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Special Topic: Logical System Control

Observability and approximate observability of Boolean control networks from finite offline data

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Boolean networks (BNs) employ binary values and logic functions to model complex discrete systems [1]. The binary values representing system states are called nodes, and the system's dynamical behavior is specified by logical functions where each node is governed by its own logical update function. Through the semi-tensor product (STP) [2], BNs are transformed into algebraic state-space representations, enabling significant progress in stability, controllability, and observability analysis. Observability is a fundamental concept in control theory, characterizing the possibility of uniquely recovering initial states from measured outputs. Observability in BNs has been extensively investigated through finite automata and matrix criteria [3,4].

The application of data-driven methods in BNs can be categorized into two distinct approaches. The first involves acquiring data through online interaction with models, such as reinforcement learning. The second approach focuses on the analysis of offline datasets. Inspired by advances in data-driven control for nonlinear systems [5], the data informativity approach has been adopted to provide a unified perspective for the control of BNs. This study presents the following main contributions.

- An error-based structural observability formulation is developed with rigorous proof that Boolean control networks (BCNs) converge to this bound within finite time, accompanied by a computational algorithm with worst case time complexity $O(MN^2T^*)$.
- Within the data informativity paradigm, data observability bounds are constructed directly from finite offline datasets without requiring explicit model identification, with critical thresholds and effectiveness metrics established for approximating structural observability limits.

Notations: The set of positive integers is $\mathbb{N}_+$; $\mathbb{N}_{a\to b} = \{a, a+1, \ldots, b\}$. $\mathbb{R}_{n\times m}$ ($\mathbb{L}_{n\times m}$) denotes the set of $n \times m$ real (logical) matrices. $\mathfrak{D} := \{0, 1\}$. The $n \times n$ identity matrix is denoted by I_n; the a-th column of I_n is designated by δ_n^a; the set $\Delta_n := \{\delta_n^a \mid a \in \mathbb{N}_{1\to n}\}$.

Consider a BCN comprising n state nodes, m input nodes, and p output observations: $\mathbf{x}_i(t+1) = f_i(\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_m(t))$, $\mathbf{y}_j(t) = h_j(\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t))$, where $\mathbf{x}_i(t) \in \mathfrak{D}$ for $i \in \mathbb{N}_{1\to n}$ denotes the state variables at discrete time $t \in \mathbb{N}_+ \cup \{0\}$, $\mathbf{u}_k(t) \in \mathfrak{D}$ for $k \in \mathbb{N}_{1\to m}$ is the control inputs, $\mathbf{y}_j(t) \in \mathfrak{D}$ for $j \in \mathbb{N}_{1\to p}$ constitutes the output observations, and $f_i : \mathfrak{D}^{n+m} \to \mathfrak{D}$ for $i \in \mathbb{N}_{1\to n}$ and $h_j : \mathfrak{D}^n \to \mathfrak{D}$ for $j \in \mathbb{N}_{1\to p}$.

Let x_i, u_j, and y_k denote the vector forms of $\mathbf{x}_i$, $\mathbf{u}_j$, and $\mathbf{y}_k$, respectively [2]. Define $x := \ltimes_{i=1}^n x_i \in \Delta_N$, $u := \ltimes_{k=1}^m u_k \in \Delta_M$, and $y := \ltimes_{j=1}^p y_j \in \Delta_P$, where $N = 2^n$, $M = 2^m$ and $P = 2^p$. Then, using STP method, the following compact representation is obtained:

$$x(t+1) = Lu(t)x(t), \ y(t) = Hx(t). \tag{1}$$

In the following, $\mathcal{B}(L, H)$ denotes a BCN with the structure matrix pair (L, H). Consider a collection of q finite experimental sequences defined as $\mathcal{Z} := \{(u_i(t), x_i(t), y_i(t), x_i(t+1)) \mid t \in \mathbb{N}_{0\to T_i-1}, \ i \in \mathbb{N}_{1\to q}\}$, where $u_i(t) \in \Delta_M$, $x_i(t) \in \Delta_N$, and $y_i(t) \in \Delta_P$ represent the input, state, and output vectors at time t for the i-th trajectory, respectively. The consolidated dataset is formulated as $\mathcal{D} := \{(u, x, y, x^+) | \exists i \in \mathbb{N}_{1\to q}, \exists t \in \mathbb{N}_{0\to T_i-1}, (u, x, y, x^+) = (u_i(t), x_i(t), y_i(t), x_i(t+1))\}$, where $u \in \Delta_M$, $x, x^+ \in \Delta_N$, and $y \in \Delta_P$ denote the input, current state, next state, and output vectors, respectively.

Consequently, the cardinality of the dataset $\mathcal{D}$, denoted by $C_{\mathcal{D}}$, is bounded by the total number of collected time samples, i.e., $C_{\mathcal{D}} = |\mathcal{D}| \leqslant \sum_{i=1}^q T_i$. Each sample $(u, x, y, x^+) \in \mathcal{D}$ constrains one column of the structure matrix pair (L, H) of BCN (1): $x^+ = Lux$, $y = Hx$. When a specific sample $(\delta_M^j, \delta_N^i, \delta_P^k, \delta_N^{i'})$ appears in the dataset $\mathcal{D}$, the corresponding column of L is uniquely determined by $x^+ = \delta_M^{i'}$, i.e., $\mathrm{Col}[(j-1)\cdot 2^n + i](A) = \delta_N^{i'}$. Conversely, if the sample $(\delta_M^j, \delta_N^i, \delta_P^k, \delta_N^{i'})$ never occurs in the dataset $\mathcal{D}$, the corresponding column of L remains unconstrained. The same principle applies to matrix H. Therefore, the complete structure matrix of the BCN cannot, in general, be fully reconstructed from finite data alone.

Definition 1. A structure matrix pair (L, H) of BCN (1) is said to be compatible with the dataset $\mathcal{D}$ if each column of L and H that has been constrained by $\mathcal{D}$ coincides with the observed data, while all unconstrained columns are free. The set of all such compatible models is denoted by $\mathcal{M}(\mathcal{D})$.

Definition 2. Let $\mathcal{P}$ be a system property. The dataset $\mathcal{D}$ is said to be informative for property $\mathcal{P}$ if every structure matrix pair (L, H) in $\mathcal{M}(\mathcal{D})$ satisfies property $\mathcal{P}$.

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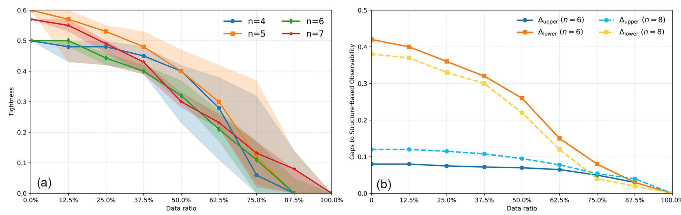

Figure 1 (Color online) Effect of data ratio on observability metrics. (a) The variation of tightness with data ratio at different network sizes; (b) gaps to structure-based observability curves for $n = 6$ and $n = 8$ showing Δ_{upper} and Δ_{lower} as functions of data ratio.

Definition 3. Consider BCN (1). The system is said to be observable if, for any two distinct initial states $x'(0) \neq x''(0)$, there exists an input sequence $u(0), u(1), \ldots, u(T-1)$ such that the corresponding output trajectories are distinguishable, i.e., $\exists k \in \{0, 1, \ldots, T\}$ s.t. $y'(k) \neq y''(k)$, where $y'(k)$ and $y''(k)$ denote the outputs at time k from initial states $x'(0)$ and $x''(0)$, respectively.

