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• Supplementary File •

Equilibrium existence and convergence of congestion games with stochastic disturbances

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Appendix A Notations used in the body of this letter

- $\mathbb{R}$: set of real numbers.
- $\mathbb{R}_{p\times q}$: set of $p\times q$ real matrices.
- $M^\top$: the transpose of matrix M.
- Given a matrix A, denote the (i,j)-th element and the i-th column of A by $[A]_{i,j}$ and $\mathrm{Col}_i(A)$, respectively.
- $\mathcal{D}_k := \{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\}$ and $\mathcal{D}_k^n := \underbrace{\mathcal{D}_k \times \cdots \times \mathcal{D}_k}_{n}$.
- $\Delta_n := \left\{\delta_n^j \,|\, j = 1, 2, \cdots, n\right\}$, where $\delta_n^j = \mathrm{Col}_j(I_n)$ and I_n denotes the n-dimensional identity matrix.
- $\delta_n\{j_1, j_2, \cdots, j_n\} := \{\delta_n^{j_1}, \delta_n^{j_2}, \cdots, \delta_n^{j_n}\}$.
- A matrix $M \in \mathbb{R}_{p\times q}$ is called a logical matrix, if $M = [\delta_p^{j_1}, \delta_p^{j_2}, \cdots, \delta_p^{j_q}]$; Express M as $M = \delta_p[j_1, j_2, \cdots, j_q]$, and denote the set of all $p\times q$ logical matrices as $\mathcal{L}_{p\times q}$.
- $A * B$: the Khatri-Rao product of matrices A and B.
- $[a : b]$: the set of integers ρ satisfying $a \leqslant \rho \leqslant b$.
- 2^M: the power set of set M.
- $|\Phi|$: the cardinality of set Φ.
- $\mathbb{P}\{\cdot\}$: the probability of certain event.

Appendix B Calculation of PNEs

Rank all possible costs of using resource l under the η-disturbance combination into the following row vector

$$\Xi_l^\eta = [\Xi_l^\eta(1), \Xi_l^\eta(2), \cdots, \Xi_l^\eta(n)] \in \mathbb{R}_{1\times n}.$$

Denote $\Xi^\eta = [\Xi_{a_1}^\eta, \Xi_{a_2}^\eta, \cdots, \Xi_{a_m}^\eta] \in \mathbb{R}_{1\times nm}$, and further build a matrix

$$\Xi = [(\Xi^1)^\top, (\Xi^2)^\top, \cdots, (\Xi^{2^q})^\top]^\top \in \mathbb{R}_{2^q\times nm}.$$

Next, we construct a set of Boolean vectors as

$$v_l(s) = [\underbrace{1, 1, \cdots, 1}_{n_l(s)}, \underbrace{0, 0, \cdots, 0}_{n-n_l(s)}],$$

and define $V(s) = [v_{a_1}(s), v_{a_2}(s), \cdots, v_{a_m}(s)]^\top$.

A straightforward calculation gives the following result.

Proposition B1. The potential function can be expressed as $P(s) = J\Xi V(s),\ s \in S$.

Construct a matrix $V = [V(\delta_\kappa^1), V(\delta_\kappa^2), \cdots, V(\delta_\kappa^\kappa)]$ and define $P = J\Xi V$, a direct calculation yields

$$\begin{aligned} P &= [J\Xi V(\delta_\kappa^1), J\Xi V(\delta_\kappa^2), \cdots, J\Xi V(\delta_\kappa^\kappa)] \\ &= [P(\delta_\kappa^1), P(\delta_\kappa^2), \cdots, P(\delta_\kappa^\kappa)], \end{aligned}$$

where $\mathrm{Col}_j(P) = P(\delta_\kappa^j)$ is the value of potential function under profile δ_κ^j.

We can obtain the desired NE set of the CGSD as

$$\mathbb{NE} = \{\delta_\kappa^j | \mathrm{Col}_j(P) \leqslant \mathrm{Col}_\theta(P), \theta = 1, 2, \cdots, \kappa\}.$$

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Appendix C Construction of algebraic state space expression for the CGSDs

Denote by $s_i(t)$ and $\xi_j(t)$ the strategy of player $i \in N$ and the state of the i-th disturbance at time t, respectively. The myopic best response for player i can be expressed as

$$O_i(t) = argmin_{s_i \in S_i} c_i(s_i, s_{-i}(t), \xi(t)),$$

where $c_i(s_i, s_{-i}(t), \xi(t)) = \sum_{l \in s_i} \Xi_l^{\eta}(n_l(s_i, s_{-i}(t)))$ is the total cost of player i when $\xi(t)$ takes the η-th disturbance combination, and $s_{-i}(t)$ is the profile except for player i. If $s_i(t) \in O_i(t)$, let $s_i(t+1) = s_i(t)$. If $|O_i(t)| \geqslant 1$ and $s_i(t) \notin O_i(t)$, then player i randomly selects a strategy from $O_i(t)$ with the same probability $p = \frac{1}{|O_i(t)|}$.

The dynamics of player i can be expressed as

$$s_i(t+1) = f_i(s(t), \xi(t)), i \in N, \tag{C1}$$

where f_i is determined by the strategy updating rule.

