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Special Topic: Logical System Control

Equilibrium existence and convergence of congestion games with stochastic disturbances

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Research on congestion games is significant as it provides a foundational model for analyzing how self-interested agents interact over shared resources. Since congestion games are potential games [1], they are widely used to solve practical issues such as spectrum resonance [2], electric vehicle charging configuration [3], and service chain composition [4]. Affected by the complex environment, disturbances are unavoidable in congestion game modeling. For instance, when modeling a traffic network with congestion games, the disturbance events can include road construction, extreme weather, or sudden changes in regulations. These disturbances can worsen congestion on certain roadways, which further changes the congestion costs of some resources. However, to the best of our knowledge, there are few results on networked congestion games with stochastic disturbances (CGSDs).

The semi-tensor product (STP) of matrices is an effective tool to analyze and control networked evolutionary games, and solves many basic issues of them, such as equilibrium analysis [5], strategy consensus [6], and stable degree analysis [7]. Particularly, it also demonstrates superiority in the analysis and control of congestion games [8, 9]. Based on this, we study the CGSDs via the STP method.

The main contributions of this study are as follows. (1) A generalized method for constructing the potential function of the CGSDs is given, based on which a formula to calculate the pure strategy Nash equilibrium (PNE) is obtained. (2) Some necessary and sufficient conditions are presented to determine whether the profiles of a CGSD converge to the desired PNEs in finite time with probability one. (3) A strategy regulation scheme is designed such that the game can globally converge to the desired PNEs.

Problem formulation.

Definition 1. A CGSD is denoted by $G = (N, M, S, \xi, \Xi_l, C)$, where

(1) $N = \{1, 2, \cdots, n\}$ is the set of players;

(2) $M = \{a_1, a_2, \cdots, a_m\}$ is the set of resources;

(3) $S = \prod_{i=1}^{n} S_i$ is the set of profiles, where $S_i \subset 2^M \setminus \emptyset$ is the strategy set of player i;

(4) $\xi = (\xi_1, \xi_2, \cdots, \xi_q) \in \mathcal{D}^q$ is the external disturbance input;

(5) $\Xi_l(k, \xi)$ is the congestion cost of resource l affected by the number of users k using resource l and disturbance ξ;

(6) $C = \{c_1, c_2, \cdots, c_n\}$ is the set of cost functions for the players.

In the models, the q disturbances are mutually independent. $\xi_i = 1, i \in \{1, 2, \cdots, q\}$ shows that the i-th disturbance event occurs, and $\xi_i = 0$ means that the i-th disturbance event does not occur. Moreover, under any disturbance combination ξ, $\Xi_l(k_1, \xi) \geqslant \Xi_l(k_2, \xi)$ for $k_1 > k_2$. This assumption indicates that under any disturbance combination, the more individuals utilizing resource l, the higher the congestion cost of using resource l for each individual.

There are 2^q disturbance combinations. Arrange them in dictionary order as $(1, 1, \cdots, 1, 1)$, $(1, 1, \cdots, 1, 0)$, $(1, 1, \cdots, 0, 1)$, $(1, 1, \cdots, 0, 0), \cdots, (0, 0, \cdots, 0, 0)$. Under the effect of different disturbance combinations, even if the number of players using resource $l \in M$ is the same, the cost that players have to pay to use resource l is not necessarily the same.

At each time, assume $\mathbb{P}\{\xi_i = 1\} = p_i, \mathbb{P}\{\xi_i = 0\} = 1 - p_i$, $0 \leqslant p_i \leqslant 1$, and let $J = \otimes_{i=1}^{q}[p_i \ 1 - p_i]$. Evidently, $[J]_{1,\eta}$ denotes the occurrence probability of the η-th disturbance combination.

For any given profile $s = (s_1, s_2, \ldots, s_n) \in S$, define $n_l(s) := |\{i | l \in s_i\}|$ as the number of players who choose resource $l \in M$. Let $\Xi_l^{\eta}(n_l(s))$ denote the congestion cost of resource l under the η-th disturbance combination. Then, we can construct a vector $\Xi_l(n_l(s)) = [\Xi_l^1(n_l(s)), \Xi_l^2(n_l(s)), \cdots, \Xi_l^{2^q}(n_l(s))]^{\top}$. The expected aggregate cost of player i under profile s is

$$Ec_i(s) = \sum_{l \in s_i} J\Xi_l(n_l(s)). \tag{1}$$

The existence of PNEs.

Definition 2. Considering a CGSD, if there exists a profile $s^* = (s_i^*, s_{-i}^*) \in S$ such that $Ec_i(s_i^*, s_{-i}^*) \leqslant Ec_i(s_i, s_{-i}^*)$ for any

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$i \in N, s_i \in S_i$ and $s_{-i} \in S_{-i}$, then $s^* = (s_i^*, s_{-i}^*)$ is called a PNE, where $S_{-i} = \prod_{j \neq i}^{n} S_j$.

A potential game has at least one PNE, so we first detect whether a CGSD is a potential game.

Definition 3. Considering a CGSD, if there exists a function $P : S \to \mathbb{R}$ such that $Ec_i(s_i, s_{-i}) - Ec_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i})$ holds for any $i \in N, s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$, then the game is a potential game and P is its potential function.

Define a function $P : S \to \mathbb{R}$ as

$$P(s) = \sum_{l \in \cup_{i=1}^{n} s_i} \sum_{j=1}^{n_l(s)} J\Xi_l(j). \tag{2}$$

Theorem 1. A CGSD is a potential game with (2) being its potential function.

In a congestion game, the minimum point of the potential function must be a PNE. Generally, the global minimum point of the potential function corresponds to the equilibrium with the lowest overall social cost. We see these PNEs as the desired equilibria. All desired PNEs can be derived via the STP method, and the calculation steps are given in the supplementary file. Let $\mathbb{NE}$ denote the desired PNE set.

Convergence of evolutionary CGSDs. It is assumed that the game is infinitely repeated and all players update their strategies according to the myopic best response adjustment rule.

Assume that $S_i = \{a_i^1, a_i^2, \cdots, a_i^{\kappa_i}\}$. Identify $a_i^j \sim \delta_{\kappa_i}^j$, then $S_i \sim \Delta_{\kappa_i}$. We call $\delta_{\kappa_i}^j$ the logical vector form of strategy a_i^j. Let $\vec{s}_i \in \Delta_{\kappa_i}$ be the vector form of strategy $s_i \in S_i$ and $\kappa = \prod_{i=1}^{n} \kappa_i$, then $(s_1, s_2, \cdots, s_n) \sim \ltimes_{i=1}^{n} \vec{s}_i \in \Delta_\kappa$.

By the STP method, we can derive the profile transition probability equation of the CGSD as

$$\mathbb{E}\vec{s}(t+1) := \bar{F}\mathbb{E}\vec{s}(t), \tag{3}$$

where $\vec{s}(t) = \ltimes_{i=1}^{n} \vec{s}_i(t) \in \Delta_\kappa$, and $\bar{F}$ is a κ-dimensional column stochastic matrix. We easily obtain $[\bar{F}]_{i,j} = (\delta_\kappa^i)^\top [\bar{F}] \delta_\kappa^j = \mathbb{P}(\vec{s}(t+1) = \delta_\kappa^i \mid \vec{s}(t) = \delta_\kappa^j)$, and $[\bar{F}^l]_{i,j} = \mathbb{P}(\vec{s}(t+l) = \delta_\kappa^i \mid \vec{s}(t) = \delta_\kappa^j)$.

Proposition 1. A fixed point of system (3) with probability one must be a PNE of the CGSD.

Let $\mathcal{NE}$ denote the logical vector form of $\mathbb{NE}$.

Definition 4. Consider system (3).

(i) An initial profile $\vec{s}(0) \in \Delta_\kappa$ is said to converge to $\mathcal{NE}$ in finite time with probability one, if there exists an integer T such that $\mathbb{P}\{\vec{s}(t) \in \mathcal{NE}\} = 1, \ \forall t \geqslant T$.

(ii) The CGSD is said to globally converge to $\mathcal{NE}$ in finite time with probability one, if each initial profile $\vec{s}(0) \in \Delta_\kappa$ can converge to $\mathcal{NE}$ in finite time with probability one.

Let $I(\mathcal{NE}) = \{\delta_\kappa^j | [\bar{F}]_{j,j} = 1, \delta_\kappa^j \in \mathcal{NE}\} := \{\delta_\kappa^{j_1^*}, \cdots, \delta_\kappa^{j_\gamma^*}\}$. Suppose that set $I(\mathcal{NE})$ is non-empty, and let $W = \sum_{i=1}^{\gamma} \delta_{j_i^*}^{\kappa}$.

Theorem 2. An initial profile $\vec{s}(0) = \delta_\kappa^j$ can converge to $I(\mathcal{NE})$ in finite time with probability one if and only if

$$W^\top \mathrm{Col}_j(\bar{F}^\kappa) = 1. \tag{4}$$

The set of profiles which converge to $I(\mathcal{NE})$ in finite time with probability one is $R(I(\mathcal{NE})) = \{\delta_\kappa^j | W^\top \mathrm{Col}_j(\bar{F}^\kappa) = 1\}$.

