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• Supplementary File •

# Communication and control co-design for heterogeneous industrial IoT over state-dependent Markov fading channels

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## Appendix A Notations used in the body of this letter

**Table A1** Key Notations

| Notations | Definitions |
|---|---|
| $I_n$ | $n$-dimensional identity matrix |
| $\delta_n^i$ | Logical vector as the $i$-th column of $I_n$ |
| $[\delta_n^{i_1} \ \cdots \ \delta_n^{i_m}]$ | Logical matrix with the $j$-th column being $\delta_n^{i_j}$ |
| $\vec{\xi}$ | Vector form of logic variable/vector $\xi$ |
| $\|x\|$ | Euclidean norm of vector $x$ |
| $\mathrm{diag}(a_1, \cdots, a_n)$ | Block-diagonal matrix whose diagonal blocks are $a_1, \cdots, a_n$ |
| $A^\top$ | Matrix transpose of $A$ |
| $[A]_{i,j}$ | $(i,j)$-th entry of matrix $A$ |
| $\rho(A)$ | Spectrum radius of matrix $A$ |
| $P_\theta \succ 0$ | Matrix $P_\theta$ is positive definite. The $n$-tuple $P = (P_{\theta_1}, \cdots, P_{\theta_n}) \succ 0$, if $P_\theta \succ 0$, for all $\theta$ |
| $P_\theta \succeq 0$ | Matrix $P_\theta$ is positive semi-definite. The $n$-tuple $P = (P_{\theta_1}, \cdots, P_{\theta_n}) \succeq 0$, if $P_\theta \succeq 0$, for all $\theta$ |
| $\mathcal{D}_n$ | Set $\{0, 1, \ldots, n-1\}$ |
| $\Delta_n$ | Set of $n$-dimensional logical vectors |
| $\mathcal{L}^{n\times m}$ | Set of $n \times m$ logical matrices |
| $\mathbb{R}^{n\times m}$ | Set of $n \times m$ real matrices |
| $\mathbb{H}^{m,n}$ | Linear space $\mathbb{H}^{m,n} = \{P = (P_{\theta_1}, \cdots, P_{\theta_n}) : P_\theta \in \mathbb{R}^{m\times m}\}$ |
| $\mathbb{H}_+^{m,n}$ | Linear space $\mathbb{H}_+^{m,n} = \{P = (P_{\theta_1}, \cdots, P_{\theta_n}) \in \mathbb{H}^{m,n} : P \succeq 0\}$ |
| $\mathbb{P}$ | Probability of a random event |
| $\mathbb{E}$ | Expectation of a random variable |
| $A \ltimes B$ | Semi-tensor product of matrices $A$ and $B$ |
| $A \otimes B$ | Kronecker product of matrices $A$ and $B$ |
| $\langle P, R\rangle$ | Inner product of $P$, $R \in \mathbb{H}^{m,n}$ defined as $\langle P, R\rangle = \sum_\theta \mathrm{tr}(P_\theta^\top R_\theta)$, $\mathrm{tr}(\cdot)$ is the trace operator |

## Appendix B Preliminaries used in the body of the letter

**Definition B1.** [1] The semi-tensor product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as

$$A \ltimes B = (A \otimes I_{r/n})(B \otimes I_{r/p}),$$

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where $r$ is the least common multiple of $n$ and $p$.

Throughout this letter, we use the semi-tensor product as the default matrix product. When $n = p$, it degenerates to the conventional matrix product, i.e., $A \ltimes B = AB$. Hence, the symbol $\ltimes$ is omitted in most places of this letter when no confusion arises.

**Proposition B1.** [1] For any vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, it holds

$$W_{[m,n]} \ltimes a \ltimes b = b \ltimes a,$$

where $W_{[m,n]} \in \mathbb{R}^{mn\times mn}$ is the swap matrix defined by $W_{[m,n]} = [I_n \otimes \delta_m^1 \cdots I_n \otimes \delta_m^m]$.

**Proposition B2.** [1] For any vector $a \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{p\times q}$, it holds

$$a \ltimes A = (I_m \otimes A)a.$$

For any element $\xi \in \mathcal{D}_n$, we can express it in the vector form as the $(n-\xi)$-th column of $I_n$, i.e.,

$$\vec{\xi} = \delta_n^{n-\xi}$$

in a higher dimensional space $\Delta_n$. For a logical vector $\xi = [\xi_1 \ \cdots \ \xi_r]^\top \in \mathcal{D}_{n_1} \times \cdots \times \mathcal{D}_{n_r}$, the vector form can be defined element-wisely using the semi-tensor product as

$$\vec{\xi} = \vec{\xi}_1 \ltimes \cdots \ltimes \vec{\xi}_r \in \Delta_{n_1\cdots n_r}.$$

**Proposition B3.** [1] For any $\xi \in \mathcal{D}_n$, it holds $\vec{\xi} \ltimes \vec{\xi} = R_{[n]}\vec{\xi}$, where $R_{[n]}$ is the power-reducing matrix defined by $R_{[n]} = \mathrm{diag}\{\delta_n^1, \cdots, \delta_n^n\}$.