Identify $\xi_i(t) = 1 \sim \delta_2^1$ and $\xi_i(t) = 0 \sim \delta_2^2$, and let $\vec{\xi}_i(t)$ be the vector expression of $\xi_i(t)$. By the STP method, we can convert (C1) into the algebraic form as

$$\vec{s}_i(t+1) = F_i \vec{\xi}(t) \vec{s}(t),\ i \in N, \tag{C2}$$

where $\vec{\xi}(t) = \ltimes_{i=1}^{q} \vec{\xi}_i(t) \in \Delta_{2^q}$ and $\vec{s}(t) = \ltimes_{i=1}^{n} \vec{s}_i(t) \in \Delta_\kappa$ are the logical vector forms of disturbance combination and profile, respectively. The matrix F_i is the structure matrix of f_i, which is a column random matrix.

Multiply the n equations of (C2), one has

$$\vec{s}(t+1) = F \vec{\xi}(t) \vec{s}(t),\ \forall \vec{\xi}(t) \vec{s}(t) \in \Delta_{2^q \kappa}, \tag{C3}$$

where $F = F_1 * F_2 * \cdots * F_n$. A simple calculation shows $\mathbb{P}(\vec{\xi}(t) = \delta_{2^q}^i) = [J]_{1,i}$. Since the external disturbance is random, we take the expectation on both sides of (C3) and obtain

$$\begin{aligned} \mathbb{E}\vec{s}(t+1) &= \sum_{i=1}^{2^q} ([J]_{1,i} F \delta_\kappa^i) \mathbb{E}\vec{s}(t) \\ &= F \ltimes J^\top \mathbb{E}\vec{s}(t) \\ &:= \bar{F} \mathbb{E}\vec{s}(t), \end{aligned} \tag{C4}$$

where $\bar{F} = F \ltimes J^\top$.

Consider system (C4).

1) A profile δ_κ^j is called a fixed point with probability one, if $\bar{F}\delta_\kappa^j = \delta_\kappa^j$;

2) A profile δ_κ^j is called a recurrent state if there exists an integer λ such that $(\delta_\kappa^j)^T \bar{F}^\lambda \delta_\kappa^j > 0$. Otherwise, it is called a transient state.

Appendix D Proofs in the body of the letter

Appendix D.1 Proof of Theorem 1

Proof. For any $i \in N, s_i, s_i' \in S_i, s_{-i} \in S_{-i}$, we can first calculate the difference of the expected cost for player i in profiles $s = (s_i, s_{-i})$ and $s' = (s_i', s_{-i})$ as

$$\begin{aligned} Ec_i(s) - Ec_i(s') &= (\sum_{l \in s_i} J\Xi_l(n_l(s)) - \sum_{l \in s_i'} J\Xi_l(n_l(s'))) \\ &= (\sum_{l \in s_i \setminus s_i'} J\Xi_l(n_l(s_{-i}) + 1) - \sum_{l \in s_i' \setminus s_i} J\Xi_l(n_l(s_{-i}) + 1)), \end{aligned}$$

where $n_l(s_{-i}) = |\{\rho \in N \setminus \{i\} | l \in s_p\}|$.

Next, we can compute $P(s) - P(s')$ as follows:

$$\begin{aligned} P(s) - P(s') &= (\sum_{l \in s_i \cup s_{-i}} \sum_{q=1}^{n_l(s)} J\Xi_l(q) - \sum_{l \in s_i' \cup s_{-i}} \sum_{q=1}^{n_l(s')} J\Xi_l(q)) \\ &= (\sum_{l \in s_{-i}} \sum_{q=1}^{n_l(s_{-i})} J\Xi_l(q) + \sum_{l \in s_i} J\Xi_l(n_l(s_{-i}) + 1) - \sum_{l \in s_{-i}} \sum_{q=1}^{n_l(s_{-i})} J\Xi_l(q) - \sum_{l \in s_i'} J\Xi_l(n_l(s_{-i}) + 1)) \\ &= (\sum_{l \in s_i \setminus s_i'} J\Xi_l(n_l(s_{-i}) + 1) - \sum_{l \in s_i' \setminus s_i} J\Xi_l(n_l(s_{-i}) + 1)). \end{aligned}$$

It follows from Definition 3 that the CGSD is a potential game with potential function (2). □

Appendix D.2 Proof of Proposition 2

Proof. Assume that profile $\delta_\kappa^{j^*} = \ltimes_{i=1}^n \delta_{\kappa_i}^{j_i^*}$ is a fixed point of system (C4) (system (3) in the body of the letter) with probability one, it follows that $\bar{F}\delta_\kappa^{j^*} = \delta_\kappa^{j^*}$. By (C3), for any $\vec{\xi}(t) \in \Delta_{2^q}$, $\delta_\kappa^{j^*} = F\vec{\xi}(t)\delta_\kappa^{j^*}$ holds. Then, according to equation (C2), one can further get $\delta_{\kappa_i}^{j_i^*} = F_i\vec{\xi}(t)\delta_{\kappa_i}^{j_i^*}$, $\forall i \in [1:n]$. Assume that $\delta_{\kappa_i}^{j_i^*}$ is the vector form of strategy s_i^*, that is, $\delta_\kappa^{j^*} \sim s^* = (s_1^*, s_2^*, \cdots, s_n^*)$, then for any $\vec{\xi}(t) \in \Delta_{2^q}$, $s_i^* \in O_i(t)$. Thus, for any $i \in [1:n]$, $\xi \in \mathcal{D}_2^q$ and $s_i \in S_i$, it holds that $c_i(s_i^*, s_{-i}^*, \xi) \leqslant c_i(s_i, s_{-i}^*, \xi)$.

