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Non-oscillation of asynchronous stochastic logical networks through negative circuits of the joint interaction graph

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Abstract Oscillation behaviors not only exert a notable impact on the stability and functionality of nonlinear systems, but also play an important role in a wide range of application fields. This paper explores the non-oscillation of asynchronous stochastic logical networks (ASLNs) through negative circuits of the joint interaction graph. First, the joint interaction graph and local interaction graph of ASLNs are established, and some basic properties of them are revealed. Based on these properties, a feasible sufficient condition is presented to verify the non-oscillation of ASLNs via the existence of negative circuits in the joint interaction graph. Second, when the joint interaction graph of ASLNs has no negative circuits, the node removal technique is introduced to find all the fixed points of ASLNs, and a new criterion is built for the global set stability in distribution of ASLNs. Finally, the effectiveness of the main results is demonstrated by using the ASLN model in the processing of Toll and EGF pathways involved in Drosophila development.

Keywords asynchronous stochastic logical networks, oscillation, set stability, negative circuits, joint interaction graph, node removal

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1 Introduction

Oscillation, as a specific dynamic behavior in natural and engineering systems, is an intuitive manifestation of systems exhibiting periodic or nearly periodic phenomena over time [1]. With respect to different types of feedback loops, the oscillations of regulatory networks were analyzed through identifying Hopf bifurcations [2]. Based on the quadratic Lyapunov function approach, the stability properties of self-oscillating solutions for piecewise affine systems were discussed in [3]. Noticeably, numerous systems are inevitably affected by various random fluctuations, environmental noise, or internal uncertainty [4]. Unlike the oscillations of deterministic systems, oscillations in stochastic systems are generally not driven by a precise periodic force but induced or enhanced by the interaction between stochastic factors and nonlinear dynamic characteristics [5]. Particularly, the circuit oscillations induced by noise play an important role in weak signal detection and frequency locking [6]. The research on oscillations in stochastic systems possesses profound theoretical significance and broad application perspectives.

As a kind of complex systems induced by the interactions between genes and proteins within cells, gene regulatory networks play a fundamental role in the biological processes of organisms [7]. Stochasticity is a salient feature naturally evolved in biological systems, which together with the inherent nonlinear regulatory relationships shapes the specific dynamic response patterns [8]. Among the various dynamic behaviors of gene regulatory networks, oscillations in the stochastic models of gene regulatory networks can serve as core clues for deciphering many life phenomena, such as circadian rhythms, cell division, and heartbeat rhythms [9]. However, once the dynamic balance of oscillations in gene regulatory networks is disrupted, potential risks may arise. For example, the uncontrolled oscillation of cell cycle-related genes may lead to the unlimited proliferation and irreversible division of tumor cells [10]. Therefore, analyzing the non-oscillations in the stochastic models of gene regulatory networks provides an important perspective for exploring the treatment of major diseases [7].

Probabilistic logical networks are a typical stochastic model of gene regulatory networks [11–13]. Over the past few decades, a series of effective approaches have been proposed to investigate the dynamic behaviors of probabilistic logical networks, such as Lyapunov function approach [14], enhanced Q-learning approach [15], semi-tensor product of matrices [16, 17] and network structure-based method [18]. A criterion was proposed in [16] to verify the stability

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in distribution of probabilistic Boolean networks (PBNs), which is theoretically perfect, but hard to use in the large-scale PBNs because of the exponentially increasing computational complexity. When there exists a mode whose network graph is acyclic, a polynomial-time criterion was established in [18] for the stability in distribution of PBNs via the node removal technique. Noticing that different time scales are often used to characterize some important living phenomena in realistic biological systems, such as the distinct time lengths required for transcriptional events, receptor internalization and cellular growth, which can be approximately depicted by the dynamics of asynchronous logical networks [19]. Over the past decade, relevant achievements in asynchronous logical networks have been emerging continuously [20, 21]. By establishing the interaction graph, the relationship between attractors and positive (negative) circuits of asynchronous logical networks was discussed in [22]. Drawing on the Thomas's rule in Boolean case [23], the distributed pinning control was proposed to investigate the non-oscillation of asynchronous Boolean networks (BNs) [24] through removing some proper arcs in the interaction graph. It is worth noticing that the results in [22, 24] are just applicable to logical networks with deterministic asynchronous update, and the study of logical networks with stochastic asynchronous update was left for future work by [24].

Considering the influence of noise and perturbation, probabilistic logical networks with stochastic asynchronous update are more effective in evaluating the steady-state distributions of biological systems [25, 26]. Using the semi-tensor product of matrices [27], the synchronization problem of PBNs with stochastic asynchronous update was explored in [28]. Since the global information of transition probability matrix was used, the computational complexity of criterion in [28] grows exponentially with the number of nodes, making it difficult to use in the large-scale PBNs with stochastic asynchronous update. Recently, some polynomial-time results have been derived for the investigation of BNs and PBNs with the help of network structure [18, 29–31], which motivates us to reduce the computational complexity of large-scale PBNs with stochastic asynchronous update based on the network structure.

In this paper, we explore the non-oscillation of asynchronous stochastic logical networks (ASLNs) from the perspective of network graph. The main novelties of this paper are listed below.

(i) By establishing the joint interaction graph that integrates the local interaction graphs of multiple modes, we reveal the relationship between the existence of negative circuits in the joint interaction graph and the existence of cyclic attractors of PBNs with stochastic asynchronous update. Additionally, the joint interaction graph technique can be extended to the general multi-valued logical networks (see Remark 5). For logical networks with a specific mode and deterministic asynchronous update, our new technique degenerates into the finding in [22].

(ii) Using the local information of the joint interaction graph, we reveal the non-oscillation of ASLNs when the joint interaction graph has no negative circuits. Compared with the traditional centralized method [28], our new result has much lower computational complexity (see Remark 3), and thus is more suitable to discuss the large-scale ASLNs.

(iii) Through utilizing the node removal [32] to each mode of ASLNs, we find all the fixed points of ASLNs and develop the new node removal technique for ASLNs. When the joint interaction graph has no negative circuits, we demonstrate the set stability in distribution of ASLNs based on all the fixed points.

The rest of this paper is organized as follows. Section 2 provides some preliminaries. In Section 3, we explore the non-oscillation and global set stability in distribution of ASLNs when the joint interaction graph has no negative circuits. Section 4 applies the main results to the processing of Toll and EGF pathways involved in Drosophila development. Section 5 is a brief conclusion.

Notations. $\mathbb{N}$ is the set of natural numbers. $\mathbb{Z}_+$ is the set of positive integers. $\mathcal{D} := \{0, 1\}$. $\mathcal{D}_k := \{0, \cdots, k-1\}$, $k \in \mathbb{Z}_+$, $k \geqslant 2$. $[a : b]_{\mathbb{N}} := \{a, a+1, \cdots, b\}$, where $a < b$ and $a, b \in \mathbb{N}$. Given $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let x_i be the i-th entry. Given a set A, let $|A|$ be the cardinality of set A.

2 Preliminaries

A PBN with n nodes and ω modes can be described as

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}\left([x_j(t)]_{j\in N(1,\sigma(t))}\right), \\ \qquad\vdots \\ x_n(t+1) = f_n^{\sigma(t)}\left([x_j(t)]_{j\in N(n,\sigma(t))}\right), \end{cases} \tag{1}$$

where $x_i(t) \in \mathcal{D}$ denotes the state of node i at time t, and $\sigma : \mathbb{N} \to [1 : \omega]_{\mathbb{N}}$ is a stochastic switching signal among different modes. When mode α is activated, $N(i, \alpha) \subseteq [1 : n]_{\mathbb{N}}$ is the index set of nodes that affect the update of node i, and $f_i^{\alpha} : \mathcal{D}^{|N(i,\alpha)|} \to \mathcal{D}$ is the update function of node i. Suppose that $\sigma(t)$ is an independent and identically distributed process, and $\mathbb{P}\{\sigma(t) = \alpha\} = p_\alpha > 0$, $t \in \mathbb{N}$, $\alpha \in [1 : \omega]_{\mathbb{N}}$, where $\sum_{\alpha=1}^{\omega} p_\alpha = 1$.

In this article, we consider PBN (1) with asynchronous stochastic update scheme. To be more specific, at each time step $t+1$, only one node is randomly selected to be updated by its update function with probability $\frac{1}{n}$, and the rest nodes keep the values at time t. For instance, when the i-th node updates at time $t+1$, the dynamics can be expressed mathematically below:

$$x_i(t+1) = \begin{cases} f_i^1\Big([x_j(t)]_{j\in N(i,1)}\Big), \text{ with probability } p_1, \\ \quad\vdots \\ f_i^\omega\Big([x_j(t)]_{j\in N(i,\omega)}\Big), \text{ with probability } p_\omega, \end{cases}$$
$$x_l(t+1) = x_l(t),\ l\in[1:n]_{\mathbb{N}}\setminus\{i\}. \tag{2}$$

Now, we give the definition of joint interaction graph for system (2).

Definition 1. The joint interaction graph of system (2), denoted by $G_F=(N,E)$, is a signed directed graph with vertex set $N=[1:n]_{\mathbb{N}}$ and arc set E. Here, (j,s,i) is called an arc of G_F, if there exist $\omega_1,\omega_2\in[1:\omega]_{\mathbb{N}}$ and $[x_l]_{l\in N\setminus\{j\}}\in\mathcal{D}^{n-1}$ such that

$$f_i^{\omega_1}\big([x_l]_{l\in N(i,\omega_1)\setminus\{j\}},1\big) - f_i^{\omega_2}\big([x_l]_{l\in N(i,\omega_2)\setminus\{j\}},0\big) = s\in\{-1,1\}.$$

The arc $(j,s,i)\in E$ is said to be positive (negative) if $s=1$ (-1).

