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An iterative learning control algorithm with a tuning parameter for discrete-time state-space linear systems

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Abstract In this paper, for repetitive discrete-time linear single-input single-output systems described by the state-space model, an iterative learning control algorithm with a tuning parameter is presented in order to utilize more historical control inputs and tracking errors. Necessary and sufficient conditions of the tuning parameter to guarantee the convergence of the tracking error are developed in terms of the spectral radius of an iterative matrix and the roots of a quadratic equation. Compared to some existing algorithms, the proposed control algorithm can improve the convergence speed of the tracking error by choosing a proper tuning parameter. Also, an explicit expression of the optimal parameter is derived to achieve the fastest convergence speed of the tracking error. Finally, the proposed theoretical results are validated by simulation.

Keywords iterative learning control, tuning parameter, convergence analysis, linear systems, convergence speed

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1 Introduction

In practical engineering, there exists a class of systems that perform work tasks at a finite fixed time interval and repeat these tasks, such as high-speed trains [1], robot manipulators [2], power converters [3], and alternating current microgrids [4]. Iterative learning control (ILC) is an intelligent method to improve the tracking performance of repetitive systems. A significant difference between the ILC method and other control methods is the evolutionary direction of the control inputs. More specifically, the control inputs in the next trial are updated by the control inputs and the tracking errors in the previous trials, such that the tracking performance of the system along the iteration direction is gradually improved [5].

The ILC method was first proposed by Uchiyama [6], and was subsequently mathematically formulated by Arimoto et al. [7] and applied to mechanical systems. After continuous development, there have been a large number of theoretical results on ILC. On the one hand, multiple types of ILC algorithms have been reported. In [8], a proportional ILC was presented for nonlinear discrete-time time-varying systems. In [9], a proportional derivative ILC was proposed for uncertain interconnected systems. In [10], an ILC scheme with a time-varying learning gain was designed for linear time-invariant systems. In [11], an ILC mechanism incorporating feedback and difference was developed for batch processes. The common feature of these ILC algorithms in [8–11] is that only the control inputs in the latest iteration are used, and the difference of these algorithms is that the tracking error information is used in different forms. In [12–14], a class of ILC algorithms with a forgetting factor was presented. Specifically, the control inputs in the latest iteration and the initial iteration were used in the algorithm of [12], the control inputs in the latest iteration were utilized in that of [13], and the control inputs in the latest two iterations were used in the algorithm of [14].

On the other hand, ILC has been applied to many types of controlled systems with different forms of description. For discrete-time linear time-invariant systems described by the state-space model, an ILC method with the intermittent data was developed in [15]. For discrete-time linear time-invariant systems described by the transfer function, robust monotonic convergence conditions were established for ILC in [16]. For a class of multiagent systems with a discrete-time unknown nonlinear function, a distributed data-driven ILC algorithm was presented

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in [17]. For continuous-time manipulator systems with unmodeled dynamics, a practical adaptive ILC mechanism was devised in [18]. It can be found that the ILC algorithm has been widely used regardless of the description form of repetitive systems. No matter whether the system model information is known or based on system data, ILC has been a common control approach to handle the tracking problem of repetitive systems.

In iterative learning control, it is crucial to ensure that the tracking error sequence is convergent in the trial-to-trial direction. In the existing literature on ILC, there are a variety of available techniques to analyze the convergence of the tracking error. To name a few, two-dimensional system theory [19], composite energy function approach [20], contraction mapping approach [21], lifting representation approach [22], and frequency-domain analysis approach [23]. Furthermore, there are various types of convergence performance of the tracking error in ILC. For instance, monotonic convergence [24] and mean square convergence [25].

When analyzing the convergence of the tracking error, the convergence speed is an essential metric to evaluate the convergence performance. However, there is little literature related to the convergence speed. In [26], the convergence speed of a model-free adaptive ILC algorithm for nonlinear systems was investigated. In [27], a data-driven mechanism was used to accelerate the convergence of the norm optimal ILC method. In [28], the trade-off between robustness and convergence speed in norm-optimal ILC was discussed. In addition, a hybrid reinforcement Q-learning approach and two hybrid iteration algorithms were developed in [29, 30], respectively. These methods in [29, 30] achieve an improvement in the convergence speed. In addition to the study of the convergence speed for asymptotic convergence, there have also been some other efforts to improve convergence performance. In [31, 32], the predefined time control and the finite-time control were studied, and the faster convergence performance was realized.

In the present paper, a novel ILC algorithm with one tuning parameter is presented to improve the convergence of the tracking error. The systems under consideration are repetitive discrete-time linear systems described by the state-space model. In the proposed algorithm, the control inputs in the current iteration are updated by using the control inputs and the tracking errors in the past two iterations. The iterative sequence of the tracking error along the iterative direction is obtained, and a necessary and sufficient condition to guarantee the convergence of the tracking error is provided in terms of the spectral radius of the iterative matrix. On this basis, an easy-to-check condition is given for choosing a proper tuning parameter to make the tracking error convergent. In addition, the expression of the optimal parameter that maximizes the convergence speed of the tracking error is provided.

The main contributions and novelties of the present paper are outlined as follows.

(1) A novel ILC algorithm with the information (control inputs and tracking errors) in the latest two iterations is proposed by introducing a tuning parameter. Compared to the algorithms in [8, 10], where only the control inputs and tracking errors in the last iteration are used, more information in past iterations is considered in the designed algorithm. The introduction of the tuning parameter can also provide an extra degree of freedom to the control algorithm.

(2) There exist some ILC algorithms in some literature, for example, [13, 33]. Compared to these algorithms, the difference of the algorithm in the current paper lies in two aspects. One is that only the information in the last iteration step is used in [13, 33]. The other is that the parameter with the function of forgetting factors in [13, 33] is confined to the interval $(0, 1)$. Such a restriction is not necessary for the tuning parameter in the current paper.

(3) An explicit expression is provided for the introduced parameter such that the considered repetitive system achieves the fastest convergence speed. However, to the best of our knowledge, such a types of results on the optimal parameters of the iterative learning control do not appear in [10, 14, 34].

The rest of this paper is organized as follows. In Section 2, the problem of this paper is formulated. In Section 3, an ILC algorithm with a tuning parameter is presented and the necessary lemmas are given. In Section 4, the convergence of the tracking error is analyzed and three theorems on the convergence related to the tuning parameter are presented. In Section 5, the optimal tuning parameter is investigated and several results on the optimal parameter for different system parameters are derived. In Section 6, three simulation examples are provided. In Section 7, the conclusion of this paper is drawn.

Notations. For a matrix G, its inverse, determinant, the ith eigenvalue and spectral radius are denoted as G^{-1}, $\det(G)$, $\lambda_i(G)$ and $\rho(G)$, respectively. The notation $\mathbb{I}[a, b]$ represents the set $\{a, a+1, \ldots, b\}$ for any two integers $a \leqslant b$. The notation $\otimes$ is the Kronecker product. In addition, I_N denotes the N-dimensional identity matrix.

2 Problem description

Consider a discrete-time linear single-input single-output (SISO) system described by the following state-space model:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $t \in \mathbb{I}[0, N]$ is the time variable and N is the duration. Additionally, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the system state, scale input and output, respectively. Besides, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ are system matrices. In particular, it is required that $CB \neq 0$, that is, the relative degree of system (1) is 1.

In practical applications, there are some systems that operate repeatedly, and these systems can be set to the same initial conditions in each operation trial [35]. In view of the repetitiveness, an iterative index is added to system (1). Thus, the repetitive discrete-time linear SISO system is described as

$$\begin{cases} x_i(t+1) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{2}$$

where $i \in \{1, 2, \ldots\}$ denotes the iterative index; $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}$ are the system state, scale input and output in the ith iteration, respectively.

The goal of iterative learning control (ILC) for repetitive systems is to make the system output $y_i(t)$ perfectly track a desired output $y_{\mathrm{d}}(t)$ through repetition in the direction of the iterative index. Since the repetitive system considered in this paper is described by the state-space model, the following assumption about $y_{\mathrm{d}}(t)$ is given.

Assumption 1 ([36])**.** For a given desired output $y_{\mathrm{d}}(t)$ over the entire duration interval $t \in \mathbb{I}[0, N]$, there exist the unique desired control input $u_{\mathrm{d}}(t)$ and desired system state $x_{\mathrm{d}}(t)$ such that

$$\begin{cases} x_{\mathrm{d}}(t+1) = Ax_{\mathrm{d}}(t) + Bu_{\mathrm{d}}(t), \\ y_{\mathrm{d}}(t) = Cx_{\mathrm{d}}(t). \end{cases} \tag{3}$$

In addition, similar to [37], we assume that the initial conditions for all iterations satisfy

$$x_i(0) = x_{\mathrm{d}}(0), \ i \in \{1, 2, \ldots\}, \tag{4}$$

where $x_{\mathrm{d}}(0)$ is the initial value of the desired state $x_{\mathrm{d}}(t)$.

