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Detailed Derivations for Paper "Communication, Sensing and Control integrated Closed-loop System: Modeling, Control Design and Resource Allocation"

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Abstract This document provides supplementary material for the paper "Communication, Sensing and Control Integrated Closed-loop System: Modeling, Control Design and Resource Allocation," including detailed proofs for Theorems 1–4 and Equations (29) and (30).

Keywords closed-loop system, wireless network, effective-capacity, model predictive control

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1 Appendix A
Proof of Theorem 1

Substituting [1, Eqn. 2] into [1, Eqn. 3], there is

$$C_i(\theta_i, W_i, \mathrm{SNR}_i, \beta_i) = -\frac{1}{\theta_i} \ln\left(\mathbb{E}\left\{\exp\left[-\theta_i W_i \log_2\left(1+\mathrm{SNR}_i\gamma^2\right)\right]\right\}\right). \tag{1}$$

Considering that γ^2 follows an exponential distribution with parameter β_i, (1) can be further derived as

$$C_i(\theta_i, W_i, \mathrm{SNR}_i, \beta_i) = W_i \log_2(\mathrm{SNR}_i\beta_i) - \frac{1}{\mathrm{SNR}_i\beta_i} - \frac{1}{\theta_i}\ln\left(\Gamma\left(1-\frac{W_i\theta_i}{\ln 2}, \frac{1}{\mathrm{SNR}_i\beta_i}\right)\right), \tag{2}$$

where $\Gamma(s,x)$ is the upper incomplete gamma function. Besides, $1-\frac{W_i\theta_i}{\ln 2} < 0$ and $\frac{1}{\mathrm{SNR}_i\beta_i} \to 0$ since the order of magnitude for bandwidth W_i is typically above 10^6, the order of SNR_i is usually above 10^3, θ is around 10^{-3} for factory scenario, and the typical value for β_i is approximately 1. Subsequently,

$$\begin{aligned}\Gamma\left(1-\frac{W_i\theta_i}{\ln 2}, \frac{1}{\mathrm{SNR}_i\beta_i}\right) &= \left(\frac{1}{\mathrm{SNR}_i\beta_i}\right)^{\left(1-\frac{W_i\theta_i}{\ln 2}\right)} E_{\frac{W_i\theta_i}{\ln 2}}\left(\frac{1}{\mathrm{SNR}_i\beta_i}\right)\\ &\approx \left(\frac{1}{\mathrm{SNR}_i\beta_i}\right)^{\left(1-\frac{W_i\theta_i}{\ln 2}\right)} \cdot \frac{-1}{\left(1-\frac{W_i\theta_i}{\ln 2}\right)},\end{aligned} \tag{3}$$

where $E_p(x)$ is the generalized exponential integral function, which is

$$E_p(x) = \int_1^\infty \frac{\mathrm{e}^{-xt}}{t^p}\,\mathrm{d}t, \tag{4}$$

and the approximation holds due to

$$E_p(0) = \frac{1}{1-p}. \tag{5}$$

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Substituting (3) into (2),

$$\begin{aligned} &C_i(\theta_i, W_i, \mathrm{SNR}_i, \beta_i) \\ =&W_i \log_2(\mathrm{SNR}_i\beta_i) - \frac{1}{\mathrm{SNR}_i\beta_i} + \frac{1}{\theta_i}\left(1 - \frac{W_i\theta_i}{\ln 2}\right)\ln\left(\mathrm{SNR}_i\beta_i\right) + \frac{1}{\theta_i}\ln\left(\frac{W_i\theta_i}{\ln 2} - 1\right) \\ =&\frac{1}{\theta_i}\ln(\mathrm{SNR}_i\beta_i)(\frac{W_i\theta_i}{\ln 2} - 1) - \frac{1}{\mathrm{SNR}_i\beta_i}. \end{aligned} \tag{6}$$

2 Appendix B Proof of Theorem 2

According to [1, Eqn. 7], the delay of a packet in a single queue follows a exponential distribution with parameter $\theta_i C_i(\theta_i, W_i, \mathrm{SNR}_i, \beta_i)$. Therefore, the probability density function of $D_i (i = u, d)$ is

$$f_{D_i}(x) = \mu_i \exp(-\mu_i x) \quad (i = u, d), \tag{7}$$

where according to Theorem 1 [1],

$$\mu_i = \theta_i C_i(\theta_i, W_i, \mathrm{SNR}_i, \beta_i) = \ln(\mathrm{SNR}_i\beta_i)(\frac{W_i\theta_i}{\ln 2} - 1) - \frac{\theta_i}{\mathrm{SNR}_i\beta_i} \quad (i = u, d). \tag{8}$$

Thus, the probability density function the closed-loop delay $D_c = D_u + D_d$ is the convolution of the probability density functions of D_u and D_d, that is

$$f_{D_c}(x) = f_{D_u}(x) * f_{D_d}(x) = -\frac{\left(\mathrm{e}^{-x\mu_u} - \mathrm{e}^{-x\mu_d}\right)\mu_u\mu_d}{\mu_u - \mu_d}. \tag{9}$$

Subsequently, the package drop probability ϵ_c satisfies

$$\epsilon_c = \mathrm{P}\left\{D_c > D_{c,\max}\right\} = 1 - \int_0^{D_{c,\max}} f_{D_c}(x)\mathrm{d}x = \frac{\mathrm{e}^{-D_{c,\mathrm{mxx}}\mu_d}\mu_u - \mathrm{e}^{-D_{c,\max}\mu_u}\mu_d}{\mu_u - \mu_d}. \tag{10}$$

Besides, the exception of the closed-loop delay when package is not dropped is

$$\begin{aligned} E\left[D_c | D_c < D_{c,\max}\right] =& \int_0^{D_{c,\ \max}} \frac{x f_{D_c}(x)}{P(D_c < D_{c,\max})}\mathrm{d}x \\ =& \frac{\mu_u^2 - \mu_d^2 + \mathrm{e}^{-D_{c,\max}\mu_u}\mu_d^2\left(1 + D_{c,\max}\mu_u\right) - \mathrm{e}^{-D_{c,\max\,x}\mu_d}\mu_u^2\left(1 + D_{c,\max}\mu_d\right)}{\mu_u\mu_d\left(\mu_u - \mu_d + \mathrm{e}^{-D_{c,\max\,x}\mu_u}\mu_d - \mathrm{e}^{-D_{c,\max}\mu_d}\mu_u\right)}. \end{aligned} \tag{11}$$

3 Appendix C Proof of Theorem 3

According to [2, Eqn. 10], the departure process of the first queue L_u is

$$L_u = \begin{cases} \lambda_u, & 0 \leqslant \theta_d \leqslant \theta_u \\ \frac{1}{\theta_d}\{\left(\theta_d - \theta_u\right)C_u(\theta_u, W_u, \mathrm{SNR}_u, \beta_u) + \lambda_u\theta_u\}, & \theta_d > \theta_u \end{cases}. \tag{12}$$

Therefore, substituting (12) into [1, Eqn. 6],

$$\lambda_d = \begin{cases} c_d\lambda_u, & 0 \leqslant \theta_d \leqslant \theta_u \\ c_d\frac{1}{\theta_d}\{\left(\theta_d - \theta_u\right)C_u(\theta_u, W_u, \mathrm{SNR}_u, \beta_u) + \lambda_u\theta_u\}, & \theta_d > \theta_u \end{cases}. \tag{13}$$

Then substituting (13) into [1, Eqn. 1], when $\theta_u > \theta_d$,

$$C_d(\theta_d, W_d, \mathrm{SNR}_d, \beta_d) \geqslant c_d\lambda_u, \tag{14}$$

and when $\theta_u < \theta_d$,

$$C_d(\theta_d, W_d, \mathrm{SNR}_d, \beta_d) \geqslant c_d\frac{1}{\theta_d}\{\left(\theta_d - \theta_u\right)C_u(\theta_u, W_u, \mathrm{SNR}_u, \beta_u) + \lambda_u\theta_u\}. \tag{15}$$

After rearranging formulas (14) and (15), Theorem 3 can be proved.

4 Appendix D Proof of (29) and (30)

Substitute [1, Eqn. 20] and [1, Eqn. 26] into the left side of [1, Eqn. 29], we can get

$$\begin{aligned}
&\mathbb{E}[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau] \\
=&\mathbb{E}[(\hat{\mathbf{X}}_{t+\tau}-\mathbf{X}_{t+\tau})^{\mathrm{T}}(\hat{\mathbf{X}}_{t+\tau}-\mathbf{X}_{t+\tau})] \\
=&\mathbb{E}\{[\widetilde{\mathbf{A}}(\hat{\mathbf{X}}_{t+\tau}-\mathbf{X}_{t+\tau})+(1-\epsilon_c-\eta)\widetilde{\mathbf{B}}\mathbf{K}\hat{\mathbf{X}}_{t+\tau}]^{\mathrm{T}}[\widetilde{\mathbf{A}}(\hat{\mathbf{X}}_{t+\tau}-\mathbf{X}_{t+\tau})+(1-\epsilon_c-\eta)\widetilde{\mathbf{B}}\mathbf{K}\hat{\mathbf{X}}_{t+\tau}]\} \\
=&\mathbb{E}\{\mathbf{e}_{\tau-1}^{\mathrm{T}}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})\mathbf{e}_{\tau-1}\}+\mathbb{E}[(1-\epsilon_c-\eta)^2\hat{\mathbf{X}}_{t+\tau-1}^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K}\hat{\mathbf{X}}_{t+\tau-1}] \\
=&\mathbb{E}\{\mathrm{Tr}[(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})\mathbf{e}_{\tau-1}\mathbf{e}_{\tau-1}^{\mathrm{T}}]\}+\mathbb{E}[(1-\epsilon_c-\eta)^2\cdot\mathrm{Tr}(\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K}\hat{\mathbf{X}}_{t+\tau-1}\hat{\mathbf{X}}_{t+\tau-1}^{\mathrm{T}})].
\end{aligned} \tag{16}$$

Since $\mathrm{Tr}(\mathbf{XY})\leqslant\mathrm{Tr}(\mathbf{X})\mathrm{Tr}(\mathbf{Y})$ if $\mathbf{X}$ and $\mathbf{Y}$ are semidefinite matrices, (16) can be further scaled as

$$\begin{aligned}
\mathbb{E}[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]\leqslant&\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})\mathbb{E}[\mathbf{e}_{\tau-1}^{\mathrm{T}}\mathbf{e}_{\tau-1}]+\mathbb{E}[(1-\epsilon_c-\eta)^2]\cdot\mathrm{Tr}(\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K})\mathbb{E}[\hat{\mathbf{X}}_{t+\tau-1}^{\mathrm{T}}\hat{\mathbf{X}}_{t+\tau-1}] \\
\leqslant&\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})\mathbb{E}[\mathbf{e}_{\tau-1}^{\mathrm{T}}\mathbf{e}_{\tau-1}]+(\epsilon_c-\epsilon_c^2)\mathrm{Tr}(\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K})\cdot\mathbf{X}_M^{\mathrm{T}}\mathbf{X}_M,
\end{aligned} \tag{17}$$

where the first inequality is due to the Cauchy-Schwarz inequality.

According to such recurrence relationship, the upper bound of $\mathbb{E}[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]$ can be obtained with $\mathbb{E}[\mathbf{e}_0^{\mathrm{T}}\mathbf{e}_0]$ given by [1, Eqn. 25], i.e. when $\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})\neq 1$,

$$\begin{aligned}
&\mathbb{E}[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]\leqslant[\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})]^\tau\mathbb{E}[\mathbf{e}_0^{\mathrm{T}}\mathbf{e}_0]+(\epsilon_c-\epsilon_c^2)\mathrm{Tr}(\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K})\mathbf{X}_M^{\mathrm{T}}\mathbf{X}_M\frac{1-[\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})]^\tau}{1-\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})} \\
=&\frac{1}{12}\frac{1}{4^r}[\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})]^\tau[(\mathbf{X}_L-\mathbf{X}_U)^{\mathrm{T}}(\mathbf{X}_L-\mathbf{X}_U)]+(\epsilon_c-\epsilon_c^2)\mathrm{Tr}(\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K})\mathbf{X}_M^{\mathrm{T}}\mathbf{X}_M\frac{1-[\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})]^\tau}{1-\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})},
\end{aligned} \tag{18}$$

and when $\mathrm{Tr}(\widetilde{\mathbf{A}}^{\mathrm{T}}\widetilde{\mathbf{A}})=1$

$$\mathbb{E}[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]\leqslant\frac{1}{12}\frac{1}{4^r}[(\mathbf{X}_L-\mathbf{X}_U)^{\mathrm{T}}(\mathbf{X}_L-\mathbf{X}_U)]+(\epsilon_c-\epsilon_c^2)\tau\cdot\mathrm{Tr}(\mathbf{K}^{\mathrm{T}}\widetilde{\mathbf{B}}^{\mathrm{T}}\widetilde{\mathbf{B}}\mathbf{K})\mathbf{X}_M^{\mathrm{T}}\mathbf{X}_M. \tag{19}$$

5 Appendix E Proof of Theorem 4

According to [1, Eqn. 20] derived in the control model, the Lyapunov function of the closed-loop system considering the effect of sensing and communication is derived as follows.

$$\begin{aligned}
\mathbb{E}[V_f(\mathbf{X}_{t+1})|\mathbf{X}_t]=&\mathbb{E}\left[\mathbf{X}_{t+1}^T\mathbf{P}\mathbf{X}_{t+1}|\mathbf{X}_t\right]=\mathbb{E}\left[(\widetilde{\mathbf{A}}\mathbf{X}_t+\eta\widetilde{\mathbf{B}}\mathbf{K}\hat{\mathbf{X}}_t)^{\mathrm{T}}\mathbf{P}(\widetilde{\mathbf{A}}\mathbf{X}_t+\eta\widetilde{\mathbf{B}}\mathbf{K}\hat{\mathbf{X}}_t)\right] \\
=&\epsilon_c|\widetilde{\mathbf{A}}\mathbf{X}_t|_{\mathbf{P}}^2+(1-\epsilon_c)|(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}\mathbf{K})X_t|_P^2+(1-\epsilon_c)\mathbb{E}_\tau[\mathbf{e}_\tau]^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{X}_t \\
&+(1-\epsilon_c)\mathbf{X}_t^{\mathrm{T}}\mathbf{PBK}\mathbb{E}_\tau[\mathbf{e}_\tau]+(1-\epsilon_c)\mathbb{E}_\tau[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\mathbf{e}_\tau],
\end{aligned} \tag{20}$$

where $\mathbb{E}_\tau[\cdot]$ represents the exception of the random variable τ.

According to [1, Eqn. 28],

$$\mathbb{E}_\tau[\mathbf{e}_\tau]=0. \tag{21}$$

Therefore, considering that with the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}_\tau[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\mathbf{e}_\tau]=&\mathbb{E}_\tau[\mathrm{Tr}\{\mathbf{e}_\tau^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\mathbf{e}_\tau\}]=\mathbb{E}_\tau[\mathrm{Tr}\{\mathbf{e}_\tau\mathbf{e}_\tau^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}]\} \\
\leqslant&\mathbb{E}_\tau[\mathrm{Tr}\{\mathbf{e}_\tau\mathbf{e}_\tau^{\mathrm{T}}\}]\mathrm{Tr}\{\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\}=\mathrm{Tr}\{\mathbb{E}[\mathbf{e}_\tau\mathbf{e}_\tau^{\mathrm{T}}]\}\mathrm{Tr}\{\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\} \\
=&\mathbb{E}_\tau[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]\mathrm{Tr}\{\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\}.
\end{aligned} \tag{22}$$

Besides, according to (25),

$$\mathbb{E}_\tau[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]\leqslant F_e(r,\epsilon_c,D_{c,\max}). \tag{23}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[V_f(\mathbf{X}_{t+1})|\mathbf{X}_t\right] \\
\leqslant&\epsilon_c|\widetilde{\mathbf{A}}\mathbf{X}_t|_{\mathbf{P}}^2+(1-\epsilon_c)|(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}\mathbf{K})X_t|_P^2+(1-\epsilon_c)F_e(r,\epsilon_c,D_{c,\max})\,\mathrm{Tr}\{\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\} \\
=&\epsilon_c|\widetilde{\mathbf{A}}\mathbf{X}_t|_{\mathbf{P}}^2+(1-\epsilon_c)|(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}\mathbf{K})X_t|_P^2+(1-\epsilon_c)F_e(r,\epsilon_c,D_{c,\max})\mathrm{Tr}\{\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{PBK}\}.
\end{aligned} \tag{24}$$

Substituting [1, Eqn. 8] into (24), and let (24) less than $\rho V_f(\mathbf{X}_t)$, we can get the sufficient condition for [1, Eqn. 32], that is

$$\epsilon_c|\widetilde{\mathbf{A}}\mathbf{X}_t|_{\mathbf{P}}^2 + (1-\epsilon_c)|(\widetilde{\mathbf{A}}+\widetilde{\mathbf{B}}\mathbf{K})X_t|_P^2 + (1-\epsilon_c)F_e(r,\epsilon_c,D_{c,\max})\mathrm{Tr}\{|\mathbf{B}\mathbf{K}|_{\mathbf{P}}^2\} \leqslant \rho|\mathbf{X}_t|_{\mathbf{P}}^2, \tag{25}$$

where $\mathbb{E}[\mathbf{e}_\tau^{\mathrm{T}}\mathbf{e}_\tau]$ is given by [1, Eqn. 29] and [1, Eqn. 30].

After rearranging (25), [1, Eqn. 32] can be obtained.

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