**Proposition B4.** [2] Let $\zeta$ be a random variable and $\xi$ be a random vector with sample spaces $\mathcal{D}_l$ and $\mathcal{D}_{n_1} \times \cdots \times \mathcal{D}_{n_r}$, respectively. Suppose the conditional probability function of $\zeta$ given that $\xi = a$ is

$$\mathbb{P}\{\zeta = b|\xi = a\} = p_{\zeta|\xi=a}(b),$$

then there exists a random logical matrix $\Lambda_{\zeta|\xi}$ taking values in $\mathcal{L}^{l\times n}$, $n = n_1 \times \cdots \times n_r$, such that

$$\vec{\zeta} = \Lambda_{\zeta|\xi} \ltimes \vec{\xi},$$

i.e., $\vec{\zeta}$ and $\Lambda_{\zeta|\xi} \ltimes \vec{\xi}$ have the same probability distribution.

The notion of stability used in this letter is the partial mean-square stability with respect to the continuous plant state $x(k) = [x_1^\top(k) \cdots x_p^\top(k)]^\top$ of system $\Sigma_x$, which is defined below.

**Definition B2.** The multi-loop control system $\Sigma_x$ with the state-dependent Markov channel (1)-(2) is said to be mean-square stable if, for any initial conditions $x(0)$, $s(0)$ and $h(0)$, it holds

$$\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|^2\} = 0.$$

**Proposition B5.** [3] Consider a positive operator $\mathbf{T} : \mathbb{H}^{m,n} \to \mathbb{H}^{m,n}$. If $\rho(\mathbf{T}) < 1$, then for any given $Q \in \mathbb{H}^{m,n}$, $Q \succ 0$, there exists a unique $P \in \mathbb{H}^{m,n}$, $P \succ 0$, such that

$$P = \mathbf{T}(P) + Q.$$

## Appendix C Remarks in the body of the letter

**Remark C1.** We assume that both the wireless uplink and downlink have the same channel characteristics. Such an assumption has been widely adopted in the study of wireless control [4]. Under this assumption, we focus on the uplink transmission, and the subsequent analysis can be extended straightforwardly to the downlink scenario.

**Remark C2.** Equation (1) in the letter takes into account the interference among transmission links by adopting a collision channel model. In this model, a packet collision occurs when two or more sensors transmit on the same frequency at the same time. If sensor $i$ transmits and the transmission is collision-free, the success of decoding the packet is given by

$$\mathbb{P}\Big\{\gamma_i(k) = 1|\phi(k) = \phi, h(k) = h\Big\} = \mathbb{P}\{\mathrm{SNR}_i(k) \geqslant \eta_i^0|\phi(k) = \phi, \phi_i \neq 0, \phi_i \neq \phi_{i'}, \forall i' \neq i\},$$

where $\mathrm{SNR}_i(k)$ and $\eta_i^0$ denote the signal-to-noise ratio and the specified decodability threshold of system $i$, respectively. It is clear that Eq. (1) captures the inter-link interference in the multi-loop system. See [5, 6] for more details.

**Remark C3.** In this letter, we consider a state-dependent fading channel whose transition probabilities depend on the position of the mobile agent. Hence, we implicitly assume that the scheduler has access to the mobile agent's state $s(k)$ and to the channel state $h(k)$ at each decision step. In practice, this assumption can be justified as follows. First, the mobile agent's position (and thus its MDP state $s(k)$) can be acquired by directly using the visual sensor to observe the position of the mobile agent. Then, the local channel state $h(k)$ can be inferred from the agent's position. Alternatively, the channel state can be estimated from a short pilot signal exchanged between the edge controller and can also be used for synchronization. This kind of pilot-based channel estimation is routinely used in industrial wireless systems and may also serve for synchronization. Such assumptions are common in the existing literature (see, e.g., [4, 6, 7]).

**Remark C4.** In Definition 1, $k$ denotes a common decision step for the co-design problem, which can be obtained by aggregating the faster wireless communication process. More specifically, one may introduce a fast time index $\ell$ for individual packet transmissions and a slow time index $k$ for the MDP state $s(k)$, corresponding to sampling instants $\{t_\ell\}_{\ell=0}^{\infty}$ and $\{t_{\tau k}\}_{k=0}^{\infty}$, respectively. Thus, each control decision interval $[t_{\tau k}, t_{\tau(k+1)})$ contains $\tau$ fast transmission slots on the order of milliseconds, while the mobile agent moves at a much slower time scale. During the interval, the agent's logical state remains in the same cell, and the shadow fading is described by the same channel state $h(k)$. The packet-success probability $q(h^j)$ in (1) is then interpreted as the probability that sensor $i$ achieves at least one successful transmission within this decision interval. As for the plant dynamics, each mode in the system $\Sigma_x$ is obtained by aggregating, over one decision interval, the switching modes of an underlying model whose state is updated at every packet transmission.

**Remark C5.** Compared with the existing results [2, 6], the main contributions of this letter are summarized as follows.

- We introduce a multi-frequency multi-level state-dependent Markov fading channel, where packet-success probabilities incorporate interference and contention among links over shared frequencies, and the channel quality on each frequency evolves as a Markov chain whose transition probabilities depend on the mobile agent's MDP state.
- We construct a merged stochastic switched system $\Sigma_y$ whose state aggregates the logical Markov state $(s(k), h(k))$ and the continuous plant state, and prove that mean-square stability of the original multi-loop system $\Sigma_x$ is equivalent to the mean-square stability of $\Sigma_y$. We then derive necessary and sufficient stability conditions in terms of the spectral radius and Lyapunov inequalities.
- We explicitly incorporate the Markovian evolution of the joint state $(s(k), h(k))$ into the merged switched system and derive a bilinear matrix inequality formulation of the co-design problem, which can be efficiently solved via a block coordinate descent scheme.

## Appendix D Proofs in the body of the letter

### Appendix D.1 Proof of Theorem 1

By the semi-tensor product representation [1] and following [2], we rewrite the MDP $\Sigma_s$ and the state-dependent Markov channel (1)-(2) as

$$\vec{s}(k+1) = M(k)\vec{a}(k)\vec{s}(k)$$

and

$$\begin{cases} \vec{\gamma}_i(k) = \Gamma_i(k)\vec{\phi}(k)\vec{h}(k), \\ \vec{h}(k+1) = H(k)\vec{s}(k)\vec{h}(k), \end{cases}$$

respectively, where $M(k)$, $\Gamma_i(k)$ and $H(k)$ are random logical matrices with sample spaces $\mathcal{L}^{n\times mn}$, $\mathcal{L}^{2\times l\bar{q}}$ (with $\bar{q} = (q+1)^p$), and $\mathcal{L}^{l\times nl}$, respectively. Meanwhile, we rewrite the control system $\Sigma_x$ as

$$x_i(k+1) = A_i\vec{\gamma}_i(k)x_i(k),\ A_i = [A_{i,1}\ A_{i,0}],\ i = 1, \cdots, p.$$

Based on Propositions B1-B3, the dynamic equation of the merged state is derived as follows:

$$\begin{aligned} y_i(k+1) &= \vec{s}(k+1) \ltimes \vec{h}(k+1) \ltimes x_i(k+1) \\ &= M(k)[\vec{a}(k)\vec{s}(k)H(k)\vec{s}(k)\vec{h}(k)][A_i\Gamma_i(k)\vec{\phi}(k)\vec{h}(k)x_i(k)] \\ &= M(k)W_{[n_i,mnl]}A_i\Gamma_i(k)(I_{\bar{q}} \otimes W_{[mn,n_il]})[I_{m\bar{q}} \otimes (W_{[l,nn_il]}H(k)R_{[nl]})]\vec{\sigma}(k)y_i(k). \end{aligned}$$

Denote

$$F_{i,\sigma(k)}(k) = M(k)W_{[n_i,mnl]}A_i\Gamma_i(k)(I_{\bar{q}} \otimes W_{[mn,n_il]})[I_{m\bar{q}} \otimes (W_{[l,nn_il]}H(k)R_{[nl]})]\vec{\sigma}(k).$$

Then, we obtain a merged stochastic switched system

$$\begin{cases} y_i(k+1) = F_{i,\sigma(k)}(k)y_i(k),\ i = 1, 2, \cdots, p, \\ y_i(0) \in \Delta_{nl} \ltimes \mathbb{R}^{n_i}, \end{cases} \tag{D1}$$

where

$$\Delta_{nl} \ltimes \mathbb{R}^{n_i} = \{\vec{s} \ltimes \vec{h} \ltimes x_i | \vec{s} \in \Delta_n, \vec{h} \in \Delta_l, x_i \in \mathbb{R}^{n_i}\}.$$

Since $\|y_i(k)\| = \|x_i(k)\|$, for any $y_i(0) = \vec{s}(k) \ltimes \vec{h}(k) \ltimes x_i(k)$ with $s(0) \in \mathcal{D}_n$ and $x_i(0) \in \mathbb{R}^{n_i}$, it holds that

$$\mathbb{E}\Big\{\|x_i(k)\|^2\Big\} = \mathbb{E}\Big\{\|y_i(k)\|^2\Big\}.$$

Then, the sufficiency is true due to the fact that $\Delta_{nl} \ltimes \mathbb{R}^{n_i} \subseteq \mathbb{R}^{nn_il}$. In addition, the necessity can be proved by the fact that $\mathrm{Span}\{\Delta_{nl} \ltimes \mathbb{R}^{n_i}\} = \mathrm{Span}\{\Delta_{nn_il}\} = \mathbb{R}^{nn_il}$.

## Appendix D.2 Proof of Theorem 2

*Step 1: Block covariance.*

For each $\theta \in \mathcal{D}_n \times \mathcal{H}$, define

$$S_{i,\theta}(k) = \mathbb{E}\Big\{y_i(k)y_i^\top(k)\mathbf{1}_{\{\theta(k)=\theta\}}\Big\},$$

where $\mathbf{1}_{\{\theta(k)=\theta\}} = 1$ if $\theta(k) = \theta$ and 0 otherwise. Then,

$$\sum_\theta S_{i,\theta}(k) = \mathbb{E}\Big\{y_i(k)y_i^\top(k)\Big\},\ \mathbb{E}\Big\{\|y_i(k)\|^2\Big\} = \mathrm{tr}\Big(\sum_\theta S_{i,\theta}(k)\Big).$$

Since every $S_{i,\theta}(k) \succeq 0$, the condition $\lim_{k\to\infty} \mathbb{E}\Big\{\|y_i(k)\|^2\Big\} = 0$ is equivalent to

$$\lim_{k\to\infty} \mathrm{tr}(S_{i,\theta}(k)) = \lim_{k\to\infty} \|S_{i,\theta}(k)\|_F = 0, \forall\theta.$$

*Step 2: One-step recursion of block covariance.*

From the law of total expectation, and the Markovian structure of $\theta(k)$, we get the recursion

$$\begin{aligned} S_{i,\theta}(k+1) &= \mathbb{E}\Big\{y_i(k+1)y_i^\top(k+1)\mathbf{1}_{\{\theta(k+1)=\theta\}}\Big\} \\ &= \sum_{\theta'}\sum_{\sigma} \mathbb{P}\{\theta(k)=\theta', \sigma(k)=\sigma\}\mathbb{E}\{F_{i,\sigma}(k)y_i(k)y_i^\top(k)F_{i,\sigma}^\top(k)\mathbf{1}_{\{\theta(k+1)=\theta\}}|\theta(k)=\theta', \sigma(k)=\sigma\} \\ &= \sum_{\theta'}\sum_{\phi,a} \Pi(\theta|\theta',a)\mathbb{P}\{\phi,a|\theta'\}\mathbb{E}\{F_{i,\sigma}(k)S_{i,\theta'}(k)F_{i,\sigma}^\top(k)\}. \end{aligned} \tag{D2}$$

Vectorizing each block, with $V_c(\cdot)$ the column-stacking operator, yields

$$V_c(S_{i,\theta}(k+1)) = \sum_{\theta'}\sum_{\phi,a} \Pi(\theta|\theta',a)\mathbb{P}\{\phi,a|\theta'\}\bar{F}_{i,(\phi,a)}V_c(S_{i,\theta'}(k)), \tag{D3}$$

where

$$\begin{aligned} \bar{F}_{i,(\phi,a)} &= \mathbb{E}\Big\{F_{i,(\phi,a)}(k)\otimes F_{i,(\phi,a)}(k)\Big\} = \sum_{\iota,\epsilon,\varsigma} m^\iota\gamma^\epsilon h^\varsigma\Big[F(i,(\phi,a),M^\iota,\Gamma^\epsilon,H^\varsigma)\otimes F(i,(\phi,a),M^\iota,\Gamma^\epsilon,H^\varsigma)\Big], \\ F(i,(\phi,a),M^\iota,\Gamma^\epsilon,H^\varsigma) &= M^\iota W_{[n_i,mnl]}A_i\Gamma^\epsilon(I_{\bar{q}}\otimes W_{[mn,n_il]})[I_{m\bar{q}}\otimes(W_{[l,nn_il]}H^\varsigma R_{[nl]})]\vec{\phi}\vec{a}. \end{aligned}$$

Here, $\mathbb{P}\{M(k)=M^\iota\} = m^\iota$, $\mathbb{P}\{\Gamma_i(k)=\Gamma^\epsilon\} = \gamma^\epsilon$, $\mathbb{P}\{H(k)=H^\varsigma\} = h^\varsigma$.

Stacking all blocks as

$$S_i(k) = (S_{i,\theta_1}(k), S_{i,\theta_2}(k), \cdots, S_{i,\theta_{nl}}(k)),\ \hat{V}_c(S_i(k)) = \Big[V_c^\top(S_{i,\theta_1}(k))\ V_c^\top(S_{i,\theta_2}(k))\cdots V_c^\top(S_{i,\theta_{nl}}(k))\Big]^\top,$$

(D3) becomes

$$\hat{V}_c(S_i(k+1)) = G_i\hat{V}_c(S_i(k)), i = 1,2,\cdots,p. \tag{D4}$$

This means that $\hat{V}_c(S_i(k))$ is the solution to the linear system

$$v_i(k+1) = G_iv_i(k),\ i = 1,2,\cdots,p, \tag{D5}$$

where the initial stacked vector lies in the cone

$$v_i(0) \in \mathcal{V}_i = \Big\{(\delta_{nl}^a \otimes I_{d^2})(y_i\otimes y_i) : a = 1,\cdots,nl, y_i \in \mathbb{R}^d\Big\},\ d = nn_il.$$

Hence, by Steps 1 and 2, system $\Sigma_y$ is mean-square stable, if and only if $\lim_{k\to\infty} v_i(k) = 0$ for all $v_i(0) \in \mathcal{V}_i$.

*Step 3: (i) $\Leftrightarrow$ (ii).*

If (ii) holds, i.e., $\rho(G_i) < 1$, $i = 1,2,\cdots,p$, then (D5) is Schur stable, and $\lim_{k\to\infty} v_i(k) = 0$ holds for every initial state; in particular for all $v_i(0) \in \mathcal{V}_i$. This implies system $\Sigma_y$ is mean-square stable, which together with Theorem 1 shows that system $\Sigma_x$ is mean-square stable.

The proof of the implication from (i) to (ii) below follows [8]. According to Theorem 1, mean-square stability of $\Sigma_x$ implies mean-square stability of the merged system $\Sigma_y$. Equivalently,

$$\lim_{k\to\infty} v_i(k) = \lim_{k\to\infty} G_i^k v_i(0) = 0,\ v_i(0) \in \mathcal{V}_i.$$

To remove the initial constraint, we now enlarge the set of admissible initial conditions. Fix $a \in \{1,\cdots,nl\}$ and consider any Hermitian matrix $H \succeq 0$. Let the spectral decomposition be

$$H = \sum_j b_j(z_j\otimes z_j),\ b_j > 0.$$

By linearity of vectorization,

$$(\delta_{nl}^a \otimes I_{d^2})V_c(H) = (\delta_{nl}^a \otimes I_{d^2}) \sum_j b_j(z_j \otimes z_j) \in \mathrm{Span}\{\mathcal{V}_i\}.$$

Hence,

$$\lim_{k\to\infty} G_i^k(\delta_{nl}^a \otimes I_{d^2})V_c(H) = 0,\ \forall a = 1, \cdots, nl,\ H \succeq 0.$$

Note that any Hermitian matrix can be written as the difference of two positive semi-definite Hermitian matrices. Therefore, by linearity, the same limit holds for every Hermitian matrix $H$. Finally, any complex matrix can be written as $H = H_R - jH_I$ with $H_R$, $H_I$= Hermitian. Thus,

$$\lim_{k\to\infty} G_i^k(\delta_{nl}^a \otimes I_{d^2})v_i(0) = 0,\ \forall a = 1, \cdots, nl,\ \text{matrices } H.$$

In other words, system (D5) is asymptotically stable for every initial state $v_i(0)$. This is equivalent to Schur stability of $G_i$, i.e., $\rho(G_i) < 1$. This completes the proof of (ii).

*Step 4: (ii) ⇔ (iii).*

According to (D2), it holds

$$S_i(k+1) = \mathbf{T}_i(S_i(k)),$$

which together with (D4) shows that

$$\hat{V}_c(\mathbf{T}_i(S_i(k))) = G_i\hat{V}_c(S_i(k)),$$

i.e., the operator $\mathbf{T}_i$ admits the matrix realization $G_i$. Hence

$$\rho(\mathbf{T}_i) = \rho(G_i). \tag{D6}$$

If (ii) holds, i.e., $\rho(G_i) < 1$, $i = 1, 2, \cdots, p$. By (D6), $\rho(\mathbf{T}_i) < 1$, $i = 1, 2, \cdots, p$. Then, according to Proposition B5, for each $i = 1, 2, \cdots, p$, for any $S_i \in \mathbb{H}^{nn_il,nl}$, $S_i \succ 0$, there exists a unique $P_i \in \mathbb{H}^{nn_il,nl}$, $P_i \succ 0$, such that $P_i = \mathbf{T}_i(P_i) + S_i$. Therefore, $\rho(G_i) < 1$, $i = 1, 2, \cdots, p$ imply the Lyapunov inequality in (iii).

If (iii) holds, then there exists $P_i \in \mathbb{H}^{nn_il,nl}$ with $P_i \succ 0$ such that $P_i - \mathbf{T}_i(P_i) \succ 0$, for $i = 1, 2, \cdots, p$. Define the operator

$$\mathbf{T}_i^*(R_i(k)) = \Big(\mathbf{T}_{i,\theta_1'}^*(R_i(k)), \mathbf{T}_{i,\theta_2'}^*(R_i(k)), \cdots, \mathbf{T}_{i,\theta_{nl}'}^*(R_i(k))\Big),$$
$$\mathbf{T}_{i,\theta'}^*(R_i) = \sum_{\theta,\phi,a} \Pi(\theta \mid \theta', a)\, \mathbb{P}\{\phi, a \mid \theta'\}\, \bar{F}_{i,(\phi,a)}^*(R_{i,\theta}),\ \bar{F}_{i,(\phi,a)}^*(R_{i,\theta}) = \mathbb{E}\Big\{ F_{i,(\phi,a)}^\top(k)\, R_{i,\theta}\, F_{i,(\phi,a)}(k) \Big\}.$$

Consider the homogeneous system

$$R_i(k+1) = \mathbf{T}_i^*(R_i(k)),\ R_i(0) \succeq 0, i = 1, 2, \cdots, p. \tag{D7}$$

Then $R_i(k) \succeq 0$ for all $k$. In addition, by the cyclic property of the trace, it holds

$$\langle \mathbf{T}_i(P_i), R_i(k)\rangle = \langle P_i, \mathbf{T}_i^*(R_i(k))\rangle. \tag{D8}$$

Define $V_i(\cdot) : \mathbb{H}_+^{nn_il,nl} \to \mathbb{R}$ by

$$V_i(R_i(k)) = \langle P_i, R_i(k)\rangle.$$

It is obvious that $V_i(\cdot)$ is continuous and $V_i(0) = 0$. Since $P_{i,\theta} \succ 0$, there exists a unique $P_{i,\theta}^{1/2} \succeq 0$ such that $P_{i,\theta} = P_{i,\theta}^{1/2}P_{i,\theta}^{1/2}$. Then, by the cyclic property of the trace, we have

$$V_i(R_i(k)) = \sum_\theta \mathrm{tr}(P_{i,\theta}R_{i,\theta}(k)) = \sum_\theta \mathrm{tr}\Big(P_{i,\theta}^{1/2}R_{i,\theta}(k)P_{i,\theta}^{1/2}\Big). \tag{D9}$$

Since $R_{i,\theta} \succeq 0$, we have $P_{i,\theta}^{1/2}R_{i,\theta}(k)P_{i,\theta}^{1/2} \succeq 0$, which means $V_i(R_i(k)) \geqslant 0$.

Let $c_0(P_i)$ and $c_1(P_i)$ denote, respectively, the minimum and maximum eigenvalues taken across all blocks of $P_i$, i.e.,

$$c_0(P_i) = \min_{\theta,a} \lambda_a(P_{i,\theta}),\ c_1(P_i) = \max_{\theta,a} \lambda_a(P_{i,\theta}),$$

where $\lambda_a(P_{i,\theta})$ denotes the $a$-th eigenvalue of $P_{i,\theta}$. Since $P_i \succ 0$, $c_0(P_i)$ and $c_1(P_i)$ are strictly positive. According to Remark 2.1 in [3], we have

$$\min_a(\lambda_a(P_{i,\theta}))\mathrm{tr}(R_{i,\theta}(k)) \leqslant \mathrm{tr}(P_{i,\theta}R_{i,\theta}(k)) \leqslant \max_a(\lambda_a(P_{i,\theta}))\mathrm{tr}(R_{i,\theta}(k)).$$