If $R(I(\mathcal{NE})) \neq \Delta_\kappa$, we consider to adjust the minimum number of profiles to ensure the system converges globally to $\mathcal{NE}$.

Let $\Omega = \Delta_\kappa \setminus R(I(\mathcal{NE}))$. If we adjust players' strategies to make all the recurrent states in Ω reach $R(I(\mathcal{NE}))$ in one step with probability one, the game can surely converge to $I(\mathcal{NE})$. Let $\mathbf{Q} = \sum_{i=1}^{\kappa} \bar{F}^i$ and $\mathcal{A}(\Omega) = \{\delta_\kappa^i \mid [\mathbf{Q}]_{i,i} > 0, \delta_\kappa^i \in \Omega\} := \{\delta_\kappa^{i_1}, \cdots, \delta_\kappa^{i_\alpha}\}$. Obviously, for each profile $\delta_k^i \in \mathcal{A}(\Omega)$, there exists an accessible path from δ_k^i to itself. Therefore, the union of all recurrent states in set Ω is $\mathcal{A}(\Omega)$.

Identify $\delta_\kappa^{i_l} \sim \delta_\alpha^l$, $l \in [1:\alpha]$, and construct a matrix $L \in \mathbb{R}_{\alpha \times \alpha}$, where $[L]_{a,b} = [\bar{F}]_{i_a, i_b}$, $a, b \in [1:\alpha]$. We further construct a matrix L_Λ with respect to $\Lambda \subseteq [1:\alpha]$ as

$$\mathrm{Col}_j(L_\Lambda) = \begin{cases} \mathbf{0}_\alpha, & \text{if } j \in \Lambda, \\ \mathrm{Col}_j(L), & \text{otherwise,} \end{cases}$$

and let $\Omega_\Lambda = \{\delta_\alpha^l \mid [\sum_{t=1}^{\alpha} (L_\Lambda)^t]_{l,l} = 0, \ l \in [1:\alpha]\}$.

Theorem 3. If there exists a set $\Lambda = \{l_1, l_2, \cdots, l_w\} \subseteq [1:\alpha]$ with the fewest elements such that $\Omega_\Lambda = \Delta_\alpha$, then by changing the one-step transition of the profiles in $\{\delta_\kappa^{i_{l_1}}, \delta_\kappa^{i_{l_2}}, \cdots, \delta_\kappa^{i_{l_w}}\}$ to make them evolve to $R(I(\mathcal{NE}))$ with probability one at next moment, the CGSD can globally converge to $I(\mathcal{NE})$. Moreover, this set has the smallest number of profiles.

Remark 1. The newly constructed matrix L can describe the evolutionary trajectories among recurrent states in the original game. Since the dimension of L is only determined by the number of recurrent states of (3), so studying the elimination of recurrent states based on L involves a significantly lower computational complexity than using $\bar{F}$.

For each profile $\delta_\kappa^j \in \{\delta_\kappa^{i_{l_1}}, \delta_\kappa^{i_{l_2}}, \cdots, \delta_\kappa^{i_{l_w}}\}$, to make it evolve to $R(I(\mathcal{NE}))$ with probability one at the next moment, the one-step transition of δ_κ^j can be modified according to Algorithm 1.

Algorithm 1 Modification of the one-step transition of δ_κ^j.

- Step 1: Find all the profiles $\vec{s} \in \Delta_k$ satisfying $(\vec{s})^\top \bar{F} \delta_\kappa^j > 0$ and define them as $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_b$;
- Step 2: Decompose these profiles as $\mathbf{s}_i = \mathbf{s}_i^1 \ltimes \mathbf{s}_i^2 \ltimes \cdots \ltimes \mathbf{s}_i^n$, and find the integers β satisfying $\mathbf{s}_1^\beta = \mathbf{s}_2^\beta = \cdots = \mathbf{s}_b^\beta$, $\beta \in [1:n]$. Define these integers as $\beta_1, \beta_2, \cdots, \beta_\tau$;
- Step 3: Find a profile $\mathbf{r}^* \in argmax_{\mathbf{r} \in R(I(\mathcal{NE}))} \mid \{\beta | \mathbf{r}^\beta = \mathbf{s}_1^\beta, \beta \in \{\beta_1, \beta_2, \cdots, \beta_\tau\}\} \mid$, where $\mathbf{r} = \mathbf{r}^1 \ltimes \mathbf{r}^2 \ltimes \cdots \ltimes \mathbf{r}^n$;
- Step 4: Change the one-step transition $\bar{F}\delta_\kappa^j$ to $\mathbf{r}^*$.

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Supporting information Appendixes A–F. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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