wedge \neg x_{15}(t),\\
&x_{14}(t+1) = x_{13}(t) \vee x_{16}(t),\\
&x_{15}(t+1) = x_{14}(t) \vee x_{17}(t) \vee u_1(t) \vee \xi_1(t),\\
&x_{16}(t+1) = \neg x_{15}(t) \vee x_{18}(t),\\
&x_{17}(t+1) = x_{16}(t) \wedge \neg x_{19}(t),\\
&x_{18}(t+1) = x_{17}(t) \wedge x_{20}(t) \wedge \xi_2(t) \wedge u_2(t),\\
&x_{19}(t+1) = x_{18(t)} \wedge x_{21}(t),\\
&x_{20}(t+1) = \neg x_{19}(t) \vee x_{22}(t),\\
&x_{21}(t+1) = x_{20}(t) \wedge \neg x_{23}(t) \vee \xi_3(t),\\
&x_{22}(t+1) = x_{21}(t) \vee x_{24}(t),\\
&x_{23}(t+1) = x_{22}(t) \wedge x_{25}(t),\\
&x_{24}(t+1) = \neg x_{23}(t) \vee x_{26}(t) \vee \xi_4(t) \wedge u_4(t),\\
&x_{25}(t+1) = x_{24}(t) \wedge \neg x_{27}(t),\\
&x_{26}(t+1) = x_{25}(t) \vee x_{28}(t),\\
&x_{27}(t+1) = x_{26}(t) \wedge x_{29}(t) \wedge u_5(t) \vee \xi_5(t),\\
&x_{28}(t+1) = \neg x_{27}(t) \vee x_{30}(t),\\
&x_{29}(t+1) = x_{28}(t) \wedge \neg x_{31}(t),\\
&x_{30}(t+1) = x_{29}(t) \vee x_{32}(t) \vee \xi_1(t) \wedge u_1(t),\\
&x_{31}(t+1) = x_{30}(t) \wedge x_{33}(t),\\
&x_{32}(t+1) = \neg x_{31}(t) \vee x_{34}(t),\\
&x_{33}(t+1) = x_{32}(t) \wedge \neg x_{35}(t) \wedge u_2(t) \vee \xi_2(t),\\
&x_{34}(t+1) = x_{33}(t) \vee x_{36}(t),\\
&x_{35}(t+1) = x_{34}(t) \wedge x_{37}(t),\\
&x_{36}(t+1) = \neg x_{35}(t) \vee x_{38}(t) \vee \xi_3(t) \wedge u_3(t),\\
&x_{37}(t+1) = x_{36}(t) \wedge \neg x_{39}(t),\\
&x_{38}(t+1) = x_{37}(t) \vee x_{40}(t),\\
&x_{39}(t+1) = x_{38}(t) \wedge x_1(t) \wedge u_4(t) \vee \xi_4(t),\\
&x_{40}(t+1) = \neg x_{39}(t) \vee x_2(t) \vee \xi_5(t) \wedge u_5(t),\\
&y_1(t) = x_2(t) \vee x_6(t) \vee x_{18}(t),\\
&y_2(t) = x_8(t) \wedge x_{15}(t) \wedge \neg x_{28}(t),\\
&y_3(t) = \neg x_5(t) \wedge x_{20}(t) \wedge \neg x_{35}(t).
\end{aligned}
$$

We address the DDP for the example system in two steps.
Step 1: Checking the solvability of DDP.
The output y_j, $j = 1, 2, 3$, of the BCN can be rewritten as

$$
\begin{aligned}
&y_1(t) = x_1(t-1) \vee x_4(t-1) \vee x_5(t-1) \vee x_8(t-1) \vee u_3(t) \vee (x_{17}(t-1) \wedge x_{20}(t-1) \wedge \xi_2(t-1) \wedge u_2(t-1)),\\
&y_2(t) = (\neg x_7(t-1) \vee x_{10}(t-1)) \wedge (x_{14}(t-1) \vee x_{17}(t-1) \vee u_1(t-1) \vee \xi_1(t-1)) \wedge (\neg x_{27}(t-1) \vee x_{30}(t-1)),\\
&y_3(t) = \neg(x_4(t-1) \wedge \neg x_7(t-1)) \wedge (\neg x_{19}(t-1) \vee x_{22}(t-1)) \wedge \neg(x_{34}(t-1) \wedge x_{37}(t-1)).
\end{aligned}
$$