Remark 1. It is worth noting that for node i, even though node $j\notin N(i,\alpha)$ is satisfied for any $\alpha\in[1:\omega]_{\mathbb{N}}$, there may exist an arc (j,s,i) in the joint interaction graph.

Definition 2. Let $(j_1,s_{j_1,i_1},i_1)\in E,\cdots,(j_r,s_{j_r,i_r},i_r)\in E$ be given, where $r\in\mathbb{N}$ and $r\geqslant 2$. $\{(j_1,s_{j_1,i_1},i_1),\cdots,(j_r,s_{j_r,i_r},i_r)\}$ is said to be a path of length r and sign $s=\prod_{l=1}^r s_{j_l,i_l}$ if $i_l=j_{l+1}$, $\forall\ l\in[1:r-1]_{\mathbb{N}}$. When $i_r=j_1$ and $s=1$ (-1), it is called a positive (negative) circuit. Particularly, (j,s,j) is said to be a positive (negative) circuit, if $s=1(-1)$.

Remark 2. The interaction graph proposed in [22] is just suitable for deterministic asynchronous logical network with a single mode. When the stochastic switching signal is involved, this interaction graph is no longer applicable. To address this, we combine the interaction information of multiple modes and propose the joint interaction graph, which depicts all possible interaction relationships between any two nodes when the stochastic switching signal is performed.

Next, we introduce the asynchronous state transition graph and attractors for system (2).

For any $x=(x_1,\cdots,x_n)\in\mathcal{D}^n$ and $\alpha\in[1:\omega]_{\mathbb{N}}$, let

$$F_i^\alpha(x) = \big(x_1,\cdots,x_{i-1},f_i^\alpha(x),x_{i+1},\cdots,x_n\big),\ i\in[1:n]_{\mathbb{N}} \tag{3}$$

and

$$I_{F^\alpha}(x) = \big\{i\in[1:n]_{\mathbb{N}} : f_i^\alpha(x)\neq x_i\big\}. \tag{4}$$

The asynchronous state transition graph of system (2), denoted by $\Gamma_F=(\mathcal{N},\mathcal{E})$, is a directed graph whose vertex set is $\mathcal{N}=\mathcal{D}^n$ and arc set is

$$\mathcal{E} = \Big\{\big(x,F_i^\alpha(x)\big) : x\in\mathcal{D}^n, i\in I_{F^\alpha}(x), \alpha\in[1:\omega]_{\mathbb{N}}\Big\}.$$

We define any arc $\big(x,F_i^\alpha(x)\big)\in\mathcal{E}$ as $x\xrightarrow{\alpha,i}F_i^\alpha(x)$. Furthermore, $x^0\xrightarrow{\omega_0,i_0}x^1\xrightarrow{\omega_1,i_1}\cdots\xrightarrow{\omega_{r-1},i_{r-1}}x^r$ is said to be an elementary path of Γ_F with length $r\in\mathbb{Z}_+$ if $x^{k+1}=F_{i_k}^{\omega_k}(x^k)$, $k\in[0:r-1]_{\mathbb{N}}$ and $x^p\neq x^q$, $\forall\ p,q\in[0:r]_{\mathbb{N}},\ p\neq q$.

A nonempty set $\mathcal{T}\subseteq\mathcal{D}^n$ is said to be a trap domain of system (2) if $\mathbb{P}\big\{x(1)\in\mathcal{T}\mid x(0)=x_0\big\}=1$ holds for any $x_0\in\mathcal{T}$. A trap domain $\mathcal{A}\subseteq\mathcal{D}^n$ is called an attractor, if any nonempty proper subset of $\mathcal{A}$ is not a trap domain. An attractor $\mathcal{A}$ with $|\mathcal{A}|=1$ is said to be a fixed point; otherwise, $\mathcal{A}$ is called a cyclic attractor. If $\mathcal{A}$ is an attractor of system (2), then $\mathcal{A}$ is also called an attractor of Γ_F. If system (2) has no cyclic attractors, then it is said to achieve the non-oscillation.

Finally, we use a simple example to demonstrate the joint interaction graph.

Example 1. Consider the following PBN with 2 modes:

$$x_i(t+1) = f_i^{\sigma(t)}\Big([x_j(t)]_{j\in N(i,\sigma(t))}\Big),\ i\in[1:6]_{\mathbb{N}}, \tag{5}$$

Table 1 Truth table of update function for node 3.

(x_1, x_3, x_4)	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)
f_3^1	1	0	1	0	1	0	1	0
f_3^2	1	1	1	0	1	1	0	0

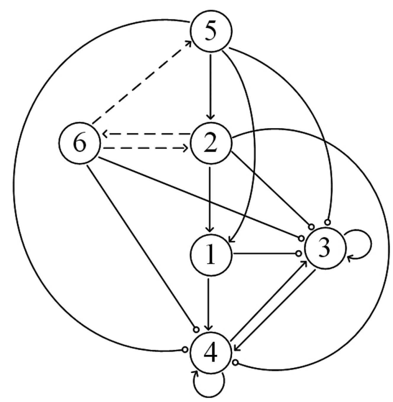

Figure 1 The joint interaction graph of system (5) with asynchronous stochastic update, where $j \to i$, $j \dashrightarrow i$, and $j \multimap i$ denote $(j, 1, i) \in E$, $(j, -1, i) \in E$, and $(j, \pm 1, i) \in E$, respectively.

where

$$\begin{cases} f_1^1\Big([x_j(t)]_{j\in N(1,1)}\Big) = f_1^2\Big([x_j(t)]_{j\in N(1,2)}\Big) = x_2(t) \wedge x_5(t), \\ f_2^1\Big([x_j(t)]_{j\in N(2,1)}\Big) = f_2^2\Big([x_j(t)]_{j\in N(2,2)}\Big) = x_5(t) \wedge \neg x_6(t), \\ f_3^1\Big([x_j(t)]_{j\in N(3,1)}\Big) = x_4(t), f_3^2\Big([x_j(t)]_{j\in N(3,2)}\Big) = \big(x_1(t) \wedge x_4(t)\big) \vee x_3(t), \\ f_4^1\Big([x_j(t)]_{j\in N(4,1)}\Big) = \big(x_1(t) \wedge x_3(t)\big) \vee x_4(t), f_4^2\Big([x_j(t)]_{j\in N(4,2)}\Big) = x_4(t), \\ f_5^1\Big([x_j(t)]_{j\in N(5,1)}\Big) = f_5^2\Big([x_j(t)]_{j\in N(5,2)}\Big) = \neg x_6(t), \\ f_6^1\Big([x_j(t)]_{j\in N(6,1)}\Big) = f_6^2\Big([x_j(t)]_{j\in N(6,2)}\Big) = \neg x_2(t). \end{cases}$$

Assume that $\mathbb{P}\{\sigma(t) = 1\} = 0.6$ and $\mathbb{P}\{\sigma(t) = 2\} = 0.4$. To present the joint interaction graph $G_F = (N, E)$ of system (5) with asynchronous stochastic update, we take node 3 as an example. The truth table of update function for node 3 is shown in Table 1.

Since $f_3^1(1,1,0) - f_3^2(0,1,0) = -1$ and $f_3^2(1,1,0) - f_3^1(0,1,0) = 1$, one has $(1,-1,3) \in E$ and $(1,1,3) \in E$. According to $f_3^2(1,1,0) - f_3^1(1,0,0) = 1$ and $f_3^{\omega_1}(x_1,1,x_4) - f_3^{\omega_2}(x_1,0,x_4) \geqslant 0$, $\forall\ \omega_1,\omega_2 \in [1:2]_{\mathbb{N}}$, $\forall\ (x_1,x_4) \in \mathcal{D}^2$, one concludes that $(3,1,3) \in E$ and $(3,-1,3) \notin E$. Similarly, $(4,1,3) \in E$ and $(4,-1,3) \notin E$. Because of $f_3^1(1,1,0) - f_3^2(1,1,0) = -1$, there must exist two states $(1,1,1,0,1,1)$ and $(1,0,1,0,1,1)$ such that $f_3^1(1,1,1,0,1,1) - f_3^2(1,0,1,0,1,1) = f_3^1(1,1,0) - f_3^2(1,1,0) = -1$ and $f_3^2(1,1,1,0,1,1) - f_3^1(1,0,1,0,1,1) = f_3^2(1,1,0) - f_3^1(1,1,0) = 1$ hold. Hence, both $(2,1,3)$ and $(2,-1,3)$ belong to E. Similarly, $(j,1,3) \in E$ and $(j,-1,3) \in E$, where $j \in [5:6]_{\mathbb{N}}$. In a summary, we derive 10 arcs for node 3, that is, $(1,\pm1,3), (2,\pm1,3), (3,1,3), (4,1,3), (5,\pm1,3), (6,\pm1,3)$.

In a similar manner, one can derive all the arcs for other nodes. Then, the joint interaction graph is shown in Figure 1.

3 Main results

In this section, we first explore the relationship between the existence of negative circuits in the joint interaction graph and the existence of cyclic attractors for system (2). Then, we use the node removal technique to find all the fixed points of system (2) when the joint interaction graph has no negative circuits.

For any $x \in \mathcal{D}^n$, we define

$$\tilde{f}_i^\alpha(x) = f_i^\alpha(x) - x_i,\ \alpha \in [1:\omega]_{\mathbb{N}},\ i \in [1:n]_{\mathbb{N}}. \tag{6}$$

Obviously, $\tilde{f}_i^\alpha(x) \in \{-1, 0, 1\}$. The main tool used in this part is the following notion of local interaction graph.

Definition 3. Given $x \in \mathcal{D}^n$, the local interaction graph of system (2), denoted by $G_F(x) = (N, E(x))$, is a signed directed graph with vertex set $N = [1:n]_{\mathbb{N}}$ and arc set $E(x)$. Here, (j, s, i) is called an arc of $G_F(x)$, if there exist $\omega_1, \omega_2 \in [1:\omega]_{\mathbb{N}}$ such that

$$\tilde{f}_i^{\omega_1}(x) \neq \tilde{f}_i^{\omega_2}\big(F_j^{\omega_1}(x)\big) \tag{7}$$

and

$$s = \tilde{f}_j^{\omega_1}(x)\tilde{f}_i^{\omega_2}\big(F_j^{\omega_1}(x)\big) \in \{-1, 1\}. \tag{8}$$

To explore the relationship between $G_F(x)$ and G_F, we give the concept of subgraph. A graph $G' = (V', E')$ is called a subgraph of $G_F = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$. Now, we give the following results to reveal the properties of local interaction graph.

Proposition 1. For any $x \in \mathcal{D}^n$, $G_F(x)$ is a subgraph of G_F.

Proof. We just need to prove $E(x) \subseteq E$. Suppose $(j, s, i) \in E(x)$.

We first consider the case $j \neq i$. According to (8), one has $\tilde{f}_j^{\omega_1}(x) \neq 0$. We assume that $\tilde{f}_j^{\omega_1}(x) = 1$, that is, $f_j^{\omega_1}(x) - x_j = 1$. The case of $\tilde{f}_j^{\omega_1}(x) = -1$ is similar, which is omitted. In this case, $x_j = 0$. For any $p \in \mathcal{D}$, we define

$$x(j, p) = (x_1, \cdots, x_{j-1}, x_j + p, x_{j+1}, \cdots, x_n).$$

Then, $x(j, 1) = F_j^{\omega_1}(x)$ and $s = \tilde{f}_i^{\omega_2}\big(x(j, 1)\big)$. Based on (7), it holds that

$$\tilde{f}_i^{\omega_1}\big(x(j, 0)\big) \neq \tilde{f}_i^{\omega_2}\big(x(j, 1)\big) = s.$$

If $s = 1$, then $\tilde{f}_i^{\omega_1}\big(x(j, 0)\big) < \tilde{f}_i^{\omega_2}\big(x(j, 1)\big)$. Consequently,

$$\begin{aligned} \tilde{f}_i^{\omega_2}\big(x(j, 1)\big) - \tilde{f}_i^{\omega_1}\big(x(j, 0)\big) &= f_i^{\omega_2}\big(x(j, 1)\big) - x_i - \big(f_i^{\omega_1}\big(x(j, 0)\big) - x_i\big) \\ &= f_i^{\omega_2}\big(x(j, 1)\big) - f_i^{\omega_1}\big(x(j, 0)\big) = 1, \end{aligned}$$

which together with Definition 1 implies that the joint interaction graph G_F has a positive arc from j to i, that is, $(j, s, i) \in E$. If $s = -1$, then $\tilde{f}_i^{\omega_1}\big(x(j, 0)\big) > \tilde{f}_i^{\omega_2}\big(x(j, 1)\big)$. Similarly, we can prove that $(j, s, i) \in E$.

We derive from $\tilde{f}_j^{\omega_1}(x) = -1$ that $x_j = 1$. For any $p \in \mathcal{D}$, we define $x(j, -p) = (x_1, \cdots, x_{j-1}, x_j - p, x_{j+1}, \cdots, x_n)$. Then, $x(j, -1) = F_j^{\omega_1}(x)$ and $s = -\tilde{f}_i^{\omega_2}\big(x(j, -1)\big)$. From (7), one has $\tilde{f}_i^{\omega_1}\big(x(j, 0)\big) \neq \tilde{f}_i^{\omega_2}\big(x(j, -1)\big) = -s$. If $s = 1$, then $\tilde{f}_i^{\omega_1}\big(x(j, 0)\big) > \tilde{f}_i^{\omega_2}\big(x(j, -1)\big)$, and

$$\begin{aligned} \tilde{f}_i^{\omega_1}\big(x(j, 0)\big) - \tilde{f}_i^{\omega_2}\big(x(j, -1)\big) &= f_i^{\omega_1}\big(x(j, 0)\big) - x_i - \big(f_i^{\omega_2}\big(x(j, -1)\big) - x_i\big) \\ &= f_i^{\omega_1}\big(x(j, 0)\big) - f_i^{\omega_2}\big(x(j, -1)\big) = 1. \end{aligned}$$

According to Definition 1, the joint interaction graph G_F has a positive arc from j to i, that is, $(j, s, i) \in E$. If $s = -1$, we can similarly prove $(j, s, i) \in E$.

Now, we consider the case $j = i$. By Definition 3, one has $s = \tilde{f}_i^{\omega_1}(x)\tilde{f}_i^{\omega_2}\big(F_i^{\omega_1}(x)\big)$ and $\tilde{f}_i^{\omega_1}(x) \neq \tilde{f}_i^{\omega_2}\big(F_i^{\omega_1}(x)\big)$, which means $s = -1$. Assume that $\tilde{f}_i^{\omega_1}(x) = 1$. The case of $\tilde{f}_i^{\omega_1}(x) = -1$ is similar. Then, $f_i^{\omega_1}(x) - x_i = 1$ and $\tilde{f}_i^{\omega_2}\big(F_i^{\omega_1}(x)\big) = \tilde{f}_i^{\omega_2}\big(x(i, 1)\big) = f_i^{\omega_2}\big(x(i, 1)\big) - (x_i + 1) = -1$. Therefore, it holds that

$$f_i^{\omega_2}\big(x(i, 1)\big) < x_i + 1 = f_i^{\omega_1}(x) = f_i^{\omega_1}\big(x(i, 0)\big),$$

which together with Definition 1 shows that the joint interaction graph G has a negative arc from i to i, that is, $(i, s, i) \in E$.

To sum up, $G_F(x)$ is a subgraph of G_F.

With the help of (4) and the asynchronous state transition graph $\Gamma_F = (\mathcal{N}, \mathcal{E})$, we further analyze the properties of $G_F(x)$.

Proposition 2. Let an elementary path $x^0 \xrightarrow{\omega_0, i_0} x^1 \xrightarrow{\omega_1, i_1} \cdots \xrightarrow{\omega_{r-1}, i_{r-1}} x^r$ of Γ_F be given, where $r \in \mathbb{Z}_+$, $\{\omega_0, \cdots, \omega_{r-1}\} \subseteq [1:\omega]_{\mathbb{N}}$ and $\{i_0, \cdots, i_{r-1}\} \subseteq [1:n]_{\mathbb{N}}$. If there exists $\omega_r \in [1:\omega]_{\mathbb{N}}$ and $i \in I_{F^{\omega_r}}(x^r)$ such that $\tilde{f}_i^{\omega_p}(x^p) \neq \tilde{f}_i^{\omega_r}(x^r)$ holds for any $p \in [0:r-1]_{\mathbb{N}}$, then there exists $j \in I_{F^{\omega_0}}(x^0)$ such that $\bigcup_{p=0}^{r-1} G_F(x^p)$ has a path from j to i of sign $\tilde{f}_j^{\omega_0}(x^0)\tilde{f}_i^{\omega_r}(x^r)$.

Proof. The proof proceeds by induction on the length r of the elementary path.

When $r=1$, since $x^0 \xrightarrow{\omega_0,i_0} x^1$ is an elementary path of Γ_F, one has $i_0 \in I_{F^{\omega_0}}(x^0)$. Noticing that there exists $\omega_1 \in [1:\omega]_{\mathbb{N}}$ and $i \in I_{F^{\omega_1}}(x^1)$ such that $\tilde{f}_i^{\omega_0}(x^0) \neq \tilde{f}_i^{\omega_1}(x^1)$, we conclude that

$$\tilde{f}_i^{\omega_0}(x^0) \neq \tilde{f}_i^{\omega_1}\big(F_{i_0}^{\omega_0}(x^0)\big),\ i_0 \in I_{F^{\omega_0}}(x^0).$$

According to Definition 3, $G_F(x^0)$ has an arc from i_0 to i of sign $\tilde{f}_{i_0}^{\omega_0}(x^0)\tilde{f}_i^{\omega_1}(x^1)$.

Assume that the conclusion holds for any integer $r \in [1:m-1]_{\mathbb{N}}$, $m \in [2:r-1]_{\mathbb{N}}$. Now, we consider the case of $r=m$. Since $x^{m-1} \xrightarrow{\omega_{m-1},i_{m-1}} x^m$ is an elementary path of Γ_F with length 1, similar to the proof of $r=1$, there exists $\omega_m \in [1:\omega]_{\mathbb{N}}$, $i \in I_{F^{\omega_m}}(x^m)$ and $i_{m-1} \in I_{F^{\omega_{m-1}}}(x^{m-1})$ such that $G_F(x^{m-1})$ has an arc from i_{m-1} to i of sign

$$s_{i_{m-1},i} = \tilde{f}_{i_{m-1}}^{\omega_{m-1}}(x^{m-1})\tilde{f}_i^{\omega_m}(x^m).$$

Find the smallest integer $q \in [0:m-1]_{\mathbb{N}}$ such that

$$\tilde{f}_{i_{m-1}}^{\omega_q}(x^q) = \tilde{f}_{i_{m-1}}^{\omega_{m-1}}(x^{m-1}). \tag{9}$$

If $q=0$, that is, $\tilde{f}_{i_{m-1}}^{\omega_0}(x^0) = \tilde{f}_{i_{m-1}}^{\omega_{m-1}}(x^{m-1}) \neq 0$, then $i_{m-1} \in I_{F^{\omega_0}}(x^0)$. Therefore, one concludes that there exists $i_{m-1} \in I_{F^{\omega_0}}(x^0)$ such that $G_F(x^{m-1})$ has a path from i_{m-1} to i of sign

$$s_{i_{m-1},i} = \tilde{f}_{i_{m-1}}^{\omega_{m-1}}(x^{m-1})\tilde{f}_i^{\omega_m}(x^m) = \tilde{f}_{i_{m-1}}^{\omega_0}(x^0)\tilde{f}_i^{\omega_m}(x^m).$$

If $q>0$, then

$$\tilde{f}_{i_{m-1}}^{\omega_l}(x^l) \neq \tilde{f}_{i_{m-1}}^{\omega_q}(x^q),\ \forall\, l \in [0:q-1]_{\mathbb{N}}.$$

As a result, for the elementary path $x^0 \xrightarrow{\omega_0,i_0} x^1 \xrightarrow{\omega_1,i_1} \cdots \xrightarrow{\omega_{q-1},i_{q-1}} x^q$, there exists $\omega_q \in [1:\omega]_{\mathbb{N}}$ and $i_{m-1} \in I_{F^{\omega_q}}(x^q)$ such that $\tilde{f}_{i_{m-1}}^{\omega_l}(x^l) \neq \tilde{f}_{i_{m-1}}^{\omega_q}(x^q)$ holds for any $l \in [0:q-1]_{\mathbb{N}}$. Since $q<m$, by induction, there exists $j \in I_{F^{\omega_0}}(x^0)$ such that $\bigcup_{l=0}^{q-1} G_F(x^l)$ has a path from j to i_{m-1} of sign

$$s_{j,i_{m-1}} = \tilde{f}_j^{\omega_0}(x^0)\tilde{f}_{i_{m-1}}^{\omega_q}(x^q).$$

Since $G_F(x^{m-1})$ has a path from i_{m-1} to i of sign $s_{i_{m-1},i}$, one claims from (9) that $\bigcup_{l=0}^{m-1} G_F(x^l)$ has a path from j to i of sign

$$\begin{aligned} s_{j,i} &= s_{j,i_{m-1}} s_{i_{m-1},i} \\ &= \tilde{f}_j^{\omega_0}(x^0)\tilde{f}_{i_{m-1}}^{\omega_q}(x^q)\tilde{f}_{i_{m-1}}^{\omega_{m-1}}(x^{m-1})\tilde{f}_i^{\omega_m}(x^m) \\ &= \tilde{f}_j^{\omega_0}(x^0)\big[\tilde{f}_{i_{m-1}}^{\omega_q}(x^q)\big]^2\tilde{f}_i^{\omega_m}(x^m) \\ &= \tilde{f}_j^{\omega_0}(x^0)\tilde{f}_i^{\omega_m}(x^m). \end{aligned}$$

The proof is complete.

Proposition 3. Let a cyclic attractor $A \subseteq \mathcal{D}^n$ be given. If there exists a state $x \in A$ such that

$$\left|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)\right| = 1, \tag{10}$$

then $\bigcup_{z\in A} G_F(z)$ has a negative circuit.

Proof. According to (10), there exists $l \in [1:\omega]_{\mathbb{N}}$ such that $|I_{F^l}(x)|=1$. We further suppose $I_{F^l}(x)=\{i\}$. If $\tilde{f}_i^l(x)>0$, then let $x^1 = F_i^l(x)$. Hence, one obtains that $x \xrightarrow{l,i} x^1$ is an arc of Γ_F and $x_i < x_i^1$. Since $x \in A$, one has $x^1 \in A$. In addition, one can find an elementary path in Γ_F, denoted by

$$x^1 \xrightarrow{\omega_1,i_1} x^2 \xrightarrow{\omega_2,i_2} \cdots \xrightarrow{\omega_{r-1},i_{r-1}} x^r,$$

where $x^r = x$ and $x^p \in A$, $\forall\, p \in [1:r]_{\mathbb{N}}$. If $\tilde{f}_i^{\omega_q}(x^q) \geqslant 0$, $\forall\, q \in [1:r-1]_{\mathbb{N}}$, then one has $x_i^q \leqslant x_i^{q+1}$, which implies that

$$x_i^1 \leqslant x_i^r = x_i.$$

This is a contradiction. Therefore, one can find the smallest integer $q \in [1 : r-1]_{\mathbb{N}}$ such that $\tilde{f}_i^{\omega_q}(x^q) < 0$. Then, for the elementary path

$$x := x^0 \xrightarrow{l,i} x^1 \xrightarrow{\omega_1, i_1} \cdots \xrightarrow{\omega_{q-1}, i_{q-1}} x^q,$$

there exists $i \in I_{F^{\omega_q}}(x^q)$ such that

$$\tilde{f}_i^{\omega_\alpha}(x^\alpha) \neq \tilde{f}_i^{\omega_q}(x^q),\ \forall\ \alpha \in [0 : q-1]_{\mathbb{N}},$$

where $\omega_0 = l$. According to Proposition 2, there exists $j \in I_{F^l}(x^0)$ such that $\bigcup_{\alpha=0}^{q-1} G_F(x^\alpha)$ has a path from j to i of sign $\tilde{f}_j^l(x^0)\tilde{f}_i^{\omega_q}(x^q)$. Because of $I_{F^l}(x^0) = \{i\}$, $\tilde{f}_i^l(x^0) > 0$ and $\tilde{f}_i^{\omega_q}(x^q) < 0$, one concludes that $\bigcup_{\alpha=0}^{q-1} G_F(x^\alpha)$ has a path from i to i of sign

$$s = \tilde{f}_j^l(x^0)\tilde{f}_i^{\omega_q}(x^q) = \tilde{f}_i^l(x^0)\tilde{f}_i^{\omega_q}(x^q) < 0,$$

that is, $\bigcup_{\alpha=0}^{q-1} G_F(x^\alpha)$ has a negative circuit. Noticing that $\{x^0, \cdots, x^{q-1}\} \subseteq A$, we derive that $\bigcup_{\alpha=0}^{q-1} G_F(x^\alpha)$ is a subgraph of $\bigcup_{x \in A} G_F(x)$, which implies that $\bigcup_{x \in A} G_F(x)$ has a negative circuit.

The proof of case $\tilde{f}_i^l(x^0) < 0$ is similar, which is omitted.

Proposition 4. Let a cyclic attractor $A \subseteq \mathcal{D}^n$ be given. If

$$|\cup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| > 1 \tag{11}$$

holds for any $x \in A$, then there exists a PBN with the following asynchronous stochastic update:

$$\begin{aligned} x_i(t+1) &= \begin{cases} h_i^1\Big([x_j(t)]_{j \in N(i,1)}\Big), & \text{with probability } p_1, \\ \vdots & \\ h_i^\omega\Big([x_j(t)]_{j \in N(i,\omega)}\Big), & \text{with probability } p_\omega, \end{cases} \\ x_l(t+1) &= x_l(t),\ l \in [1 : n]_{\mathbb{N}} \setminus \{i\},\ i \in [1 : n]_{\mathbb{N}}, \end{aligned} \tag{12}$$

having a cyclic attractor $B \subsetneqq A$, and $G_H(z)$ is a subgraph of $G_F(z)$, $\forall\ z \in \mathcal{D}^n$, where $G_H(z)$ is the local interaction graph of system (12).

Proof. Arbitrarily given $y \in A$, we derive from (11) that $I_{F^1}(y) \cup \cdots \cup I_{F^\omega}(y)$ contains at least two elements. Without loss of generality, assume that there exists $l \in [1 : \omega]_{\mathbb{N}}$ satisfying

$$1 \in I_{F^l}(y). \tag{13}$$

We construct a PBN with asynchronous stochastic update in the form of (12), where $h_1^\alpha(x) = x_1$ and $h_p^\alpha(x) = f_p^\alpha(x)$, $\forall\ \alpha \in [1 : w]_{\mathbb{N}}$, $\forall\ p \in [2 : n]_{\mathbb{N}}$.

We first prove that A is a trap domain of system (12). It is sufficient to prove that for any $x \in A$, it holds that

$$H_i^\alpha(x) \in A,\ \forall\ \alpha \in [1 : \omega]_{\mathbb{N}},\ \forall\ i \in \bigcup_{\alpha=1}^{\omega} I_{H^\alpha}(x), \tag{14}$$

where $H_i^\alpha(x) = \big(x_1, \cdots, x_{i-1}, h_i^\alpha(x), x_{i+1}, \cdots, x_n\big)$, $i \in [1 : n]_{\mathbb{N}}$ and $I_{H^\alpha}(x) = \big\{i \in [1 : n]_{\mathbb{N}} : h_i^\alpha(x) \neq x_i\big\}$. Since $h_1^\alpha(x) = x_1$, $\forall\ \alpha \in [1 : \omega]_{\mathbb{N}}$, one has $1 \notin \bigcup_{\alpha=1}^{\omega} I_{H^\alpha}(x)$. Hence,

$$F_i^\alpha(x) = H_i^\alpha(x),\ \forall\ \alpha \in [1 : \omega]_{\mathbb{N}},\ \forall\ i \in \bigcup_{\alpha=1}^{\omega} I_{H^\alpha}(x). \tag{15}$$

Considering that A is a cyclic attractor of system (2), one has

$$F_i^\alpha(x) \in A,\ \forall\ \alpha \in [1 : \omega]_{\mathbb{N}},\ \forall\ i \in \bigcup_{\alpha=1}^{\omega} I_{H^\alpha}(x),\ \forall\ x \in A,$$

which together with (15) means that

$$H_i^\alpha(x) \in A,\ \forall\ \alpha \in [1 : \omega]_{\mathbb{N}},\ \forall\ i \in \bigcup_{\alpha=1}^{\omega} I_{H^\alpha}(x),\ \forall\ x \in A.$$

Therefore, A is a trap domain of system (12). According to the definition of trap domain, one can find an attractor $B \subseteq A$ of system (12).