Thus, it holds

$$c_0(P_i)\Big(\sum_{\theta,a} \lambda_a(R_{i,\theta}(k))\Big) \leqslant V_i(R_i(k)) \leqslant c_1(P_i)\Big(\sum_{\theta,a} \lambda_a(R_{i,\theta}(k))\Big), i = 1, 2, \cdots, p. \tag{D10}$$

Since

$$\|R_i(k)\|_F^2 = \sum_\theta \mathrm{tr}(R_{i,\theta}^\top(k)R_{i,\theta}(k)) = \sum_{\theta,a} \lambda_a(R_{i,\theta}(k))^2,\ \lambda_a(R_{i,\theta}(k)) \geqslant 0,$$

$\|R_i(k)\|_F \to \infty$, if and only if $\sum_{\theta,a} \lambda_a(R_{i,\theta}(k)) \to \infty$, and $R_i(k) = 0$ if and only if $\lambda_a(R_{i,\theta}(k)) = 0$, for all $\theta, a$. Hence, $V_i(R_i(k)) \to \infty$ whenever $\|R_i(k)\|_F \to \infty$ and $V_i(R_i(k)) > 0$ for all $R_i(k) \neq 0$.

Define

$$\varepsilon_{i,\theta} = c_0\Big(P_{i,\theta}^{-1/2}(P_{i,\theta} - \mathbf{T}_{i,\theta}(P_i))P_{i,\theta}^{-1/2}\Big), \varepsilon_i = \min_\theta \varepsilon_{i,\theta} \in (0,1].$$

Then,

$$\mathbf{T}_i(P_i) \preceq (1-\varepsilon_i)P_i, \tag{D11}$$

which gives

$$V_i(R_i(k+1)) - V_i(R_i(k)) \leqslant -\varepsilon_i V_i(R_i(k)),\ V_i(R_i(k)) \leqslant (1-\varepsilon_i)^k V_i(R_i(0)).$$

Thus, $\rho(\mathbf{T}_i^*) < 1$, together with

$$\rho(\mathbf{T}_i^*) = \rho(\mathbf{T}_i) = \rho(G_i),$$

we conclude $\rho(G_i) < 1$.

## Appendix D.3 Proof of Theorem 3

*Step 1: Representation of the average expected joint cost.*

Consider the joint logical state $\theta(k) = (s(k), h(k))$. Under any stationary policy pair $(\pi_\phi, \pi_a)$, the process $\{\theta(k)\}$ is a time-homogeneous Markov chain with $\Pi(\theta'|\theta, a) = P_{s',s}(a)H_{h',h}(s)$, where $\theta = [s\ h]^\top$ and $\theta' = [s'\ h']^\top$. Following the occupation measure formulation in [9], define

$$\mu(\theta) = \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \mathbb{P}\{\theta(k) = \theta\}, \forall\ \theta. \tag{D12}$$

Then, $\mu(\theta)$ satisfies

$$\sum_\theta \mu(\theta) = 1, \mu(\theta) \geqslant 0, \tag{D13}$$

and it obeys the linear balance equations

$$\mu(\theta) = \sum_{\theta',a} \Pi(\theta|\theta', a)\mu(\theta')\pi_a(a|s'), \forall\ \theta. \tag{D14}$$

In addition, the objective function can equivalently be represented as

$$\begin{aligned}
J_{\text{joint}} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\Big\{\sum_{k=0}^{t-1} c(s(k), h(k), \phi(k), a(k))\Big\} &= \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \mathbb{E}\{c(s(k), h(k), \phi(k), a(k))\} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1}\sum_{\theta,\phi,a} c(\theta,\phi,a)\mathbb{P}\{\theta(k) = \theta, \phi(k) = \phi, a(k) = a\} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1}\sum_{\theta,\phi,a} c(\theta,\phi,a)\mathbb{P}\{\phi(k) = \phi, a(k) = a|\theta(k) = \theta\}\mathbb{P}\{\theta(k) = \theta\} \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1}\sum_{\theta} \mathbb{P}\{\theta(k) = \theta\}\sum_{\phi,a} c(\theta,\phi,a)\pi_\phi(\phi|\theta)\pi_a(a|s).
\end{aligned}$$

Then, by exchanging the finite sum and the limit, we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1}\sum_{\theta} \mathbb{P}\{\theta(k) = \theta\}\sum_{\phi,a} c(\theta,\phi,a)\pi_\phi(\phi|\theta)\pi_a(a|s) \\
&= \sum_\theta \mu(\theta)\sum_{\phi,a} c(\theta,\phi,a)\pi_\phi(\phi|\theta)\pi_a(a|s) \\
&= \sum_{\theta,\phi,a} c(\theta,\phi,a)\mu(\theta)\pi_\phi(\phi|\theta)\pi_a(a|s).
\end{aligned}$$