Following Algorithm 1, we verify whether the restricted expressions for each $y_j(t), j = 1, 2, 3,$ are equivalent. The restricted expressions are derived as follows:

$$y_1(t)\begin{cases}\xi_2(t-1)=1 \implies x_1(t-1)\vee x_4(t-1)\vee x_5(t-1)\vee x_8(t-1)\vee u_3(t)\vee(x_{17}(t-1)\wedge x_{20}(t-1)\wedge u_2(t-1)),\\ \xi_2(t-1)=0 \implies x_1(t-1)\vee x_4(t-1)\vee x_5(t-1)\vee x_8(t-1)\vee u_3(t),\end{cases} \tag{E1}$$

$$y_2(t)\begin{cases}\xi_1(t-1)=1 \implies (\neg x_7(t-1)\vee x_{10}(t-1))\wedge(\neg x_{27}(t-1)\vee x_{30}(t-1)),\\ \xi_1(t-1)=0 \implies (\neg x_7(t-1)\vee x_{10}(t-1))\wedge(x_{14}(t-1)\vee x_{17}(t-1)\vee u_1(t-1))\wedge(\neg x_{27}(t-1)\vee x_{30}(t-1)).\end{cases} \tag{E2}$$

For $y_3(t)$, since none of the disturbance variables $\xi_i(t-1), i=1,\cdots,5$, appear in the expression, the output remains unaffected by disturbances. This satisfies Condition 1 of Theorem 1.

For $y_1(t)$, equating the restricted expressions $y_1(t)|_{\xi_2(t-1)=1}=y_1(t)|_{\xi_2(t-1)=0}$ gives

$$\begin{aligned}&x_1(t-1)\vee x_4(t-1)\vee x_5(t-1)\vee x_8(t-1)\vee u_3(t-1),\\ =\;&x_1(t-1)\vee x_4(t-1)\vee x_5(t-1)\vee x_8(t-1)\vee u_3(t-1)\vee(x_{17}(t-1)\wedge x_{20}(t-1)\wedge u_2(t-1)).\end{aligned}$$

This simplifies to:

$$1=\neg(x_{17}(t-1)\wedge x_{20}(t-1)\wedge u_2(t-1)),$$

or equivalently,

$$0=x_{17}(t-1)\wedge x_{20}(t-1)\wedge u_2(t-1).$$

This matches Structure 4 in Table 1, confirming that $y_1(t)$ satisfies Theorem 1.

For $y_2(t)$, let restricted expressions $y_2(t)|_{\xi_1(t-1)=1}=y_2(t)|_{\xi_1(t-1)=0}$. Then

$$\begin{aligned}&(\neg x_7(t-1)\vee x_{10}(t-1))\wedge(\neg x_{27}(t-1)\vee x_{30}(t-1))\\ =\;&(\neg x_7(t-1)\vee x_{10}(t-1))\wedge(x_{14}(t-1)\vee x_{17}(t-1)\vee u_1(t-1))\wedge(\neg x_{27}(t-1)\vee x_{30}(t-1)),\end{aligned}$$

which simplifies to:

$$1=x_{14}(t-1)\vee x_{17}(t-1)\vee u_1(t-1).$$

This corresponds to Structure 3 in Table 1, so $y_2(t)$ also satisfies Theorem 1.

Step 2:Controller Design.

From Step 1 in this example, the equation for $y_1(t)$ yields:

$$0=x_{17}(t-1)\wedge x_{20}(t-1)\wedge u_2(t-1),$$

which matches Structure 4 in Table 1 with $E=x_{17}(t-1)\wedge x_{20}(t-1)$. Applying Rule 4, we retain the 0 entries in the simplified K-map of $\bar{E}$ (see subfigure (a) of Figure E1) and derive the simplified K-map for $u_2(t-1)$, as shown in subfigure (b).

Similarly, from Step 1 in this example, for $y_2(t)_{|\xi_1(t-1)=1}=y_2(t)_{|\xi_1(t-1)=0}$, we have:

$$1=x_{14}(t-1)\vee x_{17}(t-1)\vee u_1(t-1),$$

corresponding to Structure 3 in Table 1 with $E=x_{14}(t-1)\vee x_{17}(t-1)$. Following Rule 3, we retain the 1 entries in the simplified K-map of $\bar{E}$ (subfigure (c)) and obtain the K-map for $u_1(t-1)$ in subfigure (d).

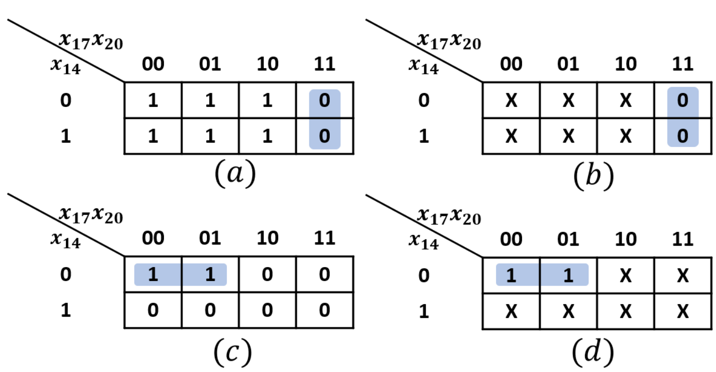

Figure E1 Controller design (u_1 and u_2) for Example 1. (a) the K-map of expression $\bar{E}=\neg(x_{17}(t-1)\wedge x_{20}(t-1))$; (b) the K-map of control $u_2(t-1)$ for $y_1(t)$; $(c)$the K-map of expression $\bar{E}=\neg(x_{14}(t-1)\vee x_{17}(t-1))$; (d) the K-map of control $u_1(t-1)$ for $y_2(t)$.

Using Figure E1, state feedback controllers for the DDP can be designed as follows:

- From subfigure (d), $u_1(t-1)$ must be 1 when both $x_{14}(t-1)$ and $x_{17}(t-1)$ are 0.
- From subfigure (b), $u_2(t-1)$ must be 0 when both $x_{17}(t-1)$ and $x_{20}(t-1)$ are 1.

For instance, the following controllers satisfy these conditions:

$$\begin{aligned}
&u_1(t-1) = \neg x_{14}(t-1),\\
&u_2(t-1) = \neg x_{17}(t-1),\\
&u_3(t-1) = 0,\\
&u_4(t-1) = 1,\\
&u_5(t-1) = 0,
\end{aligned}$$

where controls u_l, $l = 3, 4, 5$, may take any value in $\mathcal{B}$.

For Example 1 with $n = 40$ and $n_1 = 3$, solving the DDP via the semi-tensor product (STP) method entails a computational complexity of $O(2^n)$, i.e., $O(2^{40})$. In contrast, the approach proposed in this paper achieves a complexity of only $O(2^{n_1})$, i.e., $O(2^3)$, resulting in a substantial reduction in computational burden.

Example 2. Consider a BCN with 35 state variables, 5 disturbance variables, 5 control inputs and 1 output variables:

$$\begin{aligned}
&x_1(t+1) = u_1(t),\\
&x_2(t+1) = u_5(t),\\
&x_3(t+1) = x_2(t) \wedge x_5(t) \wedge u_2(t) \vee \xi_2(t),\\
&x_4(t+1) = \neg x_3(t) \vee x_6(t),\\
&x_5(t+1) = x_4(t) \wedge \neg x_7(t),\\
&x_6(t+1) = x_1(t) \vee x_4(t) \vee \xi_2(t) \vee u_3(t),\\
&x_7(t+1) = x_6(t) \vee \xi_1(t) \vee u_2(t),\\
&x_8(t+1) = \neg x_7(t) \vee x_{10}(t),\\
&x_9(t+1) = \xi_4(t),\\
&x_{10}(t+1) = x_9(t) \vee x_{12}(t),\\
&x_{11}(t+1) = x_{10}(t) \wedge x_{13}(t),\\
&x_{12}(t+1) = \xi_5(t),\\
&x_{13}(t+1) = x_{12}(t) \wedge \neg x_{15}(t),\\
&x_{14}(t+1) = x_{13}(t) \vee x_{16}(t),\\
&x_{15}(t+1) = x_{14}(t) \vee x_{17}(t) \vee u_1(t) \vee \xi_1(t),\\
&x_{16}(t+1) = \neg x_{15}(t) \vee x_{18}(t),\\
&x_{17}(t+1) = x_{16}(t) \wedge \neg x_{19}(t),\\
&x_{18}(t+1) = x_{17}(t) \wedge x_{20}(t) \wedge \xi_2(t) \wedge u_2(t),\\
&x_{19}(t+1) = x_{18(t)} \wedge x_{21}(t),\\
&x_{20}(t+1) = \neg x_{19}(t) \vee x_{22}(t),\\
&x_{21}(t+1) = x_{20}(t) \wedge \neg x_{23}(t) \vee \xi_3(t),\\
&x_{22}(t+1) = x_{21}(t) \vee x_{24}(t),\\
&x_{23}(t+1) = x_{22}(t) \wedge x_{25}(t),\\
&x_{24}(t+1) = \neg x_{23}(t) \vee x_{26}(t) \vee \xi_4(t) \wedge u_4(t),\\
&x_{25}(t+1) = x_{24}(t) \wedge \neg x_{27}(t),\\
&x_{26}(t+1) = x_{25}(t) \vee x_{28}(t),\\
&x_{27}(t+1) = x_{26}(t) \wedge x_{29}(t) \wedge u_5(t) \vee \xi_5(t),\\
&x_{28}(t+1) = \neg x_{27}(t) \vee x_{30}(t),\\
&x_{29}(t+1) = x_{28}(t) \wedge \neg x_{31}(t),\\
&x_{30}(t+1) = x_{29}(t) \vee x_{32}(t) \vee \xi_1(t) \wedge u_1(t),\\
&x_{31}(t+1) = x_{30}(t) \wedge x_{33}(t),\\
&x_{32}(t+1) = \neg x_{31}(t) \vee x_{34}(t),\\
&x_{33}(t+1) = x_{32}(t) \wedge \neg x_{35}(t) \wedge u_2(t) \vee \xi_2(t),\\
&x_{34}(t+1) = x_{33}(t) \vee x_{35}(t),\\
&x_{35}(t+1) = x_{34}(t) \wedge x_{33}(t),\\
&y_1(t) = x_1(t) \wedge x_7(t) \vee x_9(t),\\
&y_2(t) = x_2(t) \wedge x_6(t) \vee x_{12}(t).
\end{aligned}$$