Consider BCN (1), for a given initial state $x(0) \in \Delta_N$ and $T \in \mathbb{N}_+$. Let $u_{(0:T-1)} = (u(0), \ldots, u(T-1))$ be an input sequence of length T and $y_T(x(0), u_{(0:T-1)}) = (y(0), \ldots, y(T-1))$ be the output sequence from time 0 to $T-1$ under input sequence $u_{(0:T-1)}$, where $u(i) \in \Delta_M$ and $y(j) \in \Delta_P$ $i, j \in \mathbb{N}_+$. Define the indistinguishable state set of $x(0)$ as $\hat{\mathcal{X}}(x(0)) := \{\ x'(0) \in \Delta_N \mid \forall\ T > 0, \exists u_{(0:T-1)}$ such that $y_T(x(0), u_{(0:T-1)}) = y_T(x'(0), u_{(0:T-1)})\ \}$. The error associated with initial state $x(0)$ is $err(x(0)) := \max_{x'(0) \in \hat{\mathcal{X}}(x(0))} dist(x'(0), x(0))$, where $dist(x, x')$ denotes the Hamming distance between $\mathbf{x}$ and $\mathbf{x}'$, that is, $dist(x, x') = |\{i \mid \mathbf{x}_i \neq \mathbf{x}'_i\}|$, $\mathbf{x}$ and $\mathbf{x}'$ are the corresponding Boolean forms of vector x and x'. The error rate of the $\mathcal{B}(L, H)$ is then defined as

$$\operatorname{Err}(L, H) := \frac{1}{n \cdot N} \sum_{x(0) \in \Delta_N} err(x(0)). \tag{2}$$

Definition 4. Consider $\mathcal{B}(L, H)$. Given a tolerance parameter $\epsilon \in [0, 1]$, the BCN is said to be ϵ-approximately observable if the error rate satisfies $\operatorname{Err}(L, H) \leqslant \epsilon$.

Definition 5. For $\mathcal{B}(L, H)$, define the structure-based observability as $\mathcal{O}_S(L, H) = 1 - \operatorname{Err}(L, H)$, where $\operatorname{Err}(L, H)$ is the error rate introduced in (2).

For $\mathcal{B}(L, H)$, define $\Omega := \{(\delta_N^i, \delta_N^j) | 1 \leqslant i < j \leqslant N\}$. For $T \in \mathbb{N}_+ \cup \{0\}$, define $\Omega_T := \{(\delta_N^i, \delta_N^j) \in \Omega \mid \exists\ t \leqslant T$ and $u_{(0:T-1)}$ such that , $y_T(\delta_N^i, u_{(0:T-1)}) \neq y_T(\delta_N^j, u_{(0:T-1)})\}$, with $\Omega_0 := \{(\delta_N^i, \delta_N^j) \in \Omega \mid H\delta_N^i \neq H\delta_N^j\}$. Define the operator $\mathcal{Q}$ by $\mathcal{Q}(\mathcal{X}) := \{(\delta_N^i, \delta_N^j) \in \Omega | \ \exists u = \delta_M^k, (L\delta_M^k \delta_N^i, L\delta_M^k \delta_N^j) \in \mathcal{X},\ k \in \mathbb{N}_{1 \to M}.\}$

Theorem 1. For any $\mathcal{B}(L, H)$, there exists a minimal integer $T^* \leqslant N(N-1)/2$ such that $\forall T \geqslant T^*$, $\Omega_T = \Omega_{T^*}$.

Proposition 1. For $\mathcal{B}(L, H)$, the indistinguishable set of each initial state is recovered from Ω_{T^*} by $\hat{\mathcal{X}}(\delta_N^i) = \{\delta_N^i\} \cup \{\delta_N^j \mid \delta_N^j \neq \delta_N^i, (\delta_N^i, \delta_N^j) \notin \Omega_{T^*}\}, \delta_N^i, \delta_N^j \in \Delta_N$.

Remark 1. By Theorem 1, for given $\mathcal{B}(L, H)$ the recursion $\Omega_{T+1} = \Omega_T \cup \mathcal{Q}(\Omega_T)$ stabilizes at a minimal horizon T^*; hence the maximal indistinguishable set Ω_{T^*} is obtained in finite time. Proposition 1 then recovers $\operatorname{Err}(L, H)$ from the set Ω_{T^*} and yields $\mathcal{O}_S(L, H)$. Therefore $\mathcal{O}_S(L, H)$ is computable in finite time for any $\mathcal{B}(L, H)$.