For any $\xi = \delta_{2^q}^j$, $j \in [1:2^q]$, $c_i(s_i, s_{-i}, \xi) = \sum\limits_{l \in s_i} \Xi_l^j(n_l(s_i, s_{-i}))$. A direct calculation yields

$$\begin{aligned}
Ec_i(s_i^*, s_{-i}^*) &= \sum_{l \in s_i^*} J\Xi_l(n_l(s_i^*, s_{-i}^*)) \\
&= \sum_{l \in s_i^*} \sum_{j=1}^{2^q} [J]_{1,j}\Xi_l^j(n_l(s_i^*, s_{-i}^*)) \\
&= \sum_{j=1}^{2^q} [J]_{1,j} \sum_{l \in s_i^*} \Xi_l^j(n_l(s_i^*, s_{-i}^*)) \\
&\leqslant \sum_{j=1}^{2^q} [J]_{1,j} \sum_{l \in s_i} \Xi_l^j(n_l(s_i, s_{-i}^*)) \\
&= \sum_{l \in s_i} \sum_{j=1}^{2^q} [J]_{1,j}\Xi_l^j(n_l(s_i, s_{-i}^*)) \\
&= \sum_{l \in s_i} J\Xi_l(n_l(s_i, s_{-i}^*)) \\
&= Ec_i(s_i, s_{-i}^*)
\end{aligned}$$

holds for any $i \in [1:n]$ and $s_i \in S_i$. Hence, by Definition 2, the profile s^* is a PNE of the CGSD. □

Appendix D.3 Proof of Theorem 2

Proof. (Sufficiency) From the construction of $\bar{F}$, it follows that $[\bar{F}^\kappa]_{u,v} = \mathbb{P}\{\vec{s}(t+\kappa) = \delta_\kappa^u | \vec{s}(t) = \delta_\kappa^v\}$. Since $W^\top \mathrm{Col}_j(\bar{F}^\kappa) = [\bar{F}^\kappa]_{j_1^*,j} + [\bar{F}^\kappa]_{j_2^*,j} + \cdots + [\bar{F}^\kappa]_{j_\gamma^*,j}$, if $W^\top \mathrm{Col}_j(\bar{F}^\kappa) = 1$, one has $\mathbb{P}\{\vec{s}(\kappa) \in I(\mathcal{NE}) | \vec{s}(0) = \delta_\kappa^j\} = 1$. Moreover, there exist γ non-negative real numbers $a_1, a_2, \cdots, a_\gamma$ such that $\sum_{i=1}^\gamma a_i = 1$ and $\mathrm{Col}_j(\bar{F}^\kappa) = a_1\delta_\kappa^{j_1^*} + a_2\delta_\kappa^{j_2^*} + \cdots + a_\gamma\delta_\kappa^{j_\gamma^*}$. Since $\delta_{j_i^*}^{j_i^*}$, $i \in [1:\gamma]$ are fixed points with probability one, one has

$$\begin{aligned}
W^\top \mathrm{Col}_j(\bar{F}^{\kappa+1}) &= W^\top \cdot \bar{F} \cdot \mathrm{Col}_j(\bar{F}^\kappa) \\
&= W^\top(a_1\bar{F}\delta_\kappa^{j_1^*} + a_2\bar{F}\delta_\kappa^{j_2^*} + \cdots + a_\gamma\bar{F}\delta_\kappa^{j_\gamma^*}) \\
&= W^\top(a_1\delta_\kappa^{j_1^*} + a_2\delta_\kappa^{j_2^*} + \cdots + a_\gamma\delta_\kappa^{j_\gamma^*}) \\
&= W^\top \mathrm{Col}_j(\bar{F}^\kappa) = 1.
\end{aligned}$$

Obviously, $W^\top \mathrm{Col}_j(\bar{F}^{\kappa+t}) = 1$ holds for all positive integer t, which shows that $\mathbb{P}\{\vec{s}(\kappa+t) \in I(\mathcal{NE}) | \vec{s}(0) = \delta_\kappa^j\} = 1, \forall t \geqslant 0$. So δ_κ^j converges to $I(\mathcal{NE})$ in finite time with probability one.

(Necessity) We prove it by contradiction. If $W^\top \mathrm{Col}_j(\bar{F}^\kappa) = 1$ does not hold, there must exist a profile δ_κ^l such that $\mathbb{P}\{\vec{s}(\kappa) = \delta_\kappa^l \notin I(\mathcal{NE}) | \vec{s}(0) = \delta_\kappa^j\} > 0$. It shows that there exists one positive probability path with $\kappa+1$ profiles from δ_κ^j to δ_κ^l. Since there are κ pure strategy profiles in the game, there must be one time $\tau \in [0:\kappa-1]$ such that $\vec{s}(\tau) = \delta_\kappa^l$, which shows that $\mathbb{P}\{\vec{s}(\kappa) = \delta_\kappa^l | \vec{s}(\tau) = \delta_\kappa^l\} > 0$. Obviously, $\mathbb{P}\{\vec{s}(\kappa + r(\kappa-\tau)) = \delta_\kappa^l | \vec{s}(0) = \delta_\kappa^j\} > 0$ for all positive integer r, which implies that for any time t, there exists an integer $T > t$ such that $\mathbb{P}\{\vec{s}(T) \notin I(\mathcal{NE}) | \vec{s}(0) = \delta_\kappa^j\} > 0$. Then δ_κ^j can not converge to $I(\mathcal{NE})$ in finite time with probability one. The necessity is proved. □

Appendix D.4 Preliminaries for the Proof of Theorem 3

Construct a dynamic system as follows:

$$z(t+1) = Lz(t), \tag{D1}$$

where $z(t) \in \Delta_\alpha$, $L \in \mathbb{R}_{\alpha\times\alpha}$, and $0 \leqslant [L]_{i,j} \leqslant 1$ for all $i,j \in [1:\alpha]$. Obviously, $[L]_{i,j} = \mathbb{P}\{z(t+1) = \delta_\alpha^i | z(t) = \delta_\alpha^j\}$.

Lemma D1. Consider systems (C4) and (D1). For any $\delta_\kappa^{i_a} \in \mathcal{A}(\Omega)$, $\mathbb{P}\{\vec{s}(d) = \delta_\kappa^{i_a} | \vec{s}(0) = \delta_\kappa^{i_a}\} > 0$, if and only if $\mathbb{P}\{z(d) = \delta_\alpha^a | z(0) = \delta_\alpha^a\} > 0$.

Proof. (Necessity) For any $\delta_\kappa^{i_a} \in \mathcal{A}(\Omega)$, if $\mathbb{P}\{\vec{s}(d) = \delta_\kappa^{i_a} | \vec{s}(0) = \delta_\kappa^{i_a}\} > 0$, there must exist a cyclic path of length d from $\delta_\kappa^{i_a}$ to itself. Denote this cyclic path as $\delta_\kappa^{i_a} \to \delta_\kappa^{w_1} \to \cdots \to \delta_\kappa^{w_{d-1}} \to \delta_\kappa^{i_a}$. Clearly, for any $j \in [1:d-1]$, it holds that $\mathbb{P}\{\vec{s}(d) = \delta_\kappa^{w_j} | \vec{s}(0) = \delta_\kappa^{w_j}\} > 0$, so $\delta_\kappa^{w_j}$ must be a recurrent state. That is, $\delta_\kappa^{w_j} \in \{\delta_\kappa^{i_1}, \delta_\kappa^{i_2}, \cdots, \delta_\kappa^{i_\alpha}\}$. Re-denote this path as $\delta_\kappa^{i_a} \to \delta_\kappa^{i_{\tau_1}} \to \cdots \to \delta_\kappa^{i_{\tau_{d-1}}} \to \delta_\kappa^{i_a}$. We obtain that $[\bar{F}]_{i_{\tau_{j+1}}, i_{\tau_j}} > 0$ holds for all $j \in [1:d-1]$. Moreover, $[\bar{F}]_{i_{\tau_1}, i_a} > 0$, $[\bar{F}]_{i_a, i_{\tau_{d-1}}} > 0$. It follows from $[L]_{a,b} = [\bar{F}]_{i_a, i_b}$, $\forall\, a, b \in [1:\alpha]$ that $= [L]_{\tau_{j+1}, \tau_j} > 0$ for all $j \in [1:d-1]$, and $[L]_{\tau_1, a} > 0$, $[L]_{a, \tau_{d-1}} > 0$. Therefore, $\delta_\alpha^a \to \delta_\alpha^{\tau_1} \to \cdots \to \delta_\alpha^{\tau_{d-1}} \to \delta_\alpha^a$ is a cyclic path of system (D1). It is obvious that $\mathbb{P}\{z(d) = \delta_\alpha^a | z(0) = \delta_\alpha^a\} > 0$.

(Sufficiency) The proof of sufficiency is similar to that of necessity, so we do not prove it any more.

Remark 1. It can be known from the proof of Lemma D1 that there is a one-to-one correspondence between the cyclic paths of system (C4) and system (D1). To make all recurrent states in $\mathcal{A}(\Omega)$ become transient states by changing the one-step transition of some states, we only need to change the dynamics of system (D1) to make all its recurrent states disappear. In general, $\alpha \ll \kappa$. Since $\bar{F} \in \mathbb{R}_{\kappa \times \kappa}$ and $L \in \mathbb{R}_{\alpha \times \alpha}$, studying the elimination of recurrent states in system (D1) involves a significantly lower computational complexity than conducting the same research on the original system.

Remark 2. For any $\delta_\kappa^{i_a} \in \mathcal{A}(\Omega)$, there exists a positive integer $d \leqslant \alpha$ such that $\mathbb{P}\{\vec{s}(d) = \delta_\kappa^{i_a} | \vec{s}(0) = \delta_\kappa^{i_a}\} > 0$. So in system (D1), for any $\delta_\alpha^a \in \Delta_\alpha$, there exists a positive integer d such that $[L^d]_{a,a} = \mathbb{P}\{z(d) = \delta_\alpha^a | z(0) = \delta_\alpha^a\} > 0$. A simple calculation yields $[L^d]_{a,a} = \mathrm{Row}_a(L^{d-1})\mathrm{Col}_a(L)$, one has $\mathrm{Col}_a(L) \neq \mathbf{0}_\alpha$, $\forall\, a \in [1:\alpha]$. In other words, matrix L has no zero columns.

We further construct a dynamic system as

$$z(t+1) = L_\Lambda z(t),\ \Lambda \subseteq [1:\alpha]. \tag{D2}$$

Compared with system (D1), in system (D2), only the one-step dynamics of states in Λ have been changed. For any $\delta_\alpha^a \in \Lambda$, since $L_a \delta_\alpha^a = \mathbf{0}_\alpha$, it shows that $\mathbb{P}\{z(t+1) \in \Delta_\alpha | z(t) = \delta_\alpha^a\} = 0$. Correspondingly, in system (C4), it holds that $\mathbb{P}\{\vec{s}(t+1) \in \mathcal{A}(\Omega) | \vec{s}(t) = \delta_\kappa^{i_a}\} = 0$.

Appendix D.5 Proof of Theorem 3

Proof. If $\delta_\alpha^j \in \Omega_\Lambda$, one has $[(L_\Lambda)^t]_{j,j} = \mathbb{P}\{z(t) = \delta_\alpha^j | z(0) = \delta_\alpha^j\} = 0$ for all $1 \leqslant t \leqslant \alpha$ in system (D2), which implies that there is no cyclic path with a length equal or less than α from δ_α^j to itself. Since there are α non-zero states in system (D2), there is no cyclic path of length exceeding α that includes state δ_α^j. However, δ_α^j is a recurrent state of system (D1), so all the paths from δ_α^j to itself must include at least a state of set Λ in system (D1). If we let $\mathrm{Col}_a(L) = \mathbf{0}_\alpha$ for all $\delta_\alpha^a \in \Lambda$, then all the cyclic paths containing states in Ω_Λ are destroyed, that is, all states in Ω_Λ will become transient states. Denote $\Lambda = \{a_1, a_2, \cdots, a_{|\Lambda|}\}$ and $\Omega_\Lambda := \{\delta_\alpha^{b_1}, \delta_\alpha^{b_2}, \cdots, \delta_\alpha^{b_{|\Omega_\Lambda|}}\}$. Correspondingly, in system (C4), if we change the one-step transition of each profile in $\widetilde{\Lambda} := \{\delta_\kappa^{i_{a_1}}, \delta_\kappa^{i_{a_2}}, \cdots, \delta_\kappa^{i_{a_{|\Lambda|}}}\}$ to make it evolve to $R(I(\mathcal{NE}))$ at next time, then all these states will converge to $I(\mathcal{NE})$ and all states in $\widetilde{\Omega}_\Lambda := \{\delta_\kappa^{i_{b_1}}, \delta_\kappa^{i_{b_2}}, \cdots, \delta_\kappa^{i_{b_{|\Omega_\Lambda|}}}\}$ will become transient states. Since no new profile can reach set $\Delta_\kappa \setminus R(I(\mathcal{NE}))$ with positive probability, no new recurrent states appear in set $\Delta_\kappa \setminus R(I(\mathcal{NE}))$. In conclusion, if $\Omega_\Lambda = \Delta_\alpha$, then by changing the one-step transition of the profiles in $\widetilde{\Lambda}$ to make them evolve to $R(I(\mathcal{NE}))$ at next time, all profiles in $\mathcal{A}(\Omega)$ become transient states. Due to Λ is a set with the fewest elements such that $\Omega_\Lambda = \Delta_\alpha$ holds, $\delta_\kappa^{i_{a_1}}, \delta_\kappa^{i_{a_2}}, \cdots, \delta_\kappa^{i_{a_{|\Lambda|}}}$ are the fewest profiles requiring regulation to make the CGSD globally converge to $I(\mathcal{NE})$.

Remark 3. Obviously, if $[L]_{a,a} > 0$, then $[L_\mathcal{A}]_{a,a} > 0$ holds for any set $\mathcal{A}$ which does not contain δ_α^a. Therefore, δ_α^a must be in the set Λ which make $\Omega_\Lambda = \Delta_\alpha$ hold. Correspondingly, the state transition of $\delta_\kappa^{i_a}$ must be regulated in the CGSD.

Appendix E Computational complexity analysis of finding Λ

For any given set $\Gamma \subseteq [1:\alpha]$, the time complexity of calculating Ω_Γ is $O(\alpha^3)$. Since set $[1:\alpha]$ has 2^α subsets, and $\Omega_\emptyset$, $\Omega_{[1:\alpha]}$ do not need to be calculated, the maximum time complexity of finding Λ is $O((2^\alpha - 2)\alpha^3)$. In fact, the set Λ satisfying condition $\Omega_\Lambda = \Delta_\alpha$ in Theorem 3 has a small cardinality in most cases. Thus, we can quickly identify Λ by verifying the power set of $[1:\alpha]$ in the order of increasing cardinality. To further reduce the time complexity, we will investigate the relationship between sets $\Omega_{\Lambda_1 \cup \Lambda_2}$ and $\Omega_{\Lambda_1} \cup \Omega_{\Lambda_2}$ in future work, where Λ_1 and Λ_2 are the subsets of $[1:\alpha]$, and propose a more efficient method for identifying set Λ.

Appendix F An example

Example 1. Consider a transportation network shown in Figure F1, where six buses need to pass through some road segments to reach their destinations D_1, D_2, D_3 and D_4 respectively from the original station S. The destinations of the first and fourth cars are both D_1, and the destinations of the second and fifth cars are both D_2. The third car ends in D_3, and the sixth car ends in D_4. Assume that the disturbance event affecting this transportation network is road construction, where $\xi = 1$ denotes ongoing construction and $\xi = 0$ indicates no construction activities. We model this transportation system as a congestion game, where $N = \{1,2,3,4,5,6\}$ and $M = \{r_1, r_2, \cdots, r_{12}\}$. Based on the above information, we can obtain $S_1 = S_4 = \{r_3, r_1r_2\}, S_2 = S_5 = \{r_1r_7, r_4r_5r_6\}$, $S_3 = \{r_1r_7r_{10}, r_4r_8r_9, r_4r_5r_6r_{10}\}$ and $S_6 = \{r_{12}, r_3r_{11}, r_1r_2r_{11}\}$. Moreover, the external disturbance variable is ξ and $\mathbb{P}\{\xi = 0\} = 0.8, \mathbb{P}\{\xi = 1\} = 0.2$. The congestion costs for roads with different disturbance values are shown in Table F1.

Let $S_1 = S_4 \sim \Delta_2$, $S_2 = S_5 \sim \Delta_2$, $S_3 \sim \Delta_3$, $S_6 \sim \Delta_3$, then $S = \prod_{i=1}^{6} S_i \sim \Delta_{144}$. Precisely, for $S_1 = \{r_3, r_1r_2\}$, one has $r_3 \sim \delta_2^1$ and $r_1r_2 \sim \delta_2^1$. Furthermore, let $\xi = 1 \sim \delta_2^1, \xi = 0 \sim \delta_2^2$. By a direct calculation, we can get $J = (0.2\ 0.8)$,

$$\Xi = \begin{pmatrix} 6 & 9 & 10 & 11 & 13 & 15 & 9 & 10 & 13 & 15 & 17 & 19 & 10 & 18 & 19 & 22 & 24 & 26 & 6 & 9 & 11 & 13 & 15 & 17 & 4 & 6 & 9 & 11 & 16 & 19 & \cdots & 20 \\ 4 & 5 & 7 & 9 & 10 & 13 & 3 & 6 & 9 & 11 & 12 & 14 & 8 & 9 & 11 & 12 & 15 & 16 & 5 & 7 & 8 & 10 & 11 & 12 & 2 & 4 & 6 & 8 & 13 & 14 & \cdots & 15 \end{pmatrix} \in \mathbb{R}_{2\times 72},$$

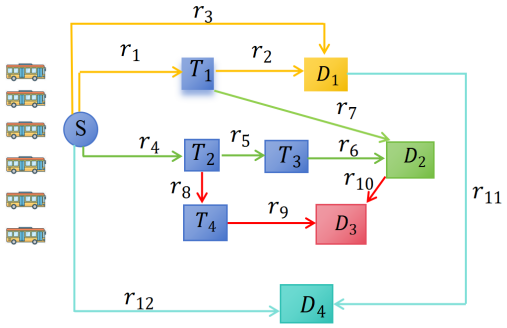

Figure F1 Route map for six buses

Table F1 Congestion costs under ξ

	$\Xi_l^1(1)$	$\Xi_l^1(2)$	$\Xi_l^1(3)$	$\Xi_l^1(4)$	$\Xi_l^1(5)$	$\Xi_l^1(6)$	$\Xi_l^2(1)$	$\Xi_l^2(2)$	$\Xi_l^2(3)$	$\Xi_l^2(4)$	$\Xi_l^2(5)$	$\Xi_l^2(6)$
r_1	6	9	10	11	13	15	4	5	7	9	10	13
r_2	9	10	13	15	17	19	3	6	9	11	12	14
r_3	10	18	19	22	24	26	8	9	11	12	15	16
r_4	6	9	11	13	15	17	5	7	8	10	11	12
r_5	4	6	9	11	16	19	2	4	6	8	13	14
r_6	6	8	13	15	17	18	5	6	9	11	14	16
r_7	6	8	9	14	17	20	4	7	8	10	13	15
r_8	7	12	13	14	16	19	6	9	10	12	13	14
r_9	8	10	12	14	15	16	4	5	6	7	9	10
r_{10}	8	12	14	16	18	19	6	9	10	11	12	14
r_{11}	6	11	13	16	19	21	4	7	8	10	11	13
r_{12}	8	12	14	15	18	20	7	9	11	13	14	15

and $V = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdots&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&\cdots&1&1&1&1&1&1&1&0&0&1&1&1&1&1&1 \\ 1&1&1&0&0&1&1&1&1&1&1&1&0&0&1&0&\cdots&1&0&0&1&0&0&1&0&0&0&1&1&1&0&0&1 \\ 0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&1&\cdots&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\cdots&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\cdots&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \in \mathbb{R}_{72\times 144}$.

According to (4), we have

$$\begin{aligned} P = [&70.4\ 80.2\ 81.2\ 67.4\ 84.8\ 76.4\ 73.2\ 81.2\ 87.8\ 68.4\ 76.4\ 81.8\ 64.4\ 74.2\ 73.4\ 66.4\ 76.2\ 73.6\ 65.4\ \ 73.4 \\ &78.8\ 65.6\ 73.6\ 77.2\ 67.4\ 77.2\ 76.4\ 76.8\ 86.6\ 79.8\ 68.4\ 76.4\ 81.8\ 71.8\ 79.8\ 83.4\ 67.4\ 77.2\ 76.4\ \ 72.6 \\ &82.4\ 79.8\ 68.4\ 76.4\ 81.8\ 71.8\ 79.8\ 83.4\ 66.4\ 76.2\ 73.6\ \ 77\ \ 86.8\ 82.8\ 65.6\ 68.8\ 77.2\ 74.8\ \ 82.8\ \ 84.6 \\ &72.6\ 82.4\ 79.8\ 88.8\ 98.6\ 94.6\ 71.8\ 79.8\ 83.4\ 86.6\ 94.6\ 96.4\ 66.8\ \ 76.8\ 87.8\ 68.4\ 76.4\ 81.8\ 82.2\ \ 79.4 \\ &102.6\ 76.2\ 81.8\ 93.8\ 65.4\ 73.4\ 78.8\ 65.6\ 73.6\ 77.2\ 73.2\ 67.2\ 79.2\ 71.6\ 77.2\ \ 88\ \ 72.8\ \ 76.4\ 81.8\ \ 71.8 \\ &79.8\ 83.4\ 76.2\ 81.8\ 93.8\ 77.8\ 83.4\ \ 94.2\ 68.4\ 76.4\ 81.8\ 71.8\ 79.8\ 83.4\ 76.2\ 81.8\ 93.8\ \ 77.8\ 83.4\ \ 94.2 \\ &65.6\ 67.8\ 77.2\ 74.8\ 82.8\ 84.6\ 71.6\ 77.2\ 88\ 79\ 84.6\ 83.8\ 82.6\ 79.8\ 83.4\ 86.6\ 94.6\ 96.4\ \ 77.8\ 83.4\ \ 94.2 \\ &90.8\ 96.4\ 105.4]. \end{aligned}$$