We address the DDP for the example system in two steps.

Step 1: Checking the solvability of DDP.

The output y_j, $j = 1, 2$, of the BCN can be rewritten as

$$y_1(t) = u_1(t-1) \wedge (x_6(t-1) \vee \xi_1(t-1) \vee u_2(t-1)) \vee \xi_4(t-1),$$
$$y_2(t) = u_5(t-1) \wedge (x_1(t-1) \vee x_4(t-1) \vee \xi_2(t-1) \vee u_3(t-1)) \vee \xi_5(t-1).$$

Following Algorithm 1, we verify whether the restricted expressions for each $y_j(t)$, $j = 1, 2$, are equivalent. The restricted expressions are derived as follows:

$$y_(t) \begin{cases} \xi_1(t-1) = 1 \implies u_1(t-1) \vee \xi_4(t-1), \\ \xi_1(t-1) = 0 \implies u_1(t-1) \wedge (x_6(t-1) \vee u_2(t-1)) \vee \xi_4(t-1), \end{cases} \tag{E3}$$

$$y_1(t) \begin{cases} \xi_4(t-1) = 1 \implies 1, \\ \xi_4(t-1) = 0 \implies u_1(t-1) \wedge (x_6(t-1) \vee \xi_1(t-1) \vee u_2(t-1)), \end{cases} \tag{E4}$$

$$y_2(t) \begin{cases} \xi_2(t-1) = 1 \implies u_5(t-1) \vee \xi_5(t-1), \\ \xi_2(t-1) = 0 \implies u_5(t-1) \wedge (x_1(t-1) \vee x_4(t-1) \vee u_3(t-1)) \vee \xi_5(t-1), \end{cases} \tag{E5}$$

$$y_2(t) \begin{cases} \xi_5(t-1) = 1 \implies 1, \\ \xi_5(t-1) = 0 \implies u_5(t-1) \wedge (x_1(t-1) \vee x_4(t-1) \vee \xi_2(t-1) \vee u_3(t-1)). \end{cases} \tag{E6}$$

For $y_1(t)$, equating the restricted expressions $y_1(t)|_{\xi_1(t-1)=1} = y_1(t)|_{\xi_1(t-1)=0}$ gives

$$u_1(t-1) \vee \xi_4(t-1) = u_1(t-1) \wedge (x_6(t-1) \vee u_2(t-1)) \vee \xi_4(t-1),$$

This simplifies to:

$$1 = x_6(t-1) \vee u_2(t-1).$$

This matches Structure 3 in Table 1, confirming that the restricted expression of $y_1(t)$ satisfies Theorem 1.

For $y_1(t)$, equating the restricted expressions $y_1(t)|_{\xi_4(t-1)=1} = y_1(t)|_{\xi_4(t-1)=0}$ gives

$$1 = u_1(t-1) \wedge (x_6(t-1) \vee \xi_1(t-1) \vee u_2(t-1)).$$

This requires $u_1(t-1) = 1$, and

$$1 = x_6(t-1) \vee \xi_1(t-1) \vee u_2(t-1). \tag{E7}$$

Since the equation still contains disturbance variables, we construct a K-map using the simplified equation in terms of $y_1(t)|_{\xi_1(t-1)=1} = y_1(t)|_{\xi_1(t-1)=0}$ to determine whether a controller exists that can make the restricted expressions involving disturbance variables equal.

