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Convergence rate of Smoluchowski-Kramers approximation for distribution dependent stochastic complex networks with jumps

Lijuan CHENG[1] & Yong REN[2*]

[1]School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang 524048, China
[2]Department of Mathematics, Hefei University, Hefei 230601, China

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The Smoluchowski-Kramers approximation originated from the work of Smoluchowski and Kramers on particle motion. The motion equation of a particle of mass m in a force field with the friction proportional to the velocity can be described as

$$m\ddot{X}_t^m = b\left(t, X_t^m\right) + \sigma\left(t, X_t^m\right)\dot{B}_t - \alpha\dot{X}_t^m, \tag{1}$$

where $X_0^m = x_0$, $\dot{X}_0^m = y_0 \in \mathbb{R}$, $b\left(t, X_t^m\right)$ is the deterministic part of the force, $\sigma\left(t, X_t^m\right)$ is the intensity of the noise, $\alpha\dot{X}_t^m$ describes the resistance (friction) to the motion, $\dot{B}_t$ is the standard Gaussian white noise, and the parameters m, α are positive real numbers. Assume that the friction coefficient α is a fixed positive constant. Without loss of generality, set $\alpha = 1$, and for $0 < m \ll 1$, X_t^m may be approximated by the solution of the equation

$$\dot{X}_t = b\left(t, X_t\right) + \sigma\left(t, X_t\right)\dot{B}_t, \quad X_0 = x_0 \in \mathbb{R}. \tag{2}$$

For any $0 < T < \infty$ and $\varepsilon > 0$, when

$$\lim_{m\to 0}\mathbb{P}\left(\max_{0\leqslant t\leqslant T}\left|X_t^m - X_t\right| > \varepsilon\right) = 0, \tag{3}$$

let X_t denote Smoluchowski-Kramers approximation of X_t^m.

As we know, Smoluchowski-Kramers approximation has attracted attention of researchers and many qualitative theories have been obtained. Recently, Son et al. [1] explored the Smoluchowski-Kramers approximation for second-order mean-field stochastic differential systems in L_p distances and in the total variation distance and gave an explicit rate of convergence. Liu et al. [2] investigated the rate of convergence based on the total variation distance for distribution dependent SDEs driven by fractional Brownian motion.

In this study, we discuss the rate of convergence in total variation distance for the Smoluchowski-Kramers approximation for the following distribution dependent stochastic complex networks (DDSCNs) driven by the Brownian motion and Poisson jumps with finite intensity measure ν, and X_t^m satisfies

$$\begin{cases} \mathrm{d}X_{kt}^m = Y_{kt}^m\mathrm{d}t, \\ m\mathrm{d}Y_{kt}^m = \Big[f_{kt}\left(X_{kt}^m, \mathscr{L}_{X_{kt}^m}\right) + \sum\limits_{j=1}^n H_{kjt}\Big(X_{kt}^m, X_{jt}^m, \\ \qquad \mathscr{L}_{X_{kt}^m X_{jt}^m}\Big)\Big]\mathrm{d}t - Y_{kt}^m\mathrm{d}t + g_{kt}\left(X_{kt}^m, \mathscr{L}_{X_{kt}^m}\right)\mathrm{d}B_t \\ \qquad + \int_{\mathbb{R}_0} h_{kt}(X_{kt-}^m, \mathscr{L}_{X_{kt-}^m}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z). \end{cases} \tag{4}$$

Also, X_t satisfies the following DDSCN:

$$\begin{aligned} \mathrm{d}X_{kt} =& \Big[f_{kt}\left(X_{kt}, \mathscr{L}_{X_{kt}}\right) + \sum_{j=1}^n H_{kjt}\Big(X_{kt}, X_{jt}, \mathscr{L}_{X_{kt}X_{jt}}\Big)\Big]\mathrm{d}t \\ &+ g_{kt}\left(X_{kt}, \mathscr{L}_{X_{kt}}\right)\mathrm{d}B_t \\ &+ \int_{\mathbb{R}_0} h_{kt}\Big(X_{kt-}, \mathscr{L}_{X_{kt-}}, z\Big)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \end{aligned} \tag{5}$$