It is easy to see that the row vector P has a minimum value 64.4, then profile δ_{144}^{13} is the desired PNE of the CGSD, that is

$$\mathcal{NE} = \{\delta_{144}^{13}\}.$$

Next, we detect whether a given initial profile can be stable to $\mathcal{NE}$ in finite time with probability one. The evolutionary equation of this CGSD can be expressed in algebraic form as

$$\vec{s}(t+1) = F\vec{\xi}(t)\vec{s}(t), \tag{F1}$$

where $F = \delta_{144}[52, 52, 52, 16, 16 \times (\frac{1}{4}) + 22 \times (\frac{1}{4}) + 88 \times (\frac{1}{4}) + 94 \times (\frac{1}{4}), 16, \cdots, 13, 13, 13 \times (\frac{1}{2}) + 49 \times (\frac{1}{2})1, 1, 13] \in \mathbb{R}_{144\times 288}$ and $\delta_{144}^{i\times\frac{a}{b}+j\times\frac{d}{c}} = \frac{b}{a}\delta_{144}^{i} + \frac{d}{c}\delta_{144}^{j}$.

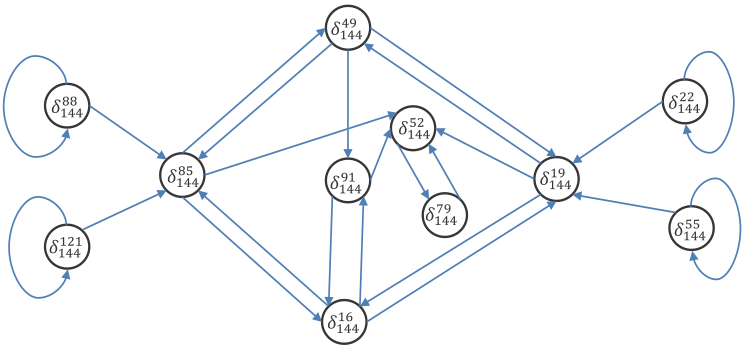

Figure F2 The evolutionary trajectories among the recurrent states in $\mathcal{A}(\Omega)$ before strategy regulation

Taking the expectation of both ends of (F1) yields

$$\mathbb{E}\vec{s}(t+1) = \bar{F}\mathbb{E}\vec{s}(t), \tag{F2}$$

where $\bar{F} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\cdots&0.2&0&0&0&0&0&1&0&0&0&1&1&0.2\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&\cdots&0.8&0.1&0&1&0&0&0&1&1&0.6&0&0&0.8\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
1&1&1&0&0&0&1&1&1&0&0&0&0&0&\cdots&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\cdots&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \in \mathbb{R}_{144\times 144}.$

We can verify that δ_{144}^{13} is a fixed point of system (F2) with probability one, that is, $I(\mathcal{NE}) = \mathcal{NE} = \{\delta_{144}^{13}\}$. Moreover, a simple calculation gives

$$R(I(\mathcal{NE})) = \{\delta_{144}^{13}, \delta_{144}^{25}, \delta_{144}^{27}, \delta_{144}^{30}, \delta_{144}^{31}, \delta_{144}^{33}, \delta_{144}^{36}, \delta_{144}^{63}, \delta_{144}^{69}, \delta_{144}^{99}, \delta_{144}^{102}, \delta_{144}^{103}, \delta_{144}^{104}, \delta_{144}^{106}, \delta_{144}^{107}, \delta_{144}^{135}, \delta_{144}^{139}, \delta_{144}^{140}\}.$$

Therefore, all the profiles in $R(I(\mathcal{NE}))$ can naturally converge to δ_{144}^{13} with probability one.

Denote $\Omega = \Delta_{144} \setminus R(I(\mathcal{NE}))$. We can calculate all the recurrent states in set Ω as

$$\mathcal{A}(\Omega) = \{\delta_{144}^{16}, \delta_{144}^{19}, \delta_{144}^{22}, \delta_{144}^{49}, \delta_{144}^{52}, \delta_{144}^{55}, \delta_{144}^{79}, \delta_{144}^{85}, \delta_{144}^{88}, \delta_{144}^{91}, \delta_{144}^{121}\}.$$

The evolutionary trajectories of these recurrent states are shown in Figure F2.