Now, we prove that B is a cyclic attractor of system (12). Arbitrarily given $x = (x_1, \cdots, x_n) \in B \subseteq A$, one has $|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| > 1$. For any $i \in \bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x) \setminus \{1\}$, there exists $\theta \in [1 : \omega]_{\mathbb{N}}$ such that

$$x_i \neq f_i^\theta(x) = h_i^\theta(x),$$

which means $x \neq H_i^\theta(x)$. Since $x \in B$ and B is an attractor, one has $H_i^\theta(x) \in B$. Therefore, $|B| \geqslant 2$, that is, B is a cyclic attractor.

Next, we prove $B \subsetneqq A$ by a reduction to absurdity. Assume $B = A$. From (13), one has

$$y \neq F_1^l(y) \in A = B.$$

Since B is a cyclic attractor of system (12), there exists a path

$$y = x^0 \xrightarrow{l,1} x^1 \xrightarrow{\omega_1, i_1} \cdots \xrightarrow{\omega_{r-1}, i_{r-1}} x^r = F_1^l(y)$$

in the state transition graph Γ_H of system (12). One derives from $h_1^\alpha(x) = x_1$, $\forall\, \alpha \in [1 : w]_{\mathbb{N}}$ that

$$x_1^0 = x_1^1 = \cdots = x_1^r,$$

which means that $y_1 = f_1^l(y)$. This is contradictory to (13). Therefore, $B \subsetneqq A$.

Finally, we prove that $G_H(z)$ is a subgraph of $G_F(z)$, $\forall z \in \mathcal{D}^n$. Suppose that $G_H(z)$ has an arc (j, s, i). By Definition 3 there exist $\omega_1, \omega_2 \in [0 : \omega]_{\mathbb{N}}$ such that

$$\tilde{h}_j^{\omega_1}(z) \neq 0$$

and

$$\tilde{h}_i^{\omega_2}\big(H_j^{\omega_1}(z)\big) \neq 0.$$

Obviously, $i \neq 1, j \neq 1$. Hence, $f_j^{\omega_1} = h_j^{\omega_1}$ and $f_i^{\omega_2} = h_i^{\omega_2}$. According to Definition 3, (j, s, i) is also an arc of $G_F(z)$.

Theorem 1. If $A \subseteq \mathcal{D}^n$ is a cyclic attractor of system (2), then $\bigcup_{x \in A} G_F(x)$ has a negative circuit.

Proof. Let $\mathcal{U}$ be the set of couples (F, A), where F represents the dynamics of system (2) and A is a cyclic attractor of Γ_F. Let $\prec$ be a well-founded strict order on $\mathcal{U}$, where $(H, B) \prec (F, A)$ if and only if $B \subsetneqq A$.

When (F, A) is a minimal element of $(\mathcal{U}, \prec)$, we suppose that $|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| > 1$ holds for any $x \in A$. Following Proposition 4, there exists (H, B) such that $(H, B) \prec (F, A)$, which contradicts the minimality of (F, A). Therefore, there exists a state $x \in A$ such that $|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| \leqslant 1$. Noticing that A is a cyclic attractor, one has $\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x) \neq \emptyset$, which means that $|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| = 1$. According to Proposition 3, $\bigcup_{x \in A} G_F(x)$ has a negative circuit.

When (F, A) is not a minimal element of $(\mathcal{U}, \prec)$, we suppose that $\bigcup_{x \in B} G_H(x)$ has a negative circuit for all $(H, B) \prec (F, A)$. If $|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| > 1$ holds for any $x \in A$, then following Proposition 4, there exists $(H, B) \prec (F, A)$ such that $G_H(z)$ is a subgraph of $G_F(z)$ for any $z \in \mathcal{D}^n$. Since $B \subsetneqq A$, one has $\bigcup_{x \in B} G_H(x)$ is a subgraph of $\bigcup_{x \in A} G_F(x)$. Hence, $\bigcup_{x \in A} G_F(x)$ has a negative circuit. Otherwise, there exists a state $x \in A$ such that $|\bigcup_{\alpha=1}^{\omega} I_{F^\alpha}(x)| = 1$, which together with Proposition 3 implies that $\bigcup_{x \in A} G_F(x)$ has a negative circuit.

Combining Proposition 1 and Theorem 1, we give the following result to reveal the relationship between non-oscillation of system (2) and negative circuits in the joint interaction graph.

Theorem 2. If the joint interaction graph G_F has no negative circuits, then system (2) achieves the non-oscillation.

Proof. We just need to prove that if system (2) has cyclic attractors, then G_F has a negative circuit.

If A is a cyclic attractor of system (2), then from Theorem 1, $\bigcup_{x \in A} G_F(x)$ has a negative circuit. According to Proposition 1, $\bigcup_{x \in A} G_F(x)$ is a subgraph of G_F. Therefore, G_F has a negative circuit.

Remark 3. The complexity of Theorem 2 consists of the following two parts.

(i) The computational complexity of establishing the joint interaction graph of system (2) from Definition 1 is $O(2^{\max_{i \in N} |\cup_{\alpha=1}^{\omega} N(i,\alpha)|} \omega^2)$.

(ii) The computational complexity of checking whether the joint interaction graph has negative circuits is $O(n|E|)$ based on the Bellman Ford algorithm [33], which is not more than $O(n^3)$ because $|E| \leqslant n(n-1)$.

To sum up, the total complexity of Theorem 2 is $O(2^{\max_{i \in N} |\cup_{\alpha=1}^{\omega} N(i,\alpha)|} \omega^2 + n^3)$.

Even though the non-oscillation of system (2) can be verified through Theorem 2, the fixed points of system (2) cannot be constructed. Next, we aim to find all the fixed points of system (2) with no negative circuits in the joint interaction graph based on the node removal technique.

We review some fundamental knowledge about node removal, which was initially introduced in [32] for the purpose of finding all the fixed points of large-scale BNs with synchronous update.

If we fix $\sigma(t) \equiv \alpha \in [1:\omega]_{\mathbb{N}}$, then PBN (1) becomes a BN, whose network graph, denoted by $G_F^{\alpha} = (N, E^{\alpha})$, is a directed graph with vertex set $N = [1:n]_{\mathbb{N}}$ and arc set E^{α}. If there exists $[x_l]_{l \in N(i,\alpha)\setminus\{j\}} \in \mathcal{D}^{|N(i,\alpha)|-1}$ such that

$$f_i^{\alpha}\big([x_l]_{l\in N(i,\alpha)\setminus\{j\}}, 1\big) - f_i^{\alpha}\big([x_l]_{l\in N(i,\alpha)\setminus\{j\}}, 0\big) \neq 0,$$

then (j,i) is called an arc of G_F^{α}. If there exists $(i,i) \in E^{\alpha}$, then we say that node i has a self-loop in the network graph. The key idea of node removal is to remove every node which does have a self-loop in the network graph. The node removal technique encompasses two algorithms: "Algorithm S" for the simplification of Boolean functions and "Algorithm R" for the reduction of Boolean variables.

For Algorithm S, the input is $F^{\alpha} = (f_1^{\alpha}, \cdots, f_n^{\alpha})$, and the output is the simplified $F^{\alpha} = (f_1^{\alpha}, \cdots, f_n^{\alpha})$ derived from the law of logical operations. The objective of Algorithm S is to eliminate the redundant variables within $f_1^{\alpha}, \cdots, f_n^{\alpha}$.

For Algorithm R, the input is the simplified $F^{\alpha} = (f_1^{\alpha}, \cdots, f_n^{\alpha})$, and a node j which does not have a self-loop in G_F^{α}. There are 3 steps in Algorithm R. Step 1: find nodes $i_1, \cdots, i_r$ such that $f_{i_1}^{\alpha}, \cdots, f_{i_r}^{\alpha}$ depend on x_j; Step 2: substitute f_j^{α} for x_j in $f_{i_1}^{\alpha}, \cdots, f_{i_r}^{\alpha}$ and use Algorithm S to get the new Boolean functions $g_{i_1}^{\alpha}, \cdots, g_{i_r}^{\alpha}$ for node $i_1, \cdots, i_r$; Step 3: delete the node j to obtain the tuple of Boolean functions of the remaining nodes, denoted by $F_{[j]}^{\alpha} = (f_{1,[j]}^{\alpha}, \cdots, f_{j-1,[j]}^{\alpha}, f_{j+1,[j]}^{\alpha}, \cdots, f_{n,[j]}^{\alpha})$, where $f_{i_l,[j]}^{\alpha} = g_{i_l}^{\alpha}$, $l \in [1:r]_{\mathbb{N}}$, and $f_{k,[j]}^{\alpha} = f_k^{\alpha}$, $k \in [1:n]_{\mathbb{N}} \setminus (\{i_1, \cdots, i_r\} \cup \{j\})$. The output is $F_{[j]}^{\alpha}$. The purpose of Algorithm R is to remove the node j which does not have a self-loop in the original BN.