where $X_{k0}^m = X_{k0} \in L^p(\Omega, \mathbb{R}^{d_k}, \mathscr{F}_0, \mathbb{P})$, $Y_{k0}^m = Y_{k0} \in L^p(\Omega, \mathbb{R}^{d_k}, \mathscr{F}_0, \mathbb{P})$, $X_{k0} = \xi_k$, $\xi \in L^p(\Omega, \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$, $p \geqslant 2$. $\mathscr{L}_{X_{kt}} := \mathbb{P}\circ X_{kt}^{-1}$ is the law of X_{kt}, $\mathscr{L}_{X_{kt}X_{jt}}$ is the joint distribution of X_{kt} and X_{jt}. $f_k : [0,T]\times\mathbb{R}^{d_k}\times\mathscr{P}\left(\mathbb{R}^{d_k}\right)\to\mathbb{R}^{d_k}$, $H_{kj} : [0,T]\times\mathbb{R}^{d_k}\times\mathbb{R}^{d_j}\times\mathscr{P}\left(\mathbb{R}^{d_k+d_j}\right)\to\mathbb{R}^{d_k}$, $g_k : [0,T]\times\mathbb{R}^{d_k}\times\mathscr{P}\left(\mathbb{R}^{d_k}\right)\to\mathbb{R}^{d_k}\otimes\mathbb{R}^{d_k}$, $h_k : [0,T]\times\mathbb{R}^{d_k}\times\mathscr{P}\left(\mathbb{R}^{d_k}\right)\times\mathbb{R}_0\to\mathbb{R}^{d_k}$ are measurable and $\sum\limits_{k=1}^n d_k = d$. $k, j \in \mathcal{N} = \{1, 2, \cdots, n\}$. B_t is one-dimensional Brownian motions on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$. $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. The compensated Poisson random measure is

$$\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t,$$

where $N(\mathrm{d}t, \mathrm{d}z)$ is a Poisson counting measure, which is independent of Brownian motion B_t, and ν is the intensity measure of N with $\nu(\mathbb{R}_0) < \infty$.

To obtain the main results, we need the following assumptions.

Assumption 1. For $t \in [0,T]$, $k, j \in \mathcal{N}$,

(1) $f_{kt}\left(x_k, \mu_k\right)$, $H_{kjt}\left(x_k, y_k, \mu_{kj}\right)$, $g_{kt}\left(x_k, \mu_k\right)$ and $h_{kt}(x_k, \mu_k, z)$ satisfy the Lipschitz conditions;

* Corresponding author (email: brightry@hotmail.com)

(2) there exists a matrix $A = (a_{kj})_{n\times n}$ such that the digraph $(\mathcal{G}, A)$ is strongly connected and for an arbitrary function $F_{kj}(x_k, x_j)$ $(k \in \mathcal{N}, j \in \mathcal{N})$, along each directed cycle $\mathcal{C}$ of the weighted digraph $(\mathcal{G}, A)$, it holds that

$$\sum_{(j,k)\in E(\mathcal{C})} F_{kj}(x_k, x_j) \leqslant 0, \tag{6}$$

for all $(x_k, x_j) \in \mathbb{R}^{n_k} \times \mathbb{R}^{n_j}$ and $k \in \mathcal{N}, j \in \mathcal{N}$.

Assumption 2. For $t \in [0,T]$, $k, j \in \mathcal{N}$, $f_{kt}(x_k, \mu_k)$, $H_{kjt}(x_k, y_k, \mu_{kj})$, $g_{kt}(x_k, \mu_k)$ and $h_{kt}(x_k, \mu_k, z)$ are twice differentiable in x_k, and the first and second order partial derivatives satisfy the Lipschitz conditions.

Lemma 1 ([3, Theorem 2.2])**.** Let $(\mathcal{G}, A)$ be a weighted digraph, where $A = (a_{kj})_{n\times n}$ $(n \geqslant 2)$. If $\mathbb{Q}$ is the set of all spanning unicyclic graphs $\mathcal{Q}$ of $(\mathcal{G}, A)$, $\mathcal{C}_{\mathcal{Q}}$ is the cycle of $\mathcal{Q}$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and c_k $(k \in \mathcal{N})$ is the cofactor of the kth diagonal element of the Laplacian matrix L for $(\mathcal{G}, A)$. Then, for an arbitrary function $F_{kj}(x_k, x_j)$ $(k \in \mathcal{N}, j \in \mathcal{N})$, it holds that

$$\sum_{k,j=1}^{n} c_k a_{kj} F_{kj}(x_k, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s).$$

In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$.

Remark 1. From Assumption 1, for each x_k, x_j and directed cycle $\mathcal{C}$, Eq. (6) holds. F_{kj} in Lemma 1 is arbitrary. Therefore, according to Assumption 1 and Lemma 1, as long as we find a suitable F_{kj}, we can solve the problem of coupling terms in stochastic complex networks.