Remark 1. In this paper, the desired output $y_{\mathrm{d}}(t)$ is assumed to be realizable. Assumption 1 is an assumption about the realizability of $y_{\mathrm{d}}(t)$, and it is usually a common assumption in ILC [36–39]. The case where the desired output is more general will be considered in our future work.

Remark 2. The same initial condition (4) is necessary for ILC to achieve a perfect tracking and is also a reasonable and common assumption in the field of ILC. It is a realistic requirement for actual systems that repeatedly perform the same task to start each task with the same initial conditions [18, 21, 35, 37, 38]. However, this assumption may not be fulfilled in practical applications due to inevitable errors and uncertainties. The relaxation/removal for the same initial condition is an important issue in the field of ILC [40, 41]. The corresponding analysis methods and techniques are quite different from the current work. Such a problem will become a theme for our future research.

In current paper, based on Assumption 1 and the same initial condition, the focus is on proposing an ILC algorithm that makes the tracking error achieve a faster convergence speed. For this end, define the tracking error as

$$e_i(t) = y_{\mathrm{d}}(t) - y_i(t). \tag{5}$$

For the system (2), the following algorithm has been proposed in [42]

$$u_{i+1}(t) = u_i(t) + Le_i(t+1), \tag{6}$$

where $L \in \mathbb{R}$ is the learning gain.

For the ILC of repetitive systems, the convergence speed of the tracking error is very crucial. In this paper, we try to present a new ILC algorithm such that the tracking error can achieve a faster convergence speed when this new algorithm is employed to the repetitive system (2).

3 An ILC algorithm with a tuning parameter

To achieve the main objective mentioned in the previous section, a novel ILC algorithm with a tuning parameter is presented in this section. By introducing any $\gamma \in \mathbb{R}$, the control input $u_{i+1}(t)$ could be transformed into

$$u_{i+1}(t) = \gamma u_{i+1}(t) + (1-\gamma)u_{i+1}(t).$$

With this, the ILC algorithm (6) can be rewritten as

$$u_{i+1}(t) = \gamma[u_i(t) + Le_i(t+1)] + (1-\gamma)[u_i(t) + Le_i(t+1)]. \tag{7}$$

In order to improve the convergence speed of the tracking error, more historical information could be used in the ILC algorithm. With this consideration, the second term in the right-hand side of (7) is replaced by $(1-\gamma)[u_{i-1}(t)+ Le_{i-1}(t+1)]$. Thus, the following ILC algorithm is proposed

$$u_{i+1}(t) = \gamma[u_i(t) + Le_i(t+1)] + (1-\gamma)[u_{i-1}(t) + Le_{i-1}(t+1)], \tag{8}$$

where $\gamma \in \mathbb{R}$ is a tuning parameter. Obviously, when $\gamma = 1$, the proposed control algorithm (8) can be simplified to the algorithm (6).

Remark 3. In the algorithm (8), the parameter γ needs to be chosen to guarantee the convergence of the tracking error of the system (2) under this algorithm. Unlike [12–14, 33] where the introduced parameter is confined to the interval $(0, 1)$, in this paper such a restriction is not needed for the tuning parameter γ. Therefore, the parameter introduced in this paper is more flexible than those in [12–14, 33]. In the next section, a necessary condition on the tuning parameters that guarantees the convergence of the tracking error is investigated.

The following results are needed in the subsequent section on the convergence analysis of the tracking error under the control algorithm (8).

Lemma 1 ([43])**.** For any $e_0 \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, the sequence e_i generated by the following iterative process:

$$e_{i+1} = Me_i + c, \ i \geqslant 0$$

is convergent if and only if $\rho(M) < 1$. In addition, the convergence speed of the iterative process is

$$V = -\ln \rho(M).$$

Lemma 2 ([44])**.** Given two square matrices $M \in \mathbb{R}^{m\times m}$ and $W \in \mathbb{R}^{n\times n}$, and a matrix P with an appropriate dimension, there holds

$$\det \begin{bmatrix} P & M \\ W & 0 \end{bmatrix} = (-1)^{mn} \det M \det W.$$

Lemma 3 ([44])**.** Given two square matrices P and Q, and two matrices M and W with appropriate dimensions, if Q is invertible, then there holds

$$\det \begin{bmatrix} P & M \\ W & Q \end{bmatrix} = \det Q \det \left(P - MQ^{-1}W\right).$$

4 Convergence analysis

In the previous section, a novel ILC algorithm in (8) has been proposed. In the current section, the convergence of the tracking error is analyzed when this control algorithm (8) is employed to the repetitive discrete-time linear SISO system (2).

For the state variable and the control input, their iteration errors in the ith iteration are defined as

$$\delta x_i(t+1) = x_{\rm d}(t+1) - x_i(t+1), \tag{9}$$

and

$$\delta u_i(t) = u_{\rm d}(t) - u_i(t). \tag{10}$$

In addition, it follows from (2), (3), (5) and (9) that

$$\begin{aligned} e_i(t+1) =& y_{\rm d}(t+1) - y_i(t+1) \\ =& Cx_{\rm d}(t+1) - Cx_i(t+1) \\ =& C(x_{\rm d}(t+1) - x_i(t+1)) \\ =& C\delta x_i(t+1). \end{aligned} \tag{11}$$

With this, it is obtained that

$$e_{i-1}(t+1) = C\delta x_{i-1}(t+1). \tag{12}$$

From (2), (3), (9) and (10), it is obtained

$$\begin{aligned} \delta x_i(t+1) =& Ax_{\rm d}(t) + Bu_{\rm d}(t) - Ax_i(t) - Bu_i(t) \\ =& A(x_{\rm d}(t) - x_i(t)) + B(u_{\rm d}(t) - u_i(t)) \\ =& A\delta x_i(t) + B\delta u_i(t). \end{aligned}$$

By using the state response of discrete-time state-space linear systems in [39], it follows that

$$\delta x_i(t+1) = A^{t+1}\delta x_i(0) + \sum_{j=0}^{t} A^{t-j}B\delta u_i(j). \tag{13}$$

Obviously, there holds $\delta x_i(0) = 0$ from (4). Thus, Eq. (13) becomes

$$\delta x_i(t+1) = \sum_{j=0}^{t} A^{t-j}B\delta u_i(j), \tag{14}$$

from which it is easily derived that

$$\delta x_{i-1}(t+1) = \sum_{j=0}^{t} A^{t-j}B\delta u_{i-1}(j). \tag{15}$$

Besides, it follows from (8) and (10) that

$$\begin{aligned} \delta u_{i+1}(t) =& u_{\rm d}(t) - u_{i+1}(t) \\ =& u_{\rm d}(t) - \gamma[u_i(t) + Le_i(t+1)] - (1-\gamma)[u_{i-1}(t) + Le_{i-1}(t+1)] \\ =& \gamma u_{\rm d}(t) + (1-\gamma)u_{\rm d}(t) - \gamma u_i(t) - \gamma Le_i(t+1) - (1-\gamma)u_{i-1}(t) - (1-\gamma)Le_{i-1}(t+1) \\ =& \gamma(u_{\rm d}(t) - u_i(t)) + (1-\gamma)(u_{\rm d}(t) - u_{i-1}(t)) - \gamma Le_i(t+1) - (1-\gamma)Le_{i-1}(t+1) \\ =& \gamma\delta u_i(t) + (1-\gamma)\delta u_{i-1}(t) - \gamma Le_i(t+1) - (1-\gamma)Le_{i-1}(t+1). \end{aligned} \tag{16}$$

Then, substituting (11) and (12) into (16), gives

$$\delta u_{i+1}(t) = \gamma\delta u_i(t) + (1-\gamma)\delta u_{i-1}(t) - \gamma LC\delta x_i(t+1) - (1-\gamma)LC\delta x_{i-1}(t+1). \tag{17}$$

Next, for any positive integer i, define

$$\xi_i = \left[\delta x_i^{\rm T}(1)\ \delta x_i^{\rm T}(2)\ \cdots\ \delta x_i^{\rm T}(N)\right]^{\rm T}, \tag{18}$$

$$\zeta_i = \left[\delta u_i^{\rm T}(0)\ \delta u_i^{\rm T}(1)\ \cdots\ \delta u_i^{\rm T}(N-1)\right]^{\rm T}, \tag{19}$$

$$\epsilon_i = \left[e_i^{\rm T}(1)\ e_i^{\rm T}(2)\ \cdots\ e_i^{\rm T}(N)\right]^{\rm T}. \tag{20}$$

With the notations in (18) and (20), it is easily obtained from (11) that

$$\epsilon_i = \Theta\xi_i, \tag{21}$$

where

$$\Theta = I_N \otimes C.$$

With the notations in (18) and (19), it follows from (14) that

$$\xi_i = S\zeta_i, \tag{22}$$

where

$$S = \begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$

From (15), it is also obtained that

$$\xi_{i-1} = S\zeta_{i-1}. \tag{23}$$

With the notations in (18) and (19), it can be denoted from (17) that

$$\zeta_{i+1} = \gamma\zeta_i + (1-\gamma)\zeta_{i-1} - W\xi_i - F\xi_{i-1}, \tag{24}$$

where

$$W = \gamma I_N \otimes LC,\ F = (1-\gamma)I_N \otimes LC.$$

Further, substituting (22) and (23) into (24), yields

$$\begin{aligned} \zeta_{i+1} =& \gamma\zeta_i + (1-\gamma)\zeta_{i-1} - WS\zeta_i - FS\zeta_{i-1} \\ =& (\gamma I_N - WS)\zeta_i + ((1-\gamma)I_N - FS)\zeta_{i-1}. \end{aligned} \tag{25}$$

With the previous preliminary, the convergence of the tracking error is provided in Theorem 1.

Theorem 1. For the repetitive discrete-time linear SISO system (2) and the ILC algorithm (8) with a tuning parameter γ, define the following two matrices:

$$R = \begin{bmatrix} \gamma - \gamma LCB & & & \\ -\gamma LCAB & \gamma - \gamma LCB & & \\ \vdots & \vdots & \ddots & \\ -\gamma LCA^{N-1}B & -\gamma LCA^{N-2}B & \cdots & \gamma - \gamma LCB \end{bmatrix},$$

$$H = \begin{bmatrix} (1-\gamma)(1-LCB) & & & \\ -(1-\gamma)LCAB & (1-\gamma)(1-LCB) & & \\ \vdots & \vdots & \ddots & \\ -(1-\gamma)LCA^{N-1}B & -(1-\gamma)LCA^{N-2}B & \cdots & (1-\gamma)(1-LCB) \end{bmatrix}.$$

Then, the tracking error along the iterative index is convergent if and only if

$$\rho(M(\gamma)) < 1, \tag{26}$$

where

$$M(\gamma) = \begin{bmatrix} R & H \\ I_N & 0 \end{bmatrix}. \tag{27}$$

Proof. For any positive integer i, combining (21) and (22), yields

$$\epsilon_{i+1} = \Theta\xi_{i+1} = \Theta S\zeta_{i+1}. \tag{28}$$

By calculation, Eq. (25) can be further written as

$$\begin{aligned}\zeta_{i+1} =&[\gamma I_N - (\gamma I_N \otimes LC)S]\zeta_i + [(1-\gamma)I_N - ((1-\gamma)I_N \otimes LC)S]\zeta_{i-1} \\ =&R\zeta_i + H\zeta_{i-1}.\end{aligned} \tag{29}$$

It follows from (28) and (29) that

$$\begin{aligned}\epsilon_{i+1} =&\Theta S(R\zeta_i + H\zeta_{i-1}) \\ =&\Theta SR\zeta_i + \Theta SH\zeta_{i-1}.\end{aligned}$$

It is easy to derive that $\Theta SR = R\Theta S$ and $\Theta SH = H\Theta S$. With these two relations, the previous relation is equivalently written as

$$\epsilon_{i+1} = R\Theta S\zeta_i + H\Theta S\zeta_{i-1}. \tag{30}$$

From (28), it is obtained that

$$\Theta S\zeta_i = \epsilon_i,\ \Theta S\zeta_{i-1} = \epsilon_{i-1}. \tag{31}$$

Therefore, it follows from (30) and (31) that

$$\epsilon_{i+1} = R\epsilon_i + H\epsilon_{i-1}. \tag{32}$$

Define

$$E_{i+1} = \begin{bmatrix} \epsilon_{i+1} \\ \epsilon_i \end{bmatrix}.$$

With the fact $\epsilon_i = \epsilon_i$, it is easily obtained from (32) that

$$E_{i+1} = \begin{bmatrix} R & H \\ I_N & 0 \end{bmatrix} E_i.$$

From Lemma 1, the tracking error is convergent along the iterative index if and only if (26) holds. The proof of this theorem is thus completed.

In Theorem 1, a convergence condition is presented in terms of the spectral radius of a matrix related to the introduced tuning parameter γ. The matrix $M(\gamma)$ is called the iterative matrix of the system composed of (2) and (8). This iterative matrix is composed of high-dimensional matrices R and H, thus it is difficult to find an appropriate parameter γ satisfying condition (26) by finding directly the eigenvalues of $M(\gamma)$. In the sequel, some further results are provided for choosing the parameter γ.

Theorem 2. For the closed-loop system composed of the repetitive discrete-time linear SISO system (2) and the ILC algorithm (8) with a tuning parameter, if the tracking error converges in the direction of the iterative index, then the parameter γ satisfies

$$1 - \frac{1}{|1-LCB|} < \gamma < 1 + \frac{1}{|1-LCB|}.$$

Proof. According to Lemma 2, from the matrix $M(\gamma)$ given in (27) we have

$$|\det(M(\gamma))| = |(-1)^{N^2}\det(H)\det(I_N)| = |(1-\gamma)(1-LCB)|^N. \tag{33}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2N}$ denote all the eigenvalues of $M(\gamma)$. Then, the determinant of $M(\gamma)$ is expressed as

$$\det(M(\gamma)) = \lambda_1\lambda_2\cdots\lambda_{2N}. \tag{34}$$

From Theorem 1, $|\lambda_j| < 1,\ j \in \mathbb{I}[1, 2N]$ holds if the tracking error is convergent. Based on this fact, it follows from (33) and (34) that

$$|\det(M(\gamma))| = |(1-\gamma)(1-LCB)|^N = |\lambda_1\lambda_2\cdots\lambda_{2N}| < 1,$$

from which the conclusion of this theorem is immediately obtained. The proof of this theorem is thus completed.

In Theorem 2, a necessary condition of the parameter γ to guarantee the convergence of the tracking error is provided. This result further specifies the interval of the tuning parameter γ. In the following, we aim to provide an easy-to-check necessary and sufficient condition for choosing a proper parameter to make the tracking error convergent.

Theorem 3. For the repetitive discrete-time linear SISO system (2) and the ILC algorithm (8) with a tuning parameter γ, the tracking error is convergent if and only if γ satisfies the condition that all roots of the quadratic equation

$$\lambda^2 = \lambda\gamma(1 - LCB) + (1 - \gamma)(1 - LCB) \tag{35}$$

with respect to λ have modulus less than 1.

Proof. Denote λ as an arbitrary eigenvalue of the matrix $M(\gamma)$ in (27). Then, there holds

$$\det(\lambda I_{2N} - M(\gamma)) = \det \begin{bmatrix} \lambda I_N - R & -H \\ -I_N & \lambda I_N \end{bmatrix} = 0. \tag{36}$$

If $\lambda = 0$ is an eigenvalue of $M(\gamma)$, then it follows from (36) that

$$\det \begin{bmatrix} -R & -H \\ -I_N & 0 \end{bmatrix} = 0,$$

which implies $\gamma = 1$. If $\gamma = 1$, the matrix $M(\gamma)$ in (27) becomes

$$M(1) = \begin{bmatrix} R|_{\gamma=1} & 0 \\ I_N & 0 \end{bmatrix}, \tag{37}$$

where

$$R|_{\gamma=1} = \begin{bmatrix} 1 - LCB & & & \\ -LCAB & 1 - LCB & & \\ \vdots & \vdots & \ddots & \\ -LCA^{N-1}B & -LCA^{N-2}B & \cdots & 1 - LCB \end{bmatrix}.$$

From (37) it can be seen that all eigenvalues of $M(1)$ are 0 and $1 - LCB$. Then, the corresponding quadratic equation can be described as $\lambda(\lambda - (1 - LCB)) = 0$, which is the same as (35) with $\gamma = 1$.

Next, the case where none of the eigenvalues of $M(\gamma)$ is 0 is considered. It follows from Lemma 3 and (36) that

$$\det(\lambda I_{2N} - M(\gamma)) = \det(\lambda I_N) \det\left(\lambda I_N - R - \frac{1}{\lambda} H I_N\right). \tag{38}$$

According to (38), since $\lambda \neq 0$, there holds $\det(\lambda I_N) \neq 0$; thus, $\det(\lambda I_{2N} - M(\gamma)) = 0$ if and only if

$$\det\left(\lambda I_N - R - \frac{1}{\lambda} H I_N\right) = 0.$$

By simple calculation, it follows that

$$\det\left(\lambda I_N - R - \frac{1}{\lambda} H I_N\right) = \left(\lambda - (\gamma - \gamma LCB) - \frac{1}{\lambda}(1 - \gamma)(1 - LCB)\right)^N = 0. \tag{39}$$

It can be observed from (39) that the matrix $M(\gamma)$ has two different eigenvalues with multiplicity N. From (39), it is obtained that

$$\lambda - (\gamma - \gamma LCB) - \frac{1}{\lambda}(1 - \gamma)(1 - LCB) = 0. \tag{40}$$

Multiplying λ on both sides of (40), yields

$$\lambda^2 - \lambda\gamma(1 - LCB) - (1 - \gamma)(1 - LCB) = 0,$$

which is (35). Thus, the condition in the form of the spectral radius in Theorem 1 is transformed to the condition that all the roots of (35) satisfy $|\lambda| < 1$. The proof is thus completed.

In Theorem 3, a necessary and sufficient condition to ensure the convergence of the tracking error is provided in terms of the roots of a quadratic equation. Compared to the condition in Theorem 1, it is much easier to find the parameter γ to satisfy this condition.

5 The choice of the optimal tuning parameter

In the previous section, several conditions for the convergence of the tracking error along the iterative index have been given. A further problem is to seek the optimal parameter to ensure that the tracking error achieves the fastest convergence speed under the proposed ILC algorithm. In the present section, such a problem is under investigation.

Obviously, the two roots of (35) are related to the tuning parameter γ. Thus, denote these two roots as $\lambda_1(\gamma)$ and $\lambda_2(\gamma)$. By the definition of the spectral radius and the result of Theorem 3, there holds

$$\rho(M(\gamma)) = \max\{|\lambda_i(\gamma)|, i = 1, 2\}.$$

The discriminant of the quadratic equation (35) with respect to λ is

$$\Delta(\gamma) = \gamma^2(1 - LCB)^2 + 4(1 - \gamma)(1 - LCB). \tag{41}$$

When $\Delta(\gamma) \geqslant 0$, the two roots of (35) are

$$\begin{cases} \lambda_1(\gamma) = \dfrac{1}{2}\gamma(1 - LCB) + \dfrac{1}{2}\sqrt{\Delta(\gamma)}, \\ \lambda_2(\gamma) = \dfrac{1}{2}\gamma(1 - LCB) - \dfrac{1}{2}\sqrt{\Delta(\gamma)}. \end{cases} \tag{42}$$

When $\Delta(\gamma) < 0$, the two roots of (35) are

$$\begin{cases} \lambda_1(\gamma) = \dfrac{1}{2}\gamma(1 - LCB) + \dfrac{1}{2}\mathrm{i}\sqrt{-\Delta(\gamma)}, \\ \lambda_2(\gamma) = \dfrac{1}{2}\gamma(1 - LCB) - \dfrac{1}{2}\mathrm{i}\sqrt{-\Delta(\gamma)}. \end{cases} \tag{43}$$

Next, we will provide expressions for the optimal parameters from two cases $1 - LCB > 0$ and $1 - LCB < 0$.

5.1 The case of $1 - LCB > 0$

In this subsection, the case of $1 - LCB > 0$ is considered.

For $\Delta(\gamma) \geqslant 0$, it is obvious from $1 - LCB > 0$ and (42) that

$$\begin{cases} |\lambda_1(\gamma)| > |\lambda_2(\gamma)|, \text{ for } \gamma \geqslant 0, \\ |\lambda_2(\gamma)| > |\lambda_1(\gamma)|, \text{ for } \gamma < 0. \end{cases}$$

Therefore, when $1 - LCB > 0$ and $\Delta(\gamma) \geqslant 0$, there holds

$$\rho(M(\gamma)) = \begin{cases} \dfrac{1}{2}\gamma(1 - LCB) + \dfrac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma \geqslant 0, \\ -\dfrac{1}{2}\gamma(1 - LCB) + \dfrac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma < 0. \end{cases} \tag{44}$$

For $\Delta(\gamma) < 0$, it is easily calculated from (43) that

$$|\lambda_1(\gamma)| = |\lambda_2(\gamma)| = \sqrt{(1 - \gamma)(LCB - 1)}.$$

Therefore, when $1 - LCB > 0$ and $\Delta(\gamma) < 0$, there holds

$$\rho(M(\gamma)) = \sqrt{(1 - \gamma)(LCB - 1)}. \tag{45}$$

Further, according to (41), the discriminant $\Delta(\gamma)$ can be written as

$$\begin{aligned} \Delta(\gamma) =& \gamma^2(1 - LCB)^2 + 4(1 - LCB) - 4\gamma(1 - LCB) \\ =& [\gamma(1 - LCB) - 2]^2 + 4[(1 - LCB) - 1], \end{aligned} \tag{46}$$

from which it is known that $\Delta(\gamma)$ is always nonnegative for all real γ when $1 - LCB \geqslant 1$ or equivalently $LCB \leqslant 0$. By combining this fact with (44), the following lemma can be immediately obtained.

Lemma 4. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with a tuning parameter γ, the matrix $M(\gamma)$ is given in (27). If $LCB \leqslant 0$, then

$$\rho(M(\gamma)) = \begin{cases} -\frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma < 0, \\ \frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma \geqslant 0, \end{cases}$$

where $\Delta(\gamma)$ is given in (41).

Next, the case where $0 < 1 - LCB < 1$ is considered. In this case, $0 < LCB < 1$ and the discriminant (46) can be written as

$$\begin{aligned} \Delta(\gamma) =& [\gamma(1-LCB)-2]^2 - 4[1-(1-LCB)] \\ =& (1-LCB)^2(\gamma-\gamma_1)(\gamma-\gamma_2) \end{aligned} \tag{47}$$

with

$$\gamma_1 = \frac{2-2\sqrt{LCB}}{1-LCB}, \tag{48}$$

$$\gamma_2 = \frac{2+2\sqrt{LCB}}{1-LCB}. \tag{49}$$

Obviously, there holds $\gamma_1 < \gamma_2$ since $0 < LCB < 1$. It is easily obtained from (47) that when $\gamma \geqslant \gamma_2$ or $\gamma \leqslant \gamma_1$, there holds $\Delta(\gamma) \geqslant 0$; when $\gamma_1 < \gamma < \gamma_2$, there holds $\Delta(\gamma) < 0$. By these two facts and the preceding expressions given in (44) and (45), the following lemma can be obtained on $\rho(M(\gamma))$ when $0 < LCB < 1$.

Lemma 5. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with a tuning parameter γ, the matrix $M(\gamma)$ is given in (27). If $0 < LCB < 1$, then

$$\rho(M(\gamma)) = \begin{cases} -\frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma < 0, \\ \frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } 0 \leqslant \gamma \leqslant \gamma_1, \\ \sqrt{(1-\gamma)(LCB-1)}, & \text{for } \gamma_1 < \gamma < \gamma_2, \\ \frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma \geqslant \gamma_2, \end{cases}$$

where $\Delta(\gamma)$ is given in (41), and γ_1 and γ_2 are given in (48) and (49), respectively.

As shown in Lemmas 4 and 5, the explicit expressions for the spectral radius of matrix $M(\gamma)$ in both cases $LCB \leqslant 0$ and $0 < LCB < 1$ are given. Based on these results, the optimal tuning parameter will be explored. For the sake of further analysis, denote

$$\begin{cases} g(\gamma) = -\frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, \\ h(\gamma) = \frac{1}{2}\gamma(1-LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, \end{cases} \tag{50}$$

where $\Delta(\gamma)$ is given in (41).

Firstly, Theorem 4 is given regarding the choice of the optimal parameter γ for the case of $LCB \leqslant 0$.

Theorem 4. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with a tuning parameter γ, the matrix $M(\gamma)$ is given in (27). When $LCB \leqslant 0$, there holds

$$\min \rho(M(\gamma)) = \sqrt{1-LCB}.$$

Proof. In the case of $LCB \leqslant 0$, the expression for the spectral radius $\rho(M(\gamma))$ is provided in Lemma 4. Additionally, the functions $g(\gamma)$ and $h(\gamma)$ defined in (50) are used later.

When $\gamma < 0$, there holds $\rho(M(\gamma)) = g(\gamma)$. Clearly, the function $g(\gamma)$ consists of $-\frac{1}{2}\gamma(1-LCB)$ and $\frac{1}{2}\sqrt{\Delta(\gamma)}$. The function $-\frac{1}{2}\gamma(1-LCB)$ is monotonically decreasing with respect to γ since $1-LCB \geqslant 1 > 0$. Moreover, it follows from (47) that the function $\frac{1}{2}\sqrt{\Delta(\gamma)}$ is monotonically decreasing with respect to γ when $\gamma < \frac{2}{1-LCB}$. Also, it is obvious that $\frac{2}{1-LCB} > 0$ due to $1 - LCB \geqslant 1$. Therefore, the function $\frac{1}{2}\sqrt{\Delta(\gamma)}$ is monotonically decreasing with respect to γ when $\gamma < 0$. Thus, when $\gamma < 0$, both $-\frac{1}{2}\gamma(1 - LCB)$ and $\frac{1}{2}\sqrt{\Delta(\gamma)}$ are monotonically decreasing with respect to γ, and thus the function $g(\gamma)$ is monotonically decreasing with respect to γ.

When $\gamma \geqslant 0$, there holds $\rho(M(\gamma)) = h(\gamma)$. Similarly, when $\gamma \geqslant \frac{2}{1-LCB}$, both $\frac{1}{2}\gamma(1 - LCB)$ and $\frac{1}{2}\sqrt{\Delta(\gamma)}$ are monotonically increasing with respect to γ due to $1 - LCB \geqslant 1 > 0$ and (47). Thus, when $\gamma \geqslant \frac{2}{1-LCB}$, the function $h(\gamma)$ is monotonically increasing with respect to γ.

Next, the monotonicity of $h(\gamma)$ is analyzed for the case of $0 \leqslant \gamma < \frac{2}{1-LCB}$. To do this, taking the derivative of the function $h(\gamma)$ with respect to γ yields

$$\begin{aligned}\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} =& \frac{1 - LCB}{2} + \frac{2\gamma(1 - LCB)^2 - 4(1 - LCB)}{4\sqrt{\Delta(\gamma)}} \\ =& \frac{(1 - LCB)\sqrt{\Delta(\gamma)} + \gamma(1 - LCB)^2 - 2(1 - LCB)}{2\sqrt{\Delta(\gamma)}}.\end{aligned} \tag{51}$$

Since $1 - LCB \geqslant 1$, it follows from (46) that

$$\begin{aligned}\Delta(\gamma) =& [\gamma(1 - LCB) - 2]^2 + 4[(1 - LCB) - 1] \\ \geqslant& [\gamma(1 - LCB) - 2]^2.\end{aligned} \tag{52}$$

Besides, there holds $\gamma(1 - LCB) < 2$ due to $\gamma < \frac{2}{1-LCB}$ and $1 - LCB \geqslant 1$. Taking the square root of both sides of (52), gives

$$\sqrt{\Delta(\gamma)} \geqslant 2 - \gamma(1 - LCB). \tag{53}$$

Multiplying both sides of (53) by $1 - LCB$, gives

$$(1 - LCB)\sqrt{\Delta(\gamma)} \geqslant 2(1 - LCB) - \gamma(1 - LCB)^2,$$

which implies that

$$(1 - LCB)\sqrt{\Delta(\gamma)} - 2(1 - LCB) + \gamma(1 - LCB)^2 \geqslant 0. \tag{54}$$

Combining (51) with (54) yields $\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} \geqslant 0$, which means that $h(\gamma)$ monotonically increases with respect to γ when $0 \leqslant \gamma < \frac{2}{1-LCB}$. In addition, it can be readily checked that $h(\gamma)$ is continuous with respect to γ. Therefore, when $\gamma \geqslant 0$, the function $h(\gamma)$ is monotonically increasing with respect to γ.

By now, it has been obtained that $\rho(M(\gamma))$ monotonically decreases when $\gamma < 0$ and monotonically increases when $\gamma \geqslant 0$. Also, the function $\rho(M(\gamma))$ in Lemma 4 is continuous by calculating the value of $\rho(M(\gamma))$ at $\gamma = 0$. Therefore, the minimal value of $\rho(M(\gamma))$ is taken at $\gamma = 0$, and is given as

$$\begin{aligned}\min \rho(M(\gamma)) =& h(\gamma)|_{\gamma=0} \\ =& 0 + \frac{1}{2}\sqrt{4(1 - LCB)} \\ =& \sqrt{1 - LCB}.\end{aligned}$$

The proof of this theorem is thus completed.

The conclusion in Theorem 4 implies that the spectral radius $\rho(M(\gamma))$ for any $\gamma \in \mathbb{R}$ is always greater than or equal to 1 when $LCB \leqslant 0$. This fact implies that there does not exist any real number γ to guarantee the convergence of the tracking error of the system (2) under the control algorithm (8) for the case of $LCB \leqslant 0$. Subsequently, the result on the choice of the optimal parameter in the case of $0 < LCB < 1$ is given in Theorem 5.

Theorem 5. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with a tuning parameter γ, the matrix $M(\gamma)$ is given in (27). When $0 < LCB < 1$, the minimal value of $\rho(M(\gamma))$ is taken as

$$\min \rho(M(\gamma)) = 1 - \sqrt{LCB}$$

at $\gamma = \gamma_1$, where γ_1 is defined in (48).

Proof. In the case of $0 < LCB < 1$, the expression for the spectral radius $\rho(M(\gamma))$ is given in Lemma 5. Analogous to the proof of Theorem 4, the functions $g(\gamma)$ and $h(\gamma)$ defined in (50) are also used. Additionally, the numbers γ_1 and γ_2 defined in (48) and (49) are also used.

When $\gamma < 0$, there holds $\rho(M(\gamma)) = g(\gamma)$. By the monotonicity analysis of $g(\gamma)$ with respect to γ in Theorem 4, when $\gamma < 0$, the function $g(\gamma)$ monotonically decreases with respect to γ.

When $0 \leqslant \gamma \leqslant \gamma_1$, there holds $\rho(M(\gamma)) = h(\gamma)$. It follows from (46) and $0 < 1 - LCB < 1$ that

$$\Delta(\gamma) < [\gamma(1 - LCB) - 2]^2. \tag{55}$$

It is obvious from (48) that $\gamma_1 < \frac{2}{1-LCB}$. Then, there holds $\gamma < \frac{2}{1-LCB}$ when $0 \leqslant \gamma \leqslant \gamma_1$, and thus $\gamma(1-LCB) < 2$ holds due to $0 < 1 - LCB < 1$. Taking the square root of both sides of (55), gives

$$\sqrt{\Delta(\gamma)} < 2 - \gamma(1 - LCB).$$

Further, there holds

$$(1 - LCB)\sqrt{\Delta(\gamma)} - 2(1 - LCB) + \gamma(1 - LCB)^2 < 0. \tag{56}$$

Combining (56) with (51) yields $\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} < 0$, which means that $h(\gamma)$ monotonically decreases with respect to γ when $0 \leqslant \gamma \leqslant \gamma_1$.

When $\gamma_1 < \gamma < \gamma_2$, there holds

$$\rho(M(\gamma)) = \sqrt{(1 - \gamma)(LCB - 1)}.$$

Clearly, this function is monotonically increasing with respect to γ since $LCB - 1 < 0$.

When $\gamma \geqslant \gamma_2$, there holds $\rho(M(\gamma)) = h(\gamma)$. It is obvious from (49) that $\gamma_2 > \frac{2}{1-LCB}$ since $0 < 1 - LCB < 1$. In addition, by the proof of Theorem 4, it is known that $h(\gamma)$ monotonically increases with respect to γ when $\gamma \geqslant \frac{2}{1-LCB}$. Therefore, the function $h(\gamma)$ monotonically increases with respect to γ when $\gamma \geqslant \gamma_2$.

By now, it has been known that $\rho(M(\gamma))$ monotonically decreases when $\gamma < 0$ and $0 \leqslant \gamma \leqslant \gamma_1$, and monotonically increases when $\gamma_1 < \gamma < \gamma_2$ and $\gamma \geqslant \gamma_2$. Also, the function $\rho(M(\gamma))$ in Lemma 5 is continuous by calculating the values of $\rho(M(\gamma))$ at $\gamma = 0$, $\gamma = \gamma_1$ and $\gamma = \gamma_2$. Thus, $\rho(M(\gamma))$ monotonically decreases when $\gamma \leqslant \gamma_1$, and monotonically increases when $\gamma > \gamma_1$. Accordingly, the minimal value of $\rho(M(\gamma))$ is taken at $\gamma = \gamma_1$, and is given as

$$\begin{aligned}\min \rho(M(\gamma)) =& h(\gamma)|_{\gamma=\gamma_1} \\ =& \frac{1}{2}\gamma_1(1 - LCB) + \frac{1}{2}\sqrt{\Delta(\gamma_1)} \\ =& \frac{1}{2}\frac{2 - 2\sqrt{LCB}}{1 - LCB}(1 - LCB) + 0 \\ =& 1 - \sqrt{LCB},\end{aligned}$$

where $\sqrt{\Delta(\gamma_1)} = 0$ holds since (47). The proof is thus completed.

By the conclusion of Theorem 5, the optimal parameter in the case of $0 < LCB < 1$ is obtained such that the tracking error of the system (2) under the control algorithm (8) has the fastest convergence speed.

5.2 The case of $1 - LCB < 0$

In this subsection, the case of $1 - LCB < 0$ is considered.

For $\Delta(\gamma) \geqslant 0$, it is obvious from $1 - LCB < 0$ and (42) that

$$\begin{cases} |\lambda_2(\gamma)| > |\lambda_1(\gamma)|, \text{ for } \gamma \geqslant 0, \\ |\lambda_1(\gamma)| > |\lambda_2(\gamma)|, \text{ for } \gamma < 0. \end{cases}$$

Therefore, when $1 - LCB < 0$ and $\Delta(\gamma) \geqslant 0$, there holds

$$\rho(M(\gamma)) = \begin{cases} -\frac{1}{2}\gamma(1 - LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma \geqslant 0, \\ \frac{1}{2}\gamma(1 - LCB) + \frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma < 0. \end{cases}$$

When $1-LCB<0$ and $\Delta(\gamma)<0$, it follows from (43) that

$$\rho(M(\gamma))=\sqrt{(1-\gamma)(LCB-1)}.$$

In addition, for the numbers γ_1 and γ_2 defined in (48) and (49), respectively, there holds $\gamma_2<0<\gamma_1$ since $1-LCB<0$. Further, it follows from (47) that when $\gamma\geqslant\gamma_1$ or $\gamma\leqslant\gamma_2$, there holds $\Delta(\gamma)\geqslant 0$; when $\gamma_2<\gamma<\gamma_1$, there holds $\Delta(\gamma)<0$. Based on these analysis, the following result can be obtained on $\rho(M(\gamma))$ when $LCB>1$.

Lemma 6. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with a tuning parameter γ, the matrix $M(\gamma)$ is given in (27). If $LCB>1$, then

$$\rho(M(\gamma))=\begin{cases}\frac{1}{2}\gamma(1-LCB)+\frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma\leqslant\gamma_2,\\ \sqrt{(1-\gamma)(LCB-1)}, & \text{for } \gamma_2<\gamma<\gamma_1,\\ -\frac{1}{2}\gamma(1-LCB)+\frac{1}{2}\sqrt{\Delta(\gamma)}, & \text{for } \gamma\geqslant\gamma_1,\end{cases}$$

where $\Delta(\gamma)$ is given in (41), and γ_1 and γ_2 are given in (48) and (49), respectively.

With the result of Lemma 6 as the basis, the choice of the optimal parameter is provided in the following theorem when $LCB>1$.

Theorem 6. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with a tuning parameter γ, the matrix $M(\gamma)$ is given in (27). When $LCB>1$, the minimal value of $\rho(M(\gamma))$ is taken as

$$\min\rho(M(\gamma))=\sqrt{LCB}-1$$

at $\gamma=\gamma_1$, where γ_1 is defined in (48).

Proof. When $LCB>1$, the expression for the spectral radius $\rho(M(\gamma))$ is given in Lemma 6. Analogous to the proofs of Theorems 4 and 5, the functions $g(\gamma)$ and $h(\gamma)$ defined in (50) are also used. Additionally, the numbers γ_1 and γ_2 defined in (48) and (49) are also used.

When $\gamma\leqslant\gamma_2$, there holds $\rho(M(\gamma))=h(\gamma)$. Similar to the monotonicity analysis of $h(\gamma)$ with respect to γ when $\gamma\geqslant\frac{2}{1-LCB}$ in Theorem 4, it can be derived that $h(\gamma)$ monotonically decreases with respect to γ when $\gamma<\frac{2}{1-LCB}$. Additionally, $\gamma_2<\frac{2}{1-LCB}$ holds since $1-LCB<0$. Therefore, the function $h(\gamma)$ monotonically decreases with respect to γ when $\gamma\leqslant\gamma_2$.

When $\gamma_2<\gamma<\gamma_1$, there holds

$$\rho(M(\gamma))=\sqrt{(1-\gamma)(LCB-1)}.$$

Clearly, this function is monotonically decreasing with respect to γ since $LCB-1>0$.

When $\gamma\geqslant\gamma_1$, there holds $\rho(M(\gamma))=g(\gamma)$. Similar to the monotonicity analysis of $g(\gamma)$ with respect to γ when $\gamma<\frac{2}{1-LCB}$ in Theorem 4, it is known that $g(\gamma)$ monotonically increases with respect to γ when $\gamma\geqslant\frac{2}{1-LCB}$. Additionally, $\gamma_1>\frac{2}{1-LCB}$ holds since $1-LCB<0$. Therefore, the function $g(\gamma)$ is monotonically increasing with respect to γ when $\gamma\geqslant\gamma_1$.

By now, it has been known that $\rho(M(\gamma))$ monotonically decreases when $\gamma\leqslant\gamma_2$ and $\gamma_2<\gamma<\gamma_1$, and monotonically increases when $\gamma\geqslant\gamma_1$. Also, the function $\rho(M(\gamma))$ in Lemma 6 is continuous by calculating the values of $\rho(M(\gamma))$ at $\gamma=\gamma_2$ and $\gamma=\gamma_1$. Thus, $\rho(M(\gamma))$ monotonically decreases when $\gamma<\gamma_1$, and monotonically increases when $\gamma\geqslant\gamma_1$. Accordingly, the minimal value of $\rho(M(\gamma))$ is taken at $\gamma=\gamma_1$, and is given as

$$\begin{aligned}\min\rho(M(\gamma))=&g(\gamma)|_{\gamma=\gamma_1}\\ =&-\frac{1}{2}\gamma_1(1-LCB)+\frac{1}{2}\sqrt{\Delta(\gamma_1)}\\ =&-\frac{1}{2}\frac{2-2\sqrt{LCB}}{1-LCB}(1-LCB)+0\\ =&\sqrt{LCB}-1,\end{aligned}$$

where $\sqrt{\Delta(\gamma_1)}=0$ holds since (47). The proof is thus completed.

From the conclusion of Theorem 6, the minimal value of $\rho(M(\gamma))$ is greater than or equal to 1 when $LCB\geqslant 4$. In this case, there does not exist a tuning parameter γ to guarantee the convergence of the tracking error of the system (2) under the control algorithm (8).

5.3 Further discussion

In the previous two subsections, the optimal parameters have been provided in Theorems 4–6 for cases $LCB \leqslant 0$, $0 < LCB < 1$ and $LCB > 1$, respectively. By summarizing these results, the following corollary can be obtained.

Corollary 1. For the repetitive discrete-time linear SISO system (2) under the ILC algorithm (8) with γ, when $0 < LCB < 4$, there exists a tuning parameter γ to guarantee the convergence of the tracking error of the corresponding closed-loop system. Furthermore, the optimal tuning parameter is $\gamma = \gamma_1$ given in (48).

It is easy to prove that the original ILC algorithm (6) can make the tracking error converge for the repetitive system (2) only if $0 < LCB < 2$ holds. Such a result has been given in [45]. Obviously, the system consisting of (2) and (6) does not work if LCB is out of this interval. However, by the above corollary, for the proposed ILC algorithm (8), a proper tuning parameter can be selected such that the system consisting of (2) and (8) can work when $0 < LCB < 4$. These facts imply that the proposed algorithm can be applied to more types of systems.

In this paper, some convergence conditions of the tracking error under the proposed ILC algorithm with a tuning parameter have been given, and explicit expressions of the optimal parameter to make the tracking error converge fastest are also derived. In the sequel, some remarks are provided to clarify the difference among the results in the current paper and some existing results.

Remark 4. In [10,14,34], the optimal solution of the introduced parameters is not considered, and the parameters need to be manually adjusted. Differently, the optimal solution of the introduced parameter is obtained in an explicit form by rigorous mathematical derivation in this paper.

Remark 5. In our previous studies [46,47], the idea of parameter introduction has been utilized to solve Lyapunov matrix equations and algebraic Riccati equations. In the current work, a tuning parameter is introduced into an existing ILC algorithm, and thus a novel ILC algorithm is presented. Compared to the results in [46, 47], the differences of the results in this paper are as follows.

(1) The studied problems are different. The aim of [46] is to present an iterative algorithm for solving the discrete periodic Lyapunov matrix equation. The aim of this paper is to present an ILC algorithm to accelerate the convergence of the tracking error.

(2) As the considered problems are different, the utilized information to construct the iterative forms is different. In [46], the estimation of the known matrices in the current and the last steps is used to update the estimation of the unknown matrices in the current step by taking advantage of the coupling structure. In this paper, both control inputs and tracking errors in past iterations are used to update the control inputs.

In the current work, the control inputs in the $(i + 1)$th iteration is updated by using the information in the ith iteration and the $(i - 1)$th iteration. By following such an idea of parameter introduction, more information can be utilized to construct ILC algorithms. Specifically, more ILC algorithms are derived based on the proposed algorithm (8). Similar to the derivation in Section 3, the ILC algorithm (8) can be written as

$$\begin{aligned} u_{i+1}(t) =& \gamma \left[u_i(t) + Le_i(t+1)\right] + (1-\gamma)\left[u_{i-1}(t) + Le_{i-1}(t+1)\right] \\ =& \gamma[u_i(t) + Le_i(t+1)] + (1-\gamma)\left[\gamma\left(u_{i-1}(t) + Le_{i-1}(t+1)\right) + (1-\gamma)\left(u_{i-1}(t) + Le_{i-1}(t+1)\right)\right]. \end{aligned} \tag{57}$$

To utilize more historical information, the term $(1-\gamma)\left(u_{i-1}(t) + Le_{i-1}(t+1)\right)$ in (57) is replaced by $(1-\gamma)(u_{i-2}(t) + Le_{i-2}(t+1))$. Thus, the following ILC algorithm is obtained

$$\begin{aligned} u_{i+1}(t) =& \gamma[u_i(t) + Le_i(t+1)] + (1-\gamma)\left[\gamma\left(u_{i-1}(t) + Le_{i-1}(t+1)\right) + (1-\gamma)\left(u_{i-2}(t) + Le_{i-2}(t+1)\right)\right] \\ =& \gamma[u_i(t) + Le_i(t+1)] + \gamma(1-\gamma)[u_{i-1}(t) + Le_{i-1}(t+1)] + (1-\gamma)^2[u_{i-2}(t) + Le_{i-2}(t+1)]. \end{aligned}$$

Along the similar line, the following ILC algorithm in a general form can be

$$u_{i+1}(t) = \gamma \sum_{l=0}^{m-1} (1-\gamma)^l [u_{i-l}(t) + Le_{i-l}(t+1)] + (1-\gamma)^m [u_{i-m}(t) + Le_{i-m}(t+1)], \tag{58}$$

where $m \geqslant 1$ is the number of past iterations used in the proposed algorithm. The convergence analysis of the tracking error under the algorithm (58) and the selection of the optimal parameter deserve further investigation.

Remark 6. In the current work, the considered system is a repetitive discrete-time linear system. In fact, the proposed ILC algorithm can also be applied to repetitive nonlinear or stochastic systems. Of course, the analysis of the convergence and the optimal parameter becomes more complicated.

Table 1 Comparison of the convergence performance for different tuning parameters.

Tuning parameter	Spectral radius	Convergence speed
$\gamma = 0.9$	0.5422	0.6121
$\gamma = 1$	0.5	0.6931
$\gamma = 1.05$	0.4720	0.7507
$\gamma = 1.1716$	0.2929	1.2279
$\gamma = 1.4$	0.4472	0.8047

Remark 7. To compute the new control input $u_{i+1}(t)$, the original algorithm (6) requires one multiplication operation and one addition operation, and the proposed algorithm (8) requires four multiplication operations and four addition operations. With the current powerful computational capabilities, the added operations of the algorithm (8) compared to the algorithm (6) do not take much computational time. Furthermore, the closed-loop system has a faster convergence speed under the algorithm (8), and thus the same tracking precision can be achieved with fewer iterations. Therefore, the presented ILC algorithm could exhibit better performance.

6 Numerical simulation

To convincingly illustrate the distinctive advantages of our presented control algorithm (8), three simulation examples are provided.

Example 1. Consider the following repetitive system that has been investigated in [39]

$$\begin{cases} x_i(t+1) = \begin{bmatrix} 0.50 & 0 & 1.00 \\ 0.15 & 0.30 & 0 \\ -0.75 & 0.25 & -0.25 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1.00 \end{bmatrix} u_i(t), \\ y_i(t) = \begin{bmatrix} 0 & 0 & 1.00 \end{bmatrix} x_i(t). \end{cases} \tag{59}$$

In [39], the control algorithm applied to (59) is

$$u_{i+1}(t) = u_i(t) + 0.5e_i(t+1). \tag{60}$$

The desired output is set to be

$$y_{\mathrm{d}}(t) = \sin(2\pi t/50) + \sin(2\pi t/5) + \sin(50\pi t).$$

Based on the control algorithm (60), the proposed novel algorithm with the parameter γ is

$$u_{i+1}(t) = \gamma[u_i(t) + 0.5e_i(t+1)] + (1-\gamma)[u_{i-1}(t) + 0.5e_{i-1}(t+1)]. \tag{61}$$

In the simulation, set the duration of each iteration to be $N = 50$ and set the number of iterations to be 180. Moreover, the initial values of the variables are set to be $x_i(0) = [0,0,0]^{\mathrm{T}}, i \in \mathbb{I}[1,180]$ and $u_1(t) = 0, t \in \mathbb{I}[0,50]$. For this example, $LCB = 0.5$. According to Theorem 5, the optimal tuning parameter γ is 1.1716. Define the root mean square of the tracking error along the iterative axis as

$$\mathrm{ER} = \sqrt{\frac{1}{N}\sum_{t=1}^{N} \|e_i(t)\|^2}.$$

When the control algorithm (61) with different tuning parameters is applied to the system (59), the curves of $\log_{10}\mathrm{ER}$ are given in Figure 1 and the comparison results of the convergence performance are summarized in Table 1. It is observed from Figure 1 and Table 1 that the tracking error under the proposed control algorithm (61) with proper parameters (e.g., $\gamma = 1.05$ and $\gamma = 1.4$) has a faster convergence speed compared to that under the original algorithm (60) (i.e., $\gamma = 1$) and the tracking error achieves the fastest convergence speed when the tuning parameter is taken as $\gamma = 1.1716$. Therefore, the presented control algorithm is effective in improving the convergence speed of the tracking error.

In addition, in order to illustrate the advantage of the proposed ILC algorithm, four existing ILC algorithms are also applied to this example system. These four algorithms are the proportional-type ILC algorithm in [8] (Algorithm 1)

$$u_{i+1}(t) = u_i(t) + Le_i(t+1), \tag{62}$$

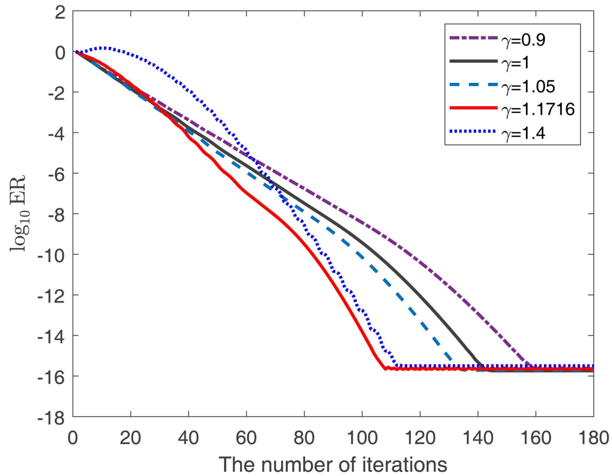

Figure 1 (Color online) Curves of the tracking error under the control algorithm (61) with different tuning parameters.

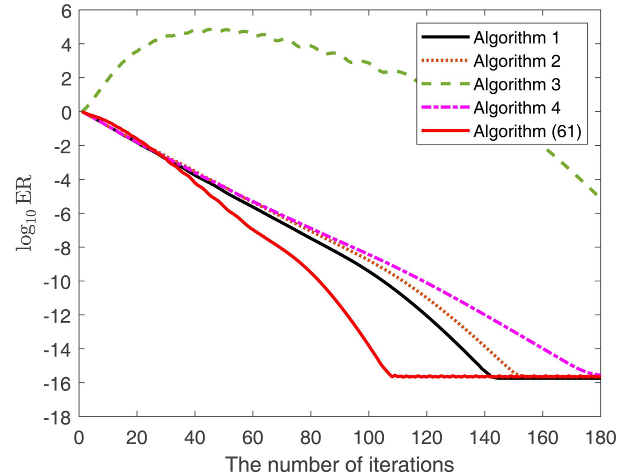

Figure 2 (Color online) Curves of the tracking error under different ILC algorithms.

the ILC algorithm with a time-varying learning gain in [10] (Algorithm 2)

$$u_{i+1}(t) = u_i(t) + \mathrm{e}^{-\alpha t} L e_i(t+1), \tag{63}$$

the derivative-type ILC algorithm in [48] (Algorithm 3)

$$u_{i+1}(t) = u_i(t) + L(e_i(t+1) - e_i(t)), \tag{64}$$

and the ILC algorithm with a fractional power update rule in [34] (Algorithm 4)

$$u_{i+1}(t) = u_i(t) + L|e_i(t+1)|^{\alpha}\mathrm{sgn}(e_i(t+1)). \tag{65}$$

For the sake of fairness, the parameters in (62)–(65) are adjusted such that the tracking errors in the corresponding closed-loop systems achieve the fastest convergence speed. Specifically, in (62), $L = 0.5$; in (63), $L = 0.5$ and $\alpha = 0.002$; in (64), $L = 0.5$; in (65), $L = 0.5$ and $\alpha = 1.01$; and in (61), $L = 0.5$ and $\gamma = 1.1716$. The curves of root mean square of the tracking error $\log_{10}\mathrm{ER}$ vs. the number of iterations are shown in Figure 2. It is clearly seen that the tracking error under the algorithm (61) in this paper possesses the fastest convergence speed.

Example 2. Consider the following repetitive linear SISO system

$$\begin{cases} x_i(t+1) = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ -0.04 & -0.2 & -1.8 \end{bmatrix} x_i(t) + \begin{bmatrix} -2 \\ 0 \\ 0.4 \end{bmatrix} u_i(t), \\ y_i(t) = \begin{bmatrix} 0.04 & 0.2 & 0 \end{bmatrix} x_i(t). \end{cases} \tag{66}$$

The desired output is set to be

$$y_{\mathrm{d}}(t) = 1 - \mathrm{e}^{-0.2(t-1)}.$$

For this example, $CB = -0.08$. When the original ILC algorithm (6) with $L = -30$

$$u_{i+1}(t) = u_i(t) - 30e_i(t+1) \tag{67}$$

is applied to the system (66), there holds $LCB = 2.4 > 2$. Therefore, the tracking error of the closed-loop system consisting of (66) and (67) is not convergent. Now, the following control algorithm with a tuning parameter γ

$$u_{i+1}(t) = \gamma[u_i(t) - 30e_i(t+1)] + (1-\gamma)[u_{i-1}(t) - 30e_{i-1}(t+1)] \tag{68}$$

is applied to the system (66). According to Theorem 2, if the tracking error is convergent, then the parameter satisfies $\gamma \in (0.2857, 1.7142)$. Further, by Theorem 6, when this tuning parameter is chosen to be $\gamma = 0.7846$, the tracking error achieves the fastest convergence speed.

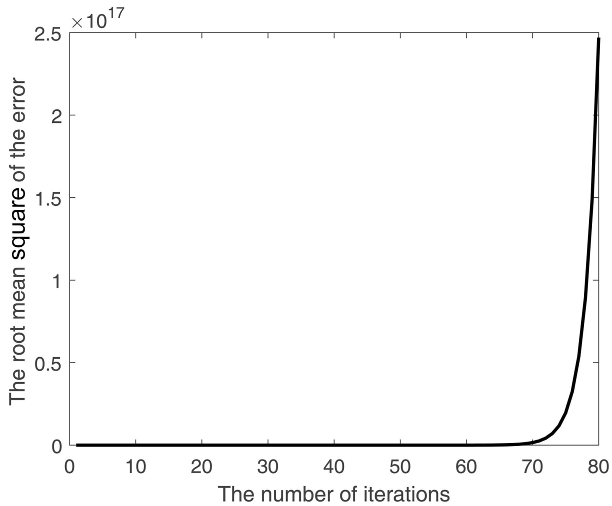

Figure 3 Curve of the root mean square of the tracking error of the system (66) under the algorithm (67).

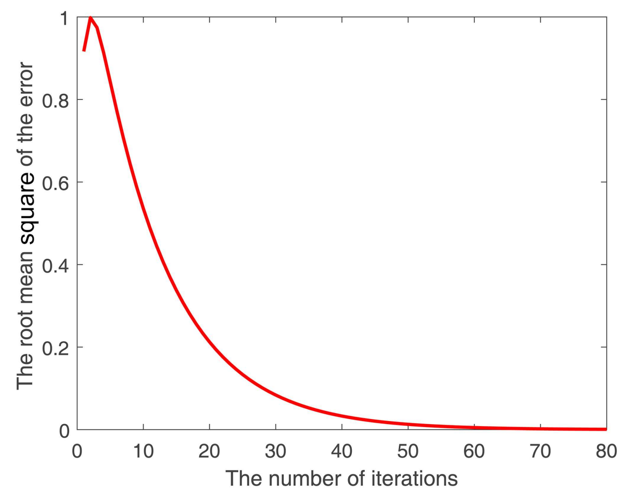

Figure 4 (Color online) Curve of the root mean square of the tracking error of the system (66) under the algorithm (68) with $\gamma = 0.7846$.

In the simulation, the duration of each iteration is set to be $N = 50$ and the number of iterations is set to be 80. In addition, the initial values are set to be $x_i(0) = [0, 0, 0]^{\mathrm{T}}, i \in \mathbb{I}[1, 80]$ and $u_1(t) = 0, t \in \mathbb{I}[0, 50]$. For the system composed of (66) and the algorithm (67), the curve of the root mean square of the tracking error vs. the number of iterations is given in Figure 3. Obviously, the tracking error is divergent. For the system composed of (66) and the algorithm (68) with $\gamma = 0.7846$, the result is depicted in Figure 4. Clearly, the tracking error is convergent. According to Figures 3 and 4, even though the tracking error of this example cannot be made to converge under the ILC algorithm (67), it can be controlled to be convergent under the ILC algorithm (68) by selecting a proper tuning parameter.

Example 3. Consider the following car suspension system, which has been studied in [49, 50]

$$\begin{cases} x_i(t+1) = \begin{bmatrix} 0.2779 & -0.006738 \\ 1 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i(t), \\ y_i(t) = \begin{bmatrix} 0.03052 & 0.005925 \end{bmatrix} x_i(t). \end{cases} \tag{69}$$

The desired output is set to be

$$y_{\mathrm{d}}(t) = \begin{cases} -20(t/40)^7 + 70(t/40)^6 - 84(t/40)^5 + 35(t/40)^4, & t \in \mathbb{I}[0, 39], \\ 1, & t \in \mathbb{I}[40, 100]. \end{cases}$$

In [50], the control algorithm applied to (69) is

$$u_{i+1}(t) = u_i(t) + 9.5e_i(t+1). \tag{70}$$

In this paper, the proposed algorithm with a tuning parameter γ is

$$u_{i+1}(t) = \gamma[u_i(t) + 9.5e_i(t+1)] + (1-\gamma)[u_{i-1}(t) + 9.5e_{i-1}(t+1)]. \tag{71}$$

In the simulation, the duration of each iteration is set to be $N = 100$ and the number of iterations is set to be 100. In addition, the initial conditions for the system state and the control input are $x_i(0) = [0, 0]^{\mathrm{T}}, i \in \mathbb{I}[1, 100]$ and $u_1(t) = 0, t \in \mathbb{I}[0, 100]$. For this example, $LCB = 0.2899$ and the optimal parameter γ is calculated as 1.3000 according to Theorem 5. When the algorithm (70) and the algorithm (71) with $\gamma = 1.3000$ are applied to the system (69), the root mean square of the tracking error ER and $\log_{10}$ ER are shown in Figures 5 and 6, respectively. It is observed that the convergence of the tracking error under the proposed algorithm (71) with $\gamma = 1.3000$ is faster than that under the algorithm (70).

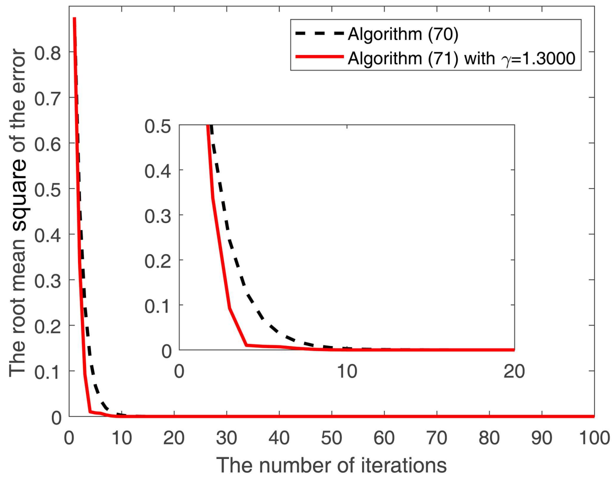

Figure 5 (Color online) Curves of ER of the system (69) under the algorithms (70) and (71).

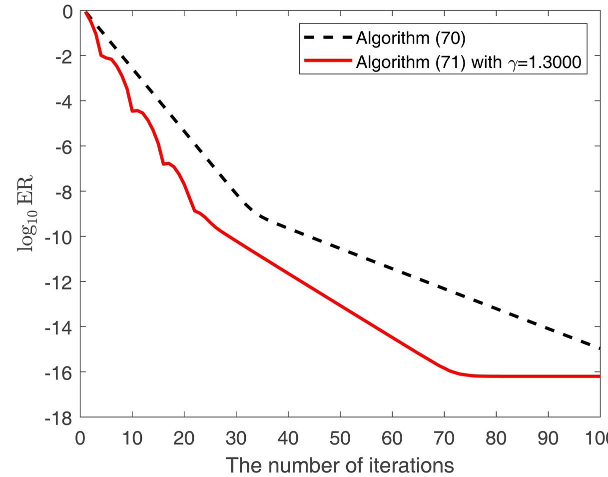

Figure 6 (Color online) Curves of $\log_{10}$ ER of the system (69) under the algorithms (70) and (71).

7 Conclusion

In the current paper, a novel ILC algorithm is presented for repetitive discrete-time state-space linear SISO systems by the method of parameter introduction, and the information of the control inputs and the tracking errors in the historical two iterations is used. Further, the convergence property of the tracking error under the presented ILC algorithm is analyzed. For this end, the relation among the iteration errors of the state variable and the control input is derived in the matrix-vector form, and further the iterative sequence of the tracking error is obtained. With such a relation as the basis, a convergence condition of the tracking error under the presented control algorithm is established in terms of the spectral radius of the iterative matrix. In addition, a necessary condition of the tuning parameter is developed to guarantee the convergence of the tracking error. By transforming the spectral radius of the iterative matrix into the roots of a quadratic equation, an easy-to-check condition is provided to choose an appropriate tuning parameter such that the tracking error is convergent. Moreover, some explicit expressions are also obtained for the optimal tuning parameter with which the tracking error of the system achieves the fastest convergence speed under the corresponding ILC algorithm.

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