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Special Topic: Logical System Control

Algebraic expressions for stochastic dynamics in populations: from strategy profile to aggregate state

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Stochastic dynamics have been commonly used to quantitatively model strategic interactions in populations [1]. By virtue of the ability to accurately characterize strategy updating and stochastic effects, these dynamics exhibit significant advantages for analysis and control [1–3]. Specifically, each player in the population selects a strategy from a finite set to engage in strategic interactions. The strategies of players undergo stochastic updating, which is termed the probabilistic strategy updating rule (PSUR). Examples include the Wright-Fisher process [1], the Fermi process [3], and the Moran process. However, quantitative analysis of such stochastic dynamics is hindered by the curse of dimensionality, which refers to the phenomenon that the state space grows exponentially with the population size [4]. A potential approach to addressing this bottleneck is aggregation [5].

In this study, we adopt an aggregation method that exploits anonymity to reduce the size of the state space. A core implication of anonymity is that the dynamics remain invariant under any permutation of player identities. Under this framework, the state space can be partitioned into a series of equivalence classes. States within each class are then aggregated into a single aggregate state. Leveraging the semi-tensor product (STP) of matrices [3], we theoretically transform the original stochastic dynamics of strategy profiles into the induced dynamics of aggregate states. The main contributions of this study are as follows: (i) an exact algebraic expression for the strategy profile dynamics under an arbitrary payoff structure and a general PSUR is formulated via STP; (ii) based on anonymity, an algebraic expression of the aggregate state dynamics associated with the aforementioned dynamics is further derived. Consequently, for a population of n players choosing among m strategies, the state space is reduced from m^n to $\frac{(n+m-1)!}{n!(m-1)!}$ through the aggregation method. This dimensionality reduction, in turn, enables efficient analysis of population dynamics and the formulation of relevant control strategies.

Problem formulation. Consider a population $N=\{1,2,\dots,n\}$ consisting of n players. Each player adopts a strategy from the finite set $\mathcal{D}=\{1,2,\dots,m\}$. By identifying strategy $k\in\mathcal{D}$ with the canonical basis vector $\delta_m^k\in\Delta_m$ (i.e., a length-m vector with 1 at position k and 0s elsewhere), the strategy of a player $i\in N$ can be expressed as a logical vector $s_i\in\Delta_m$. Accordingly, the logical vector $s=\ltimes_{i=1}^n s_i\in\Delta_{m^n}$ denotes the strategy profile. Within the population N, each player repeatedly engages in strategic interactions with other players to obtain its payoff, and subsequently updates its strategy according to a payoff-dependent PSUR.

The construction of strategy profile dynamics relies on two fundamental components: the payoff structure and the PSUR. Integrating these components yields a stochastic dynamic process characterized by the set of transition probabilities $\{P(\tilde{s}|s):\tilde{s},s\in\Delta_{m^n}\}$, where $P(\tilde{s}|s)$ denotes the probability of transitioning from strategy profile s to $\tilde{s}$. For brevity, we refer to this transition rule simply as $\{P(\tilde{s}|s)\}$. Consistent with the property of anonymity, $\{P(\tilde{s}|s)\}$ is unchanged under any relabeling of players, in the sense that

$$P\big(\sigma(\tilde{s})|\sigma(s)\big)=P\big(\tilde{s}|s\big),\quad s,\tilde{s}\in\Delta_{m^n},\ \sigma\in\mathfrak{S}_N,\tag{1}$$

where $\mathfrak{S}_N$ denotes the symmetric group on N, and $\sigma(s)=\ltimes_{i=1}^n s_{\sigma(i)}$ represents the strategy profile obtained from s by relabeling players according to σ.

Algebraic expression of strategy profile dynamics. Building upon the transition rule $\{P(\tilde{s}|s)\}$, the evolution of the strategy profile can be rigorously formulated as a discrete-time stochastic logical dynamic system. Consequently, the algebraic expression for this system is given by

$$s(t+1)=F^{\Pi}_{\omega(t)}s(t),\quad t=0,1,2,\dots,\tag{2}$$

where $\Pi\in\mathcal{M}_{m^n\times m^n}$ denotes the transition probability matrix, and $\{\omega(t)\}_{t\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables uniformly distributed on $[0,1]$. The (y,x)-th entry of Π represents the transition probability from the strategy profile $\delta_{m^n}^x$ to $\delta_{m^n}^y$, which can be expressed as

$$\Pi_{y,x}=P\big(\delta_{m^n}^y|\delta_{m^n}^x\big).$$

More formally, the k-th column of the matrix $F^{\Pi}_{\omega}\in\mathcal{L}_{m^n\times m^n}$ is formulated as

$$\mathrm{Col}_k\left(F^{\Pi}_{\omega}\right)=\begin{cases}\delta_{m^n}^{\kappa}, & \omega\in\big[\Gamma_{\kappa,k},\Gamma_{\kappa+1,k}\big),\ \kappa\in[1:m^n-1],\\ \delta_{m^n}^{m^n}, & \text{otherwise},\end{cases}$$