Lemma 2 (Kunita's first inequality [4, Theorem 4.4.23])**.** For any $p \geqslant 2$, there exists a constant $C_L > 0$ such that

$$\begin{aligned} &\mathbb{E}\sup_{0\leqslant s\leqslant t}\left|\int_0^s\int_Z H(\tau,z)\tilde{N}(\mathrm{d}\tau,\mathrm{d}z)\right|^p \\ &\leqslant C_L\left\{\mathbb{E}\left[\left(\int_0^t\int_Z |H(\tau,z)|^2\nu(\mathrm{d}z)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right. \\ &\left.+\mathbb{E}\int_0^t\int_Z |H(\tau,z)|^p\nu(\mathrm{d}z)\mathrm{d}\tau\right\}. \end{aligned} \tag{7}$$

Theorem 1 (Well-posedness)**.** If Assumption 1 holds, for $p \geqslant 2$ and $\xi \in L^p(\Omega, \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$, then DDSCN (5) has a unique solution and

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X_t|^p\right) < \infty.$$

Remark 2. The distribution of the coupling term in stochastic complex networks contains joint distribution, so Theorem 1 generalizes the result of [5, Theorem 3.1] to stochastic complex networks.

To obtain the convergence rate of the Smoluchowski-Kramers approximation, we need the following lemmas.

Lemma 3. Let $(X_t)_{t\in[0,T]}$ and $(X_t^m)_{t\in[0,T]}$ be the solution of (5) and (4), respectively. If Assumptions 1 and 2 satisfy, then

(1) $\mathbb{E}\sup_{0\leqslant t\leqslant T}|X_t^m|^p \leqslant C_{p,T,K,n}$,

(2) $\sup_{0\leqslant t\leqslant T}\mathbb{E}|X_t^m - X_t|^p \leqslant C_{p,T,K,n} m^{\frac{p}{2}}$.

Lemma 4. Under Assumptions 1 and 2, the solutions of DDSCNs (5) and (4) are Malliavin differentiable. Moreover, for $0 \leqslant r \leqslant t \leqslant T$, we get

$$\mathbb{E}\left\|D^B X_t^m - D^B X_t\right\|^2_{L^2([0,T])} \leqslant C_{p,T,K,n} m, \tag{8}$$

and

$$\mathbb{E}\left\|D^N X_t^m - D^N X_t\right\|^2_{L^2([0,T]\times\mathbb{R}_0)} \leqslant C_{p,T,K,n} m, \tag{9}$$

where $\|f\|^2_{L^2([0,T]\times\mathbb{R}_0)} = \int_0^T\int_{\mathbb{R}_0}|f(t,\mu,z)|^2\nu(\mathrm{d}z)\mathrm{d}t$, $D^B F$ and $D^N F$ denote the Malliavin derivative of F, respectively.

Lemma 5. Let Assumptions 1 and 2 hold. For $0 \leqslant t \leqslant T$, we have

$$\mathbb{E}\left\|D^B X_t\right\|^{-2p}_{L^2([0,T])} \leqslant C_{p,T,K,n} t^{-p}, \tag{10}$$

$$\mathbb{E}\left\|D^N X_t\right\|^{-2p}_{L^2([0,T]\times\mathbb{R}_0)} \leqslant C_{p,T,K,n} t^{-p}; \tag{11}$$

and

$$\mathbb{E}\left\|D_B^2 X_t\right\|^2_{(L^2([0,T]))^{\otimes 2}} \leqslant C_{p,T,K,n} t^4, \tag{12}$$

$$\mathbb{E}\left\|D_N^2 X_t\right\|^2_{(L^2([0,T]\times\mathbb{R}_0))^{\otimes 2}} \leqslant C_{p,T,K,n} t^4. \tag{13}$$

Next, we give the main result.

Theorem 2 (Smoluchowski-Kramers approximation)**.** If Assumptions 1 and 2 satisfy, then for any $0 < t \leqslant T$, we get

$$d_{\mathrm{TV}}(X_t^m, X_t) \leqslant C_{p,T,K,n} t^{-1/2} m^{1/2}, \quad m \in (0,1). \tag{14}$$

Remark 3. Theorem 2 gives the convergence rate in total variation distance of Smoluchowski-Kramers approximation for DDSCNs with jumps, based on the techniques of Malliavin calculus.

Conclusion. We consider the rate of convergence for the Smoluchowski-Kramers approximation for DDSCNs with jumps. In addition, we provide an explicit bound on the total variation distance for the rate of convergence, based on the techniques of Malliavin calculus. A similar conclusion can be reached when there are coupling terms in g and h.

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