For $y_2(t)$, let restricted expressions $y_2(t)|_{\xi_2(t-1)=1} = y_2(t)|_{\xi_2(t-1)=0}$. Then

$$u_5(t-1) \vee \xi_5(t-1) = u_5(t-1) \wedge (x_1(t-1) \vee x_4(t-1) \vee u_3(t-1)) \vee \xi_5(t-1),$$

which simplifies to:

$$1 = x_1(t-1) \vee x_4(t-1) \vee u_3(t-1).$$

This corresponds to Structure 3 in Table 1, so the pair of restricted expressions for $y_2(t)$ also satisfies Theorem 1.

For $y_2(t)$, equating the restricted expressions $y_2(t)|_{\xi_5(t-1)=1} = y_2(t)|_{\xi_5(t-1)=0}$ gives

$$1 = u_5(t-1) \wedge (x_1(t-1) \vee x_4(t-1) \vee \xi_2(t-1) \vee u_3(t-1)).$$

This implies that $u_5(t-1) = 1$, and

$$1 = x_1(t-1) \vee x_4(t-1) \vee \xi_2(t-1) \vee u_3(t-1). \tag{E8}$$

Since the equation still contains disturbance variables, we construct a K-map using the simplified equation in terms of $y_2(t)|_{\xi_2(t-1)=1} = y_2(t)|_{\xi_2(t-1)=0}$ to determine whether a controller exists that can make the restricted expressions involving disturbance variables equal.

Step 2:Controller Design.

From Step 1 in this example, the equation for $y_1(t)$ yields:

$$1 = x_6(t-1) \vee u_2(t-1).$$

which matches Structure 3 in Table 1 with $E = x_6(t-1)$. Applying Rule 3, we retain the 1 entries in the simplified K-map of $\bar{E}$ (see subfigure (a) of Figure E2) and derive the simplified K-map for $u_2(t-1)$, as shown in subfigure (b).

Similarly, from Step 1 in this example, for $y_2(t)_{|\xi_2(t-1)=1} = y_2(t)_{|\xi_2(t-1)=0}$, we have:

$$1 = x_1(t-1) \vee x_4(t-1) \vee u_3(t-1),$$

corresponding to Structure 3 in Table 1 with $E = x_1(t-1) \vee x_4(t-1)$. Following Rule 3, we retain the 1 entries in the simplified K-map of $\bar{E}$ (subfigure (c)) and obtain the K-map for $u_3(t-1)$ in subfigure (d).

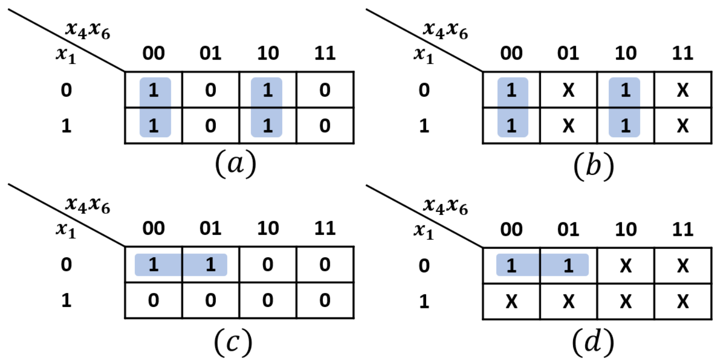

Figure E2 Controller design (u_2 and u_3) for Example 2. (a) the K-map of expression $\bar{E} = \neg x_6(t-1)$; (b) the K-map of control $u_2(t-1)$ for $y_1(t)$; (c)the K-map of expression $\bar{E} = \neg(x_1(t-1) \vee x_4(t-1))$; ($d$) the K-map of control $u_3(t-1)$ for $y_2(t)$.

Using Figure E2 and Equations (E7) and (E8), state feedback controllers for the DDP can be designed as follows:

- From subfigure (d), $u_2(t-1)$ must be 1 when $x_6(t-1)$ is 0.
- From subfigure (b), $u_4(t-1)$ must be 1 when both $x_1(t-1)$ and $x_4(t-1)$ are 0.

For instance, the following controllers satisfy these conditions:

$$\begin{aligned} u_1(t-1) &= 1, \\ u_2(t-1) &= \neg x_6(t-1), \\ u_3(t-1) &= \neg x_1(t-1), \\ u_4(t-1) &= 0, \\ u_5(t-1) &= 1, \end{aligned}$$

where control u_4, may take any value in $\mathcal{B}$.