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where $\Gamma_{\kappa,k}$ is the (κ,k)-th entry of $\Gamma = U\Pi$, and U is the strictly lower triangular matrix of ones (i.e., $U_{i,j} = 1$ if $i > j$, and 0 otherwise).

The condition (1) entails that the matrix Π satisfies the following property, which can be leveraged to aggregate these strategy profiles and reduce complexity.

Lemma 1. The transition rule satisfies (1) if and only if the transition probability matrix Π in the strategy profile dynamics (2) satisfies

$$W_\sigma^\top \Pi W_\sigma = \Pi, \quad \sigma \in \mathfrak{S}_N, \tag{3}$$

where W_σ denotes the σ-permutation matrix satisfying $\sigma(s) = W_\sigma s,\ s \in \Delta_{m^n}$.

Next, we leverage the profile-level dynamics (2) and the property (3) to derive the induced dynamics for aggregate states.

Algebraic expression of aggregate state dynamics. Define the aggregate state of the population as $\mu = [\mu_1, \dots, \mu_m]^\top \in \Upsilon_m^n$, where μ_k denotes the number of players adopting δ_m^k. Here, $\Upsilon_m^n = \{[\mu_1, \dots, \mu_m]^\top \in \mathbb{N}^m : \sum_{k=1}^m \mu_k = n\}$ represents the set of all feasible aggregate states, and its cardinality is denoted by $\tau = |\Upsilon_m^n|$. For a given strategy profile $s \in \Delta_{m^n}$, the induced aggregate state μ^s can be calculated as $\mu^s = \sum_{i\in N} s_i$. In terms of the STP, μ^s admits an equivalent matrix representation given below.

Lemma 2. For any strategy profile $s \in \Delta_{m^n}$, there exists a unique matrix $V_\mu \in \mathcal{M}_{\tau\times m^n}$ such that $\mu^s = V_\mu s$, where

$$V_\mu = \sum_{i=1}^n \mathbf{1}_{m^{i-1}}^\top \otimes I_m \otimes \mathbf{1}_{m^{n-i}}^\top$$

is referred to as the structure matrix.

Consequently, for a given s, the induced aggregate state μ^s can be computed directly via $\mu^s = V_\mu s$. By contrast, characterizing the set of all strategy profiles that induce a given aggregate state μ (i.e., the preimage of μ) is generally nontrivial. Nevertheless, every element in the preimage of μ satisfies the following property.

Lemma 3. Let $s, \tilde{s} \in \Delta_{m^n}$. The equality $\mu^{\tilde{s}} = \mu^s$ holds if and only if $\tilde{s} = W_\sigma s$ for some permutation $\sigma \in \mathfrak{S}_N$.

After sorting all aggregate states in lexicographic order, each state μ is assigned a unique index $n_\mu \in [1:\tau]$. Accordingly, the space Δ_{m^n} is partitioned into a set of equivalence classes $S = \{S_1, S_2, \dots, S_\tau\}$, where S_{n_μ} precisely constitutes the preimage of μ. By Lemma 3, the equivalence class S_{n_μ} can be explicitly expressed as

$$S_{n_\mu} = \{s \in \Delta_{m^n} : s = W_\sigma s_0^\mu,\ \sigma \in \mathfrak{S}_N\},$$

where $s_0^\mu = \ltimes_{k_1=1}^{\mu_1} \delta_m^1 \ltimes_{k_2=1}^{\mu_2} \delta_m^2 \cdots \ltimes_{k_m=1}^{\mu_m} \delta_m^m$ serves as the canonical representative of the class S_{n_μ}. Crucially, constructing s_0^μ necessitates the recovery of μ from its index n_μ. The algorithm for this reconstruction procedure is provided in Appendix G.

Observe that some permutations leave $s \in \Delta_{m^n}$ unchanged. These permutations form the stabilizer of s, defined as $\{\sigma \in \mathfrak{S}_N : W_\sigma s = s\}$. All elements of this set admit the following characterization.