Definition 6. For a given dataset $\mathcal{D}$, the corresponding set of structure matrix pairs is denoted by $\mathcal{M}(\mathcal{D})$. The data observability upper bound and data observability lower bound are defined as follows: $\mathcal{O}_D^u(\mathcal{D}) = 1 - \min_{(L,H) \in \mathcal{M}(\mathcal{D})} \operatorname{Err}(L, H)$, $\mathcal{O}_D^l(\mathcal{D}) = 1 - \max_{(L,H) \in \mathcal{M}(\mathcal{D})} \operatorname{Err}(L, H)$.

Theorem 2. For a $\mathcal{B}(L, H)$ and a tolerance parameter $\epsilon \in [0, 1]$. The BCN is ϵ-approximately observable if and only if there exists a dataset D from this BCN such that $\mathcal{O}_D^l(\mathcal{D}) \geqslant 1 - \epsilon$.

Proposition 2. For a $\mathcal{B}(L, H)$, the BCN is observable if and only if there exists a dataset $\mathcal{D}$ from this BCN such that $\mathcal{O}_D^l(\mathcal{D}) = 1$.

Proposition 3. For a $\mathcal{B}(L, H)$ and a dataset $\mathcal{D}$ sampled from this BCN, the following inequality holds: $\mathcal{O}_D^l(\mathcal{D}) \leqslant \mathcal{O}_S(L, H) \leqslant \mathcal{O}_D^u(\mathcal{D})$. Here equality holds if and only if the error rate attains the same value for all models in $\mathcal{M}(\mathcal{D})$.

Proposition 4. If datasets $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then the corresponding data observability upper and lower bounds satisfy $\mathcal{O}_D^u(\mathcal{D}_1) \geqslant \mathcal{O}_D^u(\mathcal{D}_2)$ and $\mathcal{O}_D^l(\mathcal{D}_1) \leqslant \mathcal{O}_D^l(\mathcal{D}_2)$.

Proposition 5. For BCNs with n nodes and m inputs, the minimal dataset cardinality $|\mathcal{D}|$ needed to ensure $\mathcal{O}_D^l(\mathcal{D}) = \mathcal{O}_S(L, H) = \mathcal{O}_D^u(\mathcal{D})$, can be as large as NM and can be as small as 0. Then the limits NM and 0 are obtained.

Simulation results. Randomly generated BCNs with outputs are studied. The number of nodes n is prescribed, control inputs m are randomly selected from $\{1, 2\}$, and the in-degree is uniformly chosen from $\{2, 3, 4\}$ for each node. The number of observed nodes is set to $p = \lfloor n/2 \rfloor$. For each data collection round, $(2^{n+m}/8)$ data pairs are sampled. The code is available at https://github.com/GG-ontorller/Random-BCN-with-output. Figure 1(a) is the median tightness $\mathcal{O}D^u(\mathcal{D}) - \mathcal{O}D^l(\mathcal{D})$ versus data ratio for each network size, with shaded confidence intervals. Figure 1(b) is the mean gaps Δ_{upper} and Δ_{lower} to the structure-based observability for $n = 6$ and $n = 8$, where $\Delta_{\text{upper}} = \mathcal{O}D^u(\mathcal{D}) - \mathcal{O}S(L, H)$ and $\Delta_{\text{lower}} = \mathcal{O}S(L, H) - \mathcal{O}D^l(\mathcal{D})$.

Conclusion. This work proposes a data-driven framework for assessing the observability of BCNs without model identification. A structure-based observability index converts binary notions into computable distinguishability degrees, while data observability bounds map finite datasets to values. Theoretical properties, including boundedness, monotonicity, and convergence toward the structural index, are established. One potential future work is to apply the proposed method in this study to large-scale networks.

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Supporting information Appendixes A–J. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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