To sum up, it holds

$$J_{\text{joint}} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\Big\{\sum_{k=0}^{t-1} c(s(k), h(k), \phi(k), a(k))\Big\} = \sum_{\theta,\phi,a} c(\theta,\phi,a)\mu(\theta)\pi_\phi(\phi|\theta)\pi_a(a|s). \tag{D15}$$

*Step 2: Representation of the stability constraint.*

By (iii) in Theorem 2, system $\Sigma_x$ is mean-square stable under $(\pi_\phi, \pi_a)$, if and only if for each $i = 1, 2, \cdots, p$, there exists $P_i \in \mathbb{H}^{nn_il,nl}$ with $P_i \succ 0$, such that

$$P_i - \mathbf{T}_i(P_i) \succ 0, \tag{D16}$$

where $\mathbf{T}_{i,\theta}(P_i) = \sum_{\theta',\phi,a} \Pi(\theta|\theta', a)\mathbb{P}\{\phi, a|\theta'\}\bar{F}_{i,(\phi,a)}(P_{i,\theta'})$. Expanding (D16) blockwise over $\theta$ yields

$$P_{i,\theta} - \sum_{\theta',\phi,a} \Pi(\theta \mid \theta', a)\pi_\phi(\phi|\theta')\pi_a(a|s')\bar{F}_{i,(\phi,a)}(P_{i,\theta'}) \succ 0, \forall\ i = 1, \cdots, p, \forall\ \theta.$$

This is equivalent to the condition that there exists $\varepsilon > 0$ such that the following relaxed linear matrix inequalities hold:

$$P_{i,\theta}\ \succeq\ \varepsilon I, P_{i,\theta} - \sum_{\theta',\phi,a} \Pi(\theta \mid \theta', a)\pi_\phi(\phi|\theta')\pi_a(a|s')\bar{F}_{i,(\phi,a)}(P_{i,\theta'}) \succeq \varepsilon I, \forall\ i = 1, \cdots, p, \forall\ \theta. \tag{D17}$$

*Step 3: Equivalence between optimization problem* $\mathbf{P}$ *and the original co-design problem.*

For any feasible solution $(\pi_\phi, \pi_a, \mu, \{P_{i,\theta}\})$ to $\mathbf{P}$, the balance equations in $\mathbf{P}$ and the constraints on the policy probabilities ensure that $\mu$ is a state occupation measure of the Markov chain $\{\theta\}$ under $\pi_a$, as defined in Step 1. Then, by (D15), the objective value of $\mathbf{P}$ at $(\pi_\phi, \pi_a, \mu, \{P_{i,\theta}\})$ coincides with the average expected joint cost $J_{\text{joint}}$. In addition, by Step 2 and the linear matrix inequalities in $\mathbf{P}$, the system $\Sigma_x$ is mean-square stable under $(\pi_\phi, \pi_a)$. Thus, every feasible solution to $\mathbf{P}$ induces a stationary pair $(\pi_\phi, \pi_a)$ that is feasible for the original co-design problem, and the objective value of $\mathbf{P}$ equals the average expected joint cost achieved by this policy.

Conversely, take any optimal stationary policy pair $(\pi_\phi^\star, \pi_a^\star)$ for the original co-design problem. Clearly, $\pi_\phi^\star$, $\pi_a^\star$ satisfy the constraints on the policy probabilities. Let $\mu^\star$ denote the state occupation measure of $\{\theta\}$ under $\pi_a^\star$, defined in Step 1. Then, $\mu^\star$ satisfies the linear balance equations (D13) and (D14). Since $\Sigma_x$ is mean-square stable under $(\pi_\phi^\star, \pi_a^\star)$, by the equivalence established Step 2, there exists $P_{i,\theta}^\star$ and $\varepsilon > 0$ such that linear matrix inequalities in (D17) hold. Therefore, $(\pi_\phi^\star, \pi_a^\star, \mu^\star, P_{i,\theta}^\star)$ is a feasible solution to $\mathbf{P}$. In addition, by (D15), the objective value of $\mathbf{P}$ at this solution equals the optimal average expected joint cost of the original co-design problem.

Consequently, the optimal value of $\mathbf{P}$ coincides with that of the original co-design problem, and any optimal solution to $\mathbf{P}$ provides an optimal stationary policy pair as claimed in Theorem 3.

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