Keep Algorithms S and R going. Let $F_{[j_1,\cdots,j_k]}^{\alpha}$ be the tuple of Boolean functions after the removal of nodes $j_1, \cdots, j_k$ which do not have self-loops. As was demonstrated in [32], the node removal technique is independent of the order in which the nodes are eliminated. Let $\Omega(F^{\alpha})$ denote the set of fixed points for the original BN and $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha})$ denote the set of fixed points for the BN after removing nodes $j_1, \cdots, j_k$. Then, a one-to-one correspondence between $\Omega(F^{\alpha})$ and $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha})$ was explored in [32].

Suppose that the order of nodes to be eliminated is $j_1, \cdots, j_k$, where $j_l = l$, $l \in [1:k]_{\mathbb{N}}$ (it can be guaranteed by a coordinate transformation). Let $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha}) = \{x'_{e,1}, \cdots, x'_{e,q}\}$ and $f_{j_1,[j_0]}^{\alpha} = f_{j_1}^{\alpha}$. By resorting to [32], we give Algorithm 1 to obtain $\Omega(F^{\alpha})$ from $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha}) = \{x'_{e,1}, \cdots, x'_{e,q}\}$.

Algorithm 1 Calculate $\Omega(F^{\alpha})$ from $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha})$.

Require: f_i^{α}, $i \in [1:n]_{\mathbb{N}}$, $j_1, \cdots, j_k$, $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha})$;
Ensure: $\Omega(F^{\alpha})$;
for $i = k : 1$ **do**
  $\Omega(F^{\alpha}) = \emptyset$;
  for each $x'_{e,r}$ in $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha})$ **do**
    $x'_{e,r} = (f_{j_i,[j_1,\cdots,j_{i-1}]}^{\alpha}(x'_{e,r}), x'_{e,r})$;
    $\Omega(F^{\alpha}) = \Omega(F^{\alpha}) \cup \{x'_{e,r}\}$;
  end for
  $\Omega(F_{[j_1,\cdots,j_k]}^{\alpha}) = \Omega(F^{\alpha})$;
end for

With the help of Algorithm 1, we construct the following set:

$$\Omega = \bigcap_{\alpha=1}^{\omega} \Omega(F^{\alpha}). \tag{16}$$

Now, we give the following result to reveal the property of Ω.

Theorem 3. If the joint interaction graph G_F of system (2) has no negative circuits, then $\Omega \neq \emptyset$ is the set of fixed points for system (2).

Proof. Let $\mathcal{T} = \mathcal{D}^n$. Obviously, $\mathcal{T}$ is a trap domain of system (2). If there exist no nonempty proper subsets of $\mathcal{T}$ being a trap domain, then $\mathcal{T}$ is a cyclic attractor, which is contradictory to Theorem 2. Hence, one can find a nonempty set $\mathcal{A} \subsetneqq \mathcal{T}$ such that any nonempty proper subset is not a trap domain. According to Theorem 2, $\mathcal{A}$ is not a cyclic attractor but a fixed point. For $y \in \mathcal{A}$, one has $y = F_i^{\alpha}(y)$, $\forall\, i \in [1:n]_{\mathbb{N}}$, $\forall\, \alpha \in [1:\omega]_{\mathbb{N}}$, which implies that $y \in \Omega(F^{\alpha})$, $\forall\, \alpha \in [1:\omega]_{\mathbb{N}}$. Hence, $\Omega \neq \emptyset$.

Arbitrarily given $x_e \in \Omega$, one has $x_e = \big(f_1^{\alpha}(x_e), \cdots, f_n^{\alpha}(x_e)\big)$, $\forall\, \alpha \in [1:\omega]_{\mathbb{N}}$, that is, $x_e = F^{\alpha}(x_e)$, $\forall\, \alpha \in [1:\omega]_{\mathbb{N}}$. Hence, $\mathbb{P}\big\{x(1) = x_e \mid x_e\big\} = 1$, which means that x_e is a fixed point of system (2).

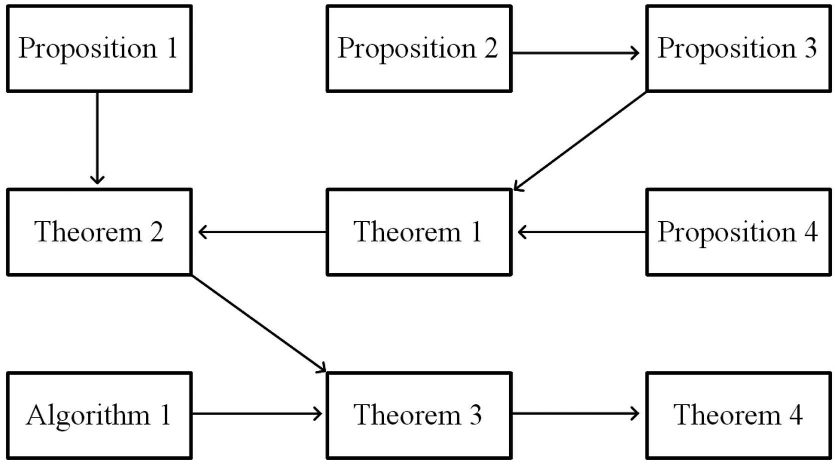

Figure 2 The outline of the key proof ideas for the propositions and theorems.

To sum up, $\Omega \neq \emptyset$ is the set of fixed points for system (2).

In the following, we further show that system (2) is stable at set Ω in distribution when the condition in Theorem 2 is satisfied.

Definition 4. Let a set $B \subseteq \mathcal{D}^n$ be given. System (2) is said to be stable at set B in distribution if

$$\lim_{t\to\infty} \mathbb{P}\big\{x(t) \in B \mid x(0)\big\} = 1 \tag{17}$$

holds for any initial state $x(0) \in \mathcal{D}^n$.

Theorem 4. If the joint interaction graph G_F of system (2) has no negative circuits, then system (2) is stable at set Ω in distribution.

Proof. If $|\Omega| = 2^n$, then $\mathbb{P}\big\{x(1) = x_0 \mid x_0\big\} = 1$ holds for any $x_0 \in \mathcal{D}^n$.

If $|\Omega| < 2^n$, we just need to prove that for any state $x_0 \in \mathcal{D}^n$, there exists a positive integer T_{x_0} such that $\mathbb{P}\big\{x(T_{x_0}) \in \Omega \mid x_0\big\} > 0$. We carry out this point by a reduction to absurdity. Divide $\mathcal{D}^n$ into two disjoint parts:

$$\begin{aligned}
&J_1 = \{x_0 \in \mathcal{D}^n : \text{there exists } t_0 \in \mathbb{Z}_+ \text{ such that } \mathbb{P}\{x(t_0) \in \Omega \mid x_0\} > 0\},\\
&J_2 = \{x_0 \in \mathcal{D}^n : \mathbb{P}\{x(t) \in \Omega \mid x_0\} = 0,\ \forall\, t \in \mathbb{Z}_+\}.
\end{aligned}$$

Obviously, $J_2 \cap \Omega = \emptyset$, which means that

$$\mathbb{P}\{x(t) \in J_2 \mid x_0\} = 1,\ \forall\, x_0 \in J_2,\ \forall\, t \in \mathbb{Z}_+.$$

Hence, J_2 is a trap domain. If $|J_2| = 1$, then $J_2 \subseteq \Omega$, which is a contradiction. If $|J_2| \geqslant 2$, then there exists an attractor $\mathcal{A} \subseteq J_2$. Since the joint interaction graph G_F has no negative circuits, one concludes from Theorem 2 that system (2) has no cyclic attractor. Hence, $\mathcal{A}$ is a fixed point and $\mathcal{A} \subseteq \Omega$, which contradicts $J_2 \cap \Omega = \emptyset$.

To sum up, $J_2 = \emptyset$ and system (2) is stable at set Ω in distribution.

In order to make the derivations of our main results clearer, we use Figure 2 to show the outline of the key proof ideas for the above propositions and theorems.

Finally, we use a simple example to show the process of finding the set Ω via the node removal technique.

Example 2. Consider system (5) with asynchronous stochastic update in Example 3. Fixing $\sigma(t) \equiv 1$, one has

$$F^1 = (f_1^1, \cdots, f_6^1) = \big(x_2 \wedge x_5, x_5 \wedge \neg x_6, x_4, (x_1 \wedge x_3) \vee x_4, \neg x_6, \neg x_2\big).$$

We perform the node removal technique according to the following steps. First, we remove the node 6 and obtain

$$\begin{aligned}
F_{[6]}^1 = \big(f_{1,[6]}^1, \cdots, f_{5,[6]}^1\big) &= \big(x_2 \wedge x_5, x_5 \wedge x_2, x_4, (x_1 \wedge x_3) \vee x_4, x_2\big)\\
&= \big(x_2 \wedge x_5, x_5 \wedge x_2, x_4, (x_1 \wedge x_3) \vee x_4, x_2\big).
\end{aligned}$$

Second, we delete the node 5 and derive

$$F_{[6,5]}^1 = \big(f_{1,[6,5]}^1, \cdots, f_{4,[6,5]}^1\big) = \big(x_2 \wedge x_2, x_2 \wedge x_2, x_4, (x_1 \wedge x_3) \vee x_4\big) = \big(x_2, x_2, x_4, (x_1 \wedge x_3) \vee x_4\big).$$

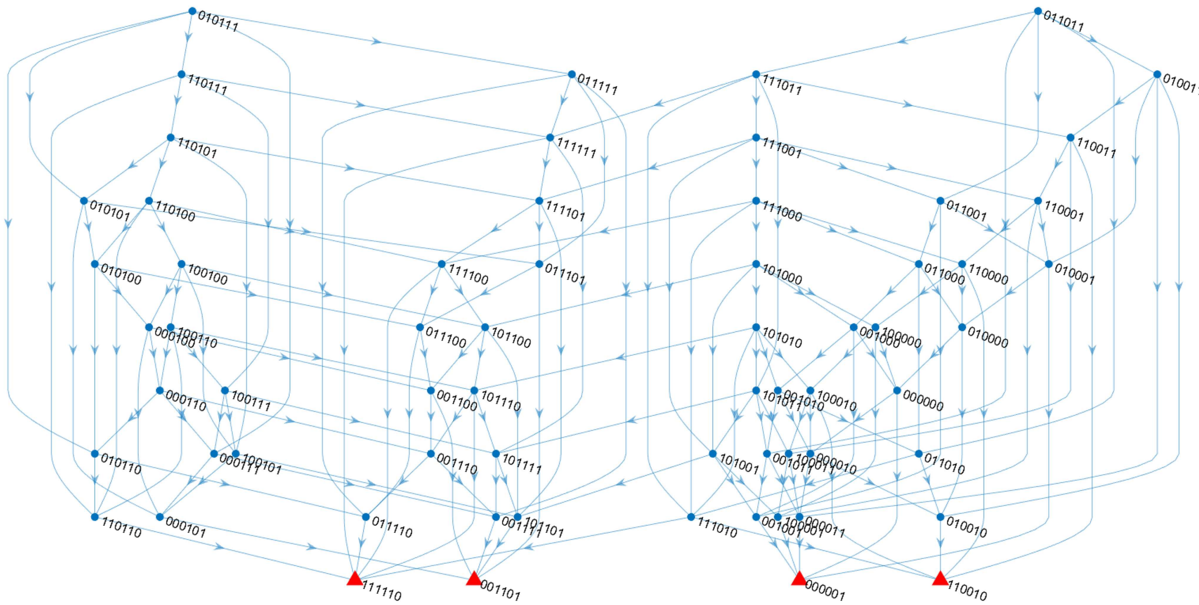

Figure 3 (Color online) The state transition graph of system (5) with asynchronous stochastic update, where the triangles represent fixed points.

Third, we remove the node 3 and acquire

$$F^1_{[6,5,3]} = \left(f^1_{1,[6,5,3]}, f^1_{2,[6,5,3]}, f^1_{4,[6,5,3]}\right) = \left(x_2, x_2, (x_1 \wedge x_4) \vee x_4\right) = \left(x_2, x_2, x_4\right).$$

Finally, we eliminate the node 1 and obtain

$$F^1_{[6,5,3,1]} = \left(f^1_{2,[6,5,3,1]}, f^1_{4,[6,5,3,1]}\right) = \left(x_2, x_4\right).$$

Therefore, one finds that

$$\Omega(F^1_{[6,5,3,1]}) = \{(1,1),(1,0),(0,1),(0,0)\},$$

which together with Algorithm 1 implies that

$$\Omega(F^1) = \{(1,1,1,1,1,0),(1,1,0,0,1,0),(0,0,1,1,0,1),(0,0,0,0,0,1)\}.$$

Similarly, one obtains

$$\begin{aligned}\Omega(F^2) \ &= \{(1,1,1,1,1,0),(1,1,1,0,1,0),(1,1,0,0,1,0),(0,0,1,1,0,1),(0,0,1,0,0,1),\\ &\quad (0,0,0,1,0,1),(0,0,0,0,0,1)\}.\end{aligned}$$

We derive from (16) that the fixed point set of system (5) with asynchronous stochastic update is

$$\Omega = \Omega(F^1) \cap \Omega(F^2) = \{(1,1,1,1,1,0),(1,1,0,0,1,0),(0,0,1,1,0,1),(0,0,0,0,0,1)\}.$$

Therefore, according to Theorem 4, system (5) with asynchronous stochastic update is stable at set Ω in distribution.

Furthermore, we give the state transition graph of system (5) with asynchronous stochastic update in Figure 3, which shows that it has no cyclic attractors and 4 fixed points. This simulation supports the results in Theorems 2 and 3.

Remark 4. The network graph of practical BNs is usually very sparse [34], and sparse directed graphs tend to have fewer cycles [35]. This means that after node removal, the number of remaining nodes K is generally much smaller than the original total number of nodes n. By using the node removal technique, the complexity of calculating Ω is significantly reduced from $\omega 2^n$ to $\omega 2^K$.

Remark 5. Although the joint interaction graph of this paper is constructed for BNs, it can also be extended to the general multi-valued logical networks. The core difference between multi-valued logical networks and BNs lies in the cardinality of the finite set which is taken by the node states. Correspondingly, in order to make this extension realistic, we adjust Definition 1 as follows. Assume that the node states take values from the finite set

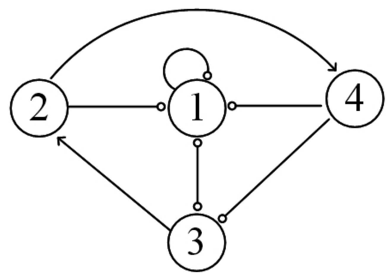

Figure 4 The joint interaction graph of system (18) with asynchronous stochastic update, where $j \to i$ and $j \multimap i$ denote $(j, 1, i) \in E$ and $(j, \pm 1, i) \in E$, respectively.

$\mathcal{D}_k$. The joint interaction graph of system (2), denoted by $G_F = (N, E)$, is a signed directed graph with vertex set $N = [1:n]_{\mathbb{N}}$ and arc set E. Here, (j, s, i) is called an arc of G_F, if there exist $\omega_1, \omega_2 \in [1:\omega]_{\mathbb{N}}$ and $[x_l]_{l \in N} \in \mathcal{D}_k^n$ with $x_j + 1 \in \mathcal{D}_k$ such that

$$\frac{f_i^{\omega_1}\big([x_l]_{l \in N(i,\omega_1)\backslash\{j\}}, x_j + 1\big) - f_i^{\omega_2}\big([x_l]_{l \in N(i,\omega_2)}\big)}{|f_i^{\omega_1}\big([x_l]_{l \in N(i,\omega_1)\backslash\{j\}}, x_j + 1\big) - f_i^{\omega_2}\big([x_l]_{l \in N(i,\omega_2)}\big)|} = s \in \{-1, 1\}.$$

Based on this new definition, one can verify the non-oscillation of multi-valued logical networks similar to the derivation of Theorem 2.

We give the following example to demonstrate the joint interaction graph of multi-valued logical networks.

Example 3. Consider the following network with 4 agents and 2 modes over the finite field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$:

$$x_i(t+1) = f_i^{\sigma(t)}\Big([x_j(t)]_{j \in N(i,\sigma(t))}\Big),\ i \in [1:4]_{\mathbb{N}}, \tag{18}$$

where

$$\begin{cases} f_1^1\Big([x_j(t)]_{j \in N(1,1)}\Big) = 4 \times_5 x_2(t) +_5 2 \times_5 x_3(t), f_1^2\Big([x_j(t)]_{j \in N(1,2)}\Big) = 3 \times_5 x_2(t) +_5 3 \times_5 x_3(t), \\ f_2^1\Big([x_j(t)]_{j \in N(2,1)}\Big) = f_2^2\Big([x_j(t)]_{j \in N(2,2)}\Big) = x_3(t), \\ f_3^1\Big([x_j(t)]_{j \in N(3,1)}\Big) = f_3^2\Big([x_j(t)]_{j \in N(3,2)}\Big) = 2 \times_5 x_1(t) +_5 4 \times_5 x_4(t), \\ f_4^1\Big([x_j(t)]_{j \in N(4,1)}\Big) = f_4^2\Big([x_j(t)]_{j \in N(4,2)}\Big) = x_2(t). \end{cases}$$

Assume that $\mathbb{P}\{\sigma(t) = 1\} = 0.42$ and $\mathbb{P}\{\sigma(t) = 2\} = 0.58$. To present the joint interaction graph $G_F = (N, E)$ of system (18) with asynchronous stochastic update, we take node 1 as an example. Since $f_1^1(0, 1) - f_1^1(0, 0) = 2$ and $f_1^1(0, 3) - f_1^1(0, 2) = -3$, one has $(3, \pm 1, 1) \in E$. According to $f_1^2(1, 0) - f_1^1(0, 0) = 3$ and $f_1^2(1, 1) - f_1^1(0, 1) = -1$, one concludes that $(2, \pm 1, 1) \in E$. Because of $f_1^1(0, 1) - f_1^2(0, 1) = -1$, there must exist two states $(1, 0, 1, 1)$ and $(0, 0, 1, 1)$ such that $f_1^1(1, 0, 1, 1) - f_1^2(0, 0, 1, 1) = -1$ and $f_1^2(1, 0, 1, 1) - f_1^1(0, 0, 1, 1) = 1$ hold. Hence, $(1, \pm 1, 1) \in E$. Similarly, $(4, \pm 1, 1) \in E$.

In a similar manner, one can derive all the arcs for other nodes. Then, the joint interaction graph is shown in Figure 4.

4 An illustrative example

As was shown in [36], each of logical models for the prominent signalling pathways involved in Drosophila development can be considered a building block to assemble more complex models for specific developmental processes. Here, we consider the BN model for the combination of Toll and EGF pathways in Drosophila development whose dynamics is shown in (19).

According to [37], disrupting the specific interactions can result in some diseases, which can be modelled as the removal of corresponding edges in the network graph. For the purpose of illustrating Theorem 4, we suppose that there exists a stochastic edge removal perturbation in the whole network, that is, edges $(16, 47)$ and $(46, 47)$ can be stochastically removed at the same time. Then, we establish a PBN with 2 modes, where $F^1 = (f_1^1, \cdots, f_{54}^1)$ is presented in (19), $F^2 = (f_1^2, \cdots, f_{54}^2)$, $f_i^2 = f_i^1$, $\forall\ i \in [1:46]_{\mathbb{N}} \cup [48:54]_{\mathbb{N}}$, and $f_{47}^2 = x_4 \wedge x_{47}$. Meanwhile, we assume that $\mathbb{P}\{\sigma(t) = j\} = p_j > 0$, where $t \in \mathbb{N}$, $j \in [1:2]_{\mathbb{N}}$, and $p_1 + p_2 = 1$.

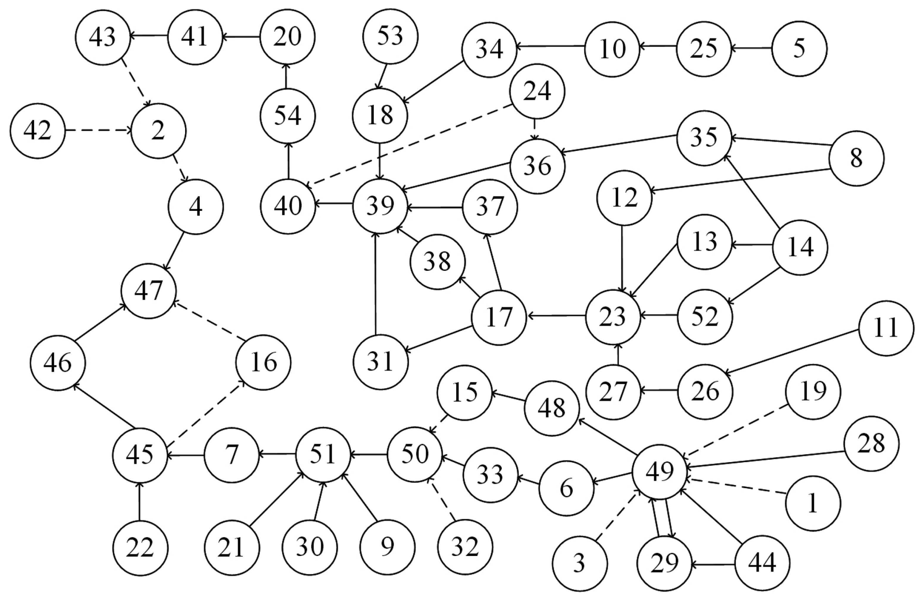

Figure 5 The partial joint interaction graph of the PBN model of Toll and EGF pathways with asynchronous stochastic update, where $j \to i$ and $j \dashrightarrow i$ denote $(j, 1, i) \in E$ and $(j, -1, i) \in E$, respectively. Here, $(j, \pm 1, 47) \in E$, $j \in [1:54]_{\mathbb{N}} \setminus \{4, 16, 46, 47\}$ are omitted.

$$
\begin{cases}
\text{Aos}: x_1(t+1) = 0, \text{Cactus}: x_2(t+1) = \neg x_{42}(t) \wedge \neg x_{43}(t), \text{Cbl}: x_3(t+1) = 0, \\
\text{Dif}: x_4(t+1) = \neg x_2(t), \text{Dv-patterning}: x_5(t+1) = 1, \text{Drk}: x_6(t+1) = x_{49}(t), \\
\text{Dsor1}: x_7(t+1) = x_{51}(t), \text{Fungi}: x_8(t+1) = 1, \text{Cnk}: x_9(t+1) = 1, \text{Gd}: x_{10}(t+1) = x_{25}(t), \\
\text{GramN-Bact}: x_{11}(t+1) = 1, \text{GNBP3}: x_{12}(t+1) = x_8(t), \text{GNBP1}: x_{13}(t+1) = x_{14}(t), \\
\text{Gramp-Bact}: x_{14}(t+1) = 1, \text{Gap1}: x_{15}(t+1) = x_{48}(t), \text{Gro}: x_{16}(t+1) = \neg x_{45}(t), \\
\text{Grass}: x_{17}(t+1) = x_{23}(t), \text{Easter}: x_{18}(t+1) = x_{34}(t) \wedge x_{53}(t), \text{Kek}: x_{19}(t+1) = 0, \\
\text{MyD88}: x_{20}(t+1) = x_{54}(t), \text{Ksr}: x_{21}(t+1) = 1, \text{Msk}: x_{22}(t+1) = 1, \\
\text{McdSP}: x_{23}(t+1) = x_{12}(t) \wedge x_{13}(t) \wedge x_{27}(t) \wedge x_{52}(t), \text{Nec}: x_{24}(t+1) = 0, \text{Ndl}: x_{25}(t+1) = x_5(t), \\
\text{DAP}: x_{26}(t+1) = x_{11}(t), \text{PGRP-SD}: x_{27}(t+1) = x_{26}(t), \text{Shc}: x_{28}(t+1) = 1, \\
\text{Vein}: x_{29}(t+1) = x_{44}(t) \wedge x_{49}(t), \text{Src42}: x_{30}(t+1) = 1, \text{Spheroide}: x_{31}(t+1) = x_{17}(t), \\
\text{Sty}: x_{32}(t+1) = 0, \text{Sos}: x_{33}(t+1) = x_6(t), \text{Snl}: x_{34}(t+1) = x_{10}(t), \\
\text{Viru-Fact}: x_{35}(t+1) = x_8(t) \wedge x_{14}(t), \text{Psh}: x_{36}(t+1) = \neg x_{24}(t) \wedge x_{35}(t), \\
\text{Sphinx}: x_{37}(t+1) = x_{17}(t), \text{Sririt}: x_{38}(t+1) = x_{17}(t), \\
\text{SPE}: x_{39}(t+1) = x_{18}(t) \wedge x_{31}(t) \wedge x_{36}(t) \wedge x_{37}(t) \wedge x_{38}(t), \text{Spz}: x_{40}(t+1) = \neg x_{24}(t) \wedge x_{39}(t), \\
\text{Tube}: x_{41}(t+1) = x_{20}(t), \text{Slmb}: x_{42}(t+1) = 1, \text{Pelle}: x_{43}(t+1) = x_{41}(t), \text{Spi}: x_{44}(t+1) = 1, \\
\text{RI}: x_{45}(t+1) = x_7(t) \wedge x_{22}(t), \text{Pnt}: x_{46}(t+1) = x_{45}(t), \\
\text{Targets}: x_{47}(t+1) = x_4(t) \wedge \neg x_{16}(t) \wedge x_{46}(t) \wedge x_{47}(t), \text{PLCg}: x_{48}(t+1) = x_{49}(t), \\
\text{Der}: x_{49}(t+1) = \neg x_1(t) \wedge \neg x_3(t) \wedge \neg x_{19}(t) \wedge x_{28}(t) \wedge x_{29}(t) \wedge x_{44}(t), \\
\text{Ras}: x_{50}(t+1) = \neg x_{15}(t) \wedge \neg x_{32}(t) \wedge x_{33}(t), \text{Raf}: x_{51}(t+1) = x_9(t) \wedge x_{21}(t) \wedge x_{30}(t) \wedge x_{50}(t), \\
\text{PGRP-SA}: x_{52}(t+1) = x_{14}(t), x_{53}(t+1) = 1, x_{54}(t+1) = x_{40}(t).
\end{cases} \tag{19}
$$

For the purpose of simplicity, the partial joint interaction graph $G(F)$ is shown in Figure 5, where the arcs $(j, \pm 1, 47) \in E$, $j \in [1:54]_{\mathbb{N}} \setminus \{4, 16, 46, 47\}$ are omitted. Obviously, $G(F)$ has no negative circuits. With Algorithms S and R, we obtain

$$
F^1_{[1,\cdots,28,30\cdots,54]} = (x_{29}), \; F^2_{[1,\cdots,28,30\cdots,46,48,\cdots,54]} = (x_{29}, x_{47}).
$$

Using Algorithm 1, one has

$$\begin{aligned}\Omega = \ &\{(0,0,0,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,0,1,1,1,1,0,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,\\&0,0,0,1,1,0,0,1,1,1),(0,0,0,1,1,0,0,1,1,1,1,1,1,1,0,1,1,1,0,1,1,1,1,0,1,1,1,1,0,1,1,0,0,1,1,1,1,\\&1,1,1,1,1,1,1,0,0,0,0,0,0,0,1,1,1)\}.\end{aligned}$$

According to Theorem 4, the PBN model of Toll and EGF pathways with asynchronous stochastic update is stable at set Ω in distribution.

5 Conclusion

In this paper, we have explored the relationship between the non-oscillation of ASLNs and the existence of negative circuits in the joint interaction graph. Based on the joint interaction graph and local interaction graph, we have presented a sufficient condition to verify the non-oscillation of ASLNs. Furthermore, we have used the node removal technique to find all the fixed points of an ASLN with no negative circuits in the joint interaction graph, and revealed the global stability at the set of all the fixed points in distribution. Finally, we have applied our new results to the Toll and EGF pathways involved in Drosophila development. Future work will apply the obtained results to explore the non-oscillation of key pathways in the Parkinson's disease, and further consider the Koopman theory for the non-oscillation of ASLNs [38].

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