Identify $\delta_{144}^{16} \sim \delta_{11}^{1}, \delta_{144}^{19} \sim \delta_{11}^{2}, \cdots, \delta_{144}^{121} \sim \delta_{11}^{11}$, then matrix L can be constructed as

$$L = \begin{pmatrix}
0&0.2&0&0&0&0&0&0.2&0&0.1&0\\
0.05&0&0.5&0.05&0&0.45&0&0&0&0&0\\
0&0&0.4&0&0&0&0&0&0&0&0\\
0&0.2&0&0&0&0&0&0.2&0&0&0\\
0&0.2&0&0&0&0&1&0.2&0&0.8&0\\
0&0&0&0&0&0.4&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0.05&0&0&0.05&0&0&0&0&0.5&0&0.5\\
0&0&0&0&0&0&0&0&0.4&0&0\\
0.85&0&0&0.85&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0.4
\end{pmatrix}.$$

Since $[L]_{a,a} > 0$ for $a = 3, 6, 9, 11$, one knows from Remark 3 that $\delta_{11}^{3}, \delta_{11}^{6}, \delta_{11}^{9}, \delta_{11}^{11}$ must be in set Λ. A simple calculation shows that $\Lambda_1 = \{1, 3, 4, 5, 6, 9, 11\}$ and $\Lambda_2 = \{1, 3, 4, 6, 7, 9, 11\}$ are both the sets with the fewest elements such that

$$\Omega_{\Lambda_1} = \Delta_{11},\ \Omega_{\Lambda_2} = \Delta_{11},$$

hence, the profiles that need to be dynamically adjusted can be chosen as δ_{144}^{16}, δ_{144}^{22}, δ_{144}^{49}, δ_{144}^{52}, δ_{144}^{55}, δ_{144}^{88}, δ_{144}^{121} or δ_{144}^{16}, δ_{144}^{22}, δ_{144}^{49}, δ_{144}^{55}, δ_{144}^{79}, δ_{144}^{88}, δ_{144}^{121}.

Then we change the one-step transition of these profiles according to Algorithm 1 to make the CGSD globally converge to the desired PNE δ_{144}^{13}. We will take δ_{144}^{16} as an example to demonstrate the specific steps.

Table F2 Strategy evolutionary changes induced by regulation

Profiles that needs to be regulated	δ_{144}^{16}	δ_{144}^{22}	δ_{144}^{49}	δ_{144}^{52}	δ_{144}^{55}	δ_{144}^{88}	δ_{144}^{121}
Subsequent profiles in the next moment	δ_{144}^{13}	δ_{144}^{13}	δ_{144}^{13}	δ_{144}^{104}	δ_{144}^{13}	δ_{144}^{13}	δ_{144}^{13}
Players requiring strategy adjustment	1,4	4,5	1,4	3	2,3,4	1,5	1,2

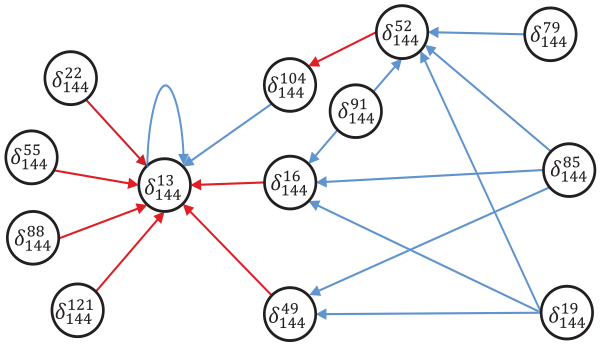

Figure F3 The evolutionary trajectories of the recurrent states after strategy regulation. The blue arrows represent the evolutionary trajectories without strategy regulation, while the red arrows denotes the evolutionary trajectories after strategy regulation.

Since $(\delta_{144}^{13})^\top \bar{F}\delta_{144}^{16} > 0$, $(\delta_{144}^{19})^\top \bar{F}\delta_{144}^{16} > 0$, $(\delta_{144}^{85})^\top \bar{F}\delta_{144}^{16} > 0$, $(\delta_{144}^{91})^\top \bar{F}\delta_{144}^{16} > 0$, profile δ_{144}^{16} has a certain probability to evolve to $\delta_{144}^{13}, \delta_{144}^{19}, \delta_{144}^{85}$, and δ_{144}^{91}. Decompose these profiles as follows:

$$\delta_{144}^{13} = \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_3^2 \ltimes \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_3^1,\ \delta_{144}^{19} = \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_3^2 \ltimes \delta_2^2 \ltimes \delta_2^1 \ltimes \delta_3^1,$$
$$\delta_{144}^{95} = \delta_2^2 \ltimes \delta_2^1 \ltimes \delta_3^2 \ltimes \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_3^1,\ \delta_{144}^{91} = \delta_2^2 \ltimes \delta_2^1 \ltimes \delta_3^2 \ltimes \delta_2^2 \ltimes \delta_2^1 \ltimes \delta_3^1.$$

In these four profiles, just the strategies of players 1 and 4 are different. This shows that when $x(t) = \delta_{144}^{16}$, players 1 and 4 have multiple strategies to choose, but players 2,3,5 and 6 have only one choice. So we can adjust the strategies of players 1 and 4, and let other players evolve naturally according to the strategy updating rule. For example, by means of rewards or other incentives, we can make $x_1(t+1) = \delta_2^1$, $x_4(t+1) = \delta_2^1$, in this case, δ_{144}^{16} will converge to PNE δ_{144}^{13} at next time.

Similarly, the dynamics of the remaining profiles can be adjusted. The specific adjustments can be shown in Table F2. Since profile δ_{144}^{52} only requires adjusting one player, we can choose to regulate it rather than δ_{144}^{79}.

The evolutionary trajectories of the recurrent states after strategy regulation are shown in Figure F3. Obviously, after strategy regulation, all recurrent states in $\mathcal{A}(\Omega)$ have been transformed into transient states, so they all evolve to the equilibrium point δ_{144}^{13} in finite time with probability one.