Lemma 4. Let $s \in \Delta_{m^n}$. A permutation $\sigma \in \mathfrak{S}_N$ satisfies $W_\sigma s = s$ if and only if

$$\sigma = \sigma_1 \circ \sigma_2 \cdots \circ \sigma_m,$$

where $\sigma_k \in \mathfrak{S}_{N\setminus N_k(s)},\ k \in \mathcal{D}$. Here, $N_k(s)$ denotes the set of δ_m^k-strategists in the strategy profile s, and $\mathfrak{S}_{N\setminus M}$ denotes the pointwise stabilizer of subset $N\setminus M$ in $\mathfrak{S}_N$.

To elaborate, the stabilizer of s is the direct product of permutations acting on each set of players who share the same strategy in s. This structural property facilitates the enumeration of strategy profiles within each equivalence class. The following proposition presents a collected matrix that algebraically sums all strategy profiles belonging to the equivalence class S_{n_μ}.

Proposition 1. Let $\Psi \in \mathcal{M}_{m^n\times\tau}$ be a collector matrix whose n_μ-th column is defined by $\mathrm{Col}_{n_\mu}(\Psi) = \sum_{s\in S_{n_\mu}} s$. Then $\mathrm{Col}_{n_\mu}(\Psi)$ admits the representation:

$$\mathrm{Col}_{n_\mu}(\Psi) = \frac{1}{\mu_1!\mu_2!\cdots\mu_m!} \sum_{\sigma\in\mathfrak{S}_N} W_\sigma s_0^\mu.$$

Moreover, Ψ is invariant under any permutation in the sense that $W_\sigma \Psi = \Psi,\ \sigma \in \mathfrak{S}_N$.

Remark 1. The normalization factor $(\mu_1!\mu_2!\cdots\mu_m!)^{-1}$ compensates for the overcounting introduced by the stabilizer in the summation, and ensures that

$$s^\top \mathrm{Col}_{n_\mu}(\Psi) = 1, \quad s \in S_{n_\mu}.$$

By identifying $n_\mu \sim \delta_\tau^{n_\mu}, n_\mu \in [1:\tau]$, each aggregate state can be equivalently expressed in a logical vector form as $x \in \Delta_\tau$. By virtue of Proposition 1, the aggregated dynamics can be formulated as the following.

Theorem 1. Consider the stochastic logical dynamics $s(t)$ with the algebraic form (2), where the transition probability matrix Π satisfies (3). The induced aggregate state of $s(t)$ with respect to the partition S is $V_\mu s(t)$, whose lexicographic index is denoted by $n_\mu(t)$. Let $x(t) = \delta_\tau^{n_\mu(t)}$ be the logical vector equivalent to $V_\mu s(t)$. Then the algebraic expression for the induced aggregate state dynamics $x(t)$ is given by

$$x(t+1) = F_{\omega(t)}^{\hat{\Pi}} x(t), \quad t = 0, 1, 2, \dots,$$

where $\hat{\Pi} = \left(\mathrm{diag}(\Psi^\top \mathbf{1}_{m^n})\right)^{-1} \Psi^\top \Pi \Psi$ is the $\tau\times\tau$ transition probability matrix.

Remark 2. By partitioning the original m^n profiles into $\tau = \frac{(n+m-1)!}{n!(m-1)!}$ aggregate classes, we obtain a new stochastic process whose state space is dramatically reduced. For example, instead of 2^n states for n players with 2 strategies, we would have only $n+1$ aggregate states.

An illustrative example is provided in Appendix K to demonstrate how the proposed aggregation method effectively mitigates the computational bottleneck noted earlier, while preserving exact dynamics.

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Supporting information Appendixes A–K. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

References

1 Tan S, Lu J, Hill D J. Towards a theoretical framework for analysis and intervention of random drift on general networks. IEEE Trans Automat Contr, 2015, 60: 576–581
2 Wu Y, Shen T. Policy iteration algorithm for optimal control of stochastic logical dynamical systems. IEEE Trans Neural Netw Learn Syst, 2018, 29: 2031–2036
3 Cheng D, He F, Qi H, et al. Modeling, analysis and control of networked evolutionary games. IEEE Trans Automat Contr, 2015, 60: 2402–2415
4 Hauser O P, Traulsen A, Nowak M A. Heterogeneity in background fitness acts as a suppressor of selection. J Theor Biol, 2014, 343: 178–185
5 Li H, Wang S, Li W. Aggregation method to strategy consensus of large-size networked evolutionary matrix games. IEEE Trans Automat Contr, 2024, 69: 3975–3981