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A quantized order estimator

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Abstract This paper considers the order estimation problem of stochastic autoregressive exogenous input (ARX) systems using quantized data. Based on the least squares algorithm and inspired by the control systems information criterion (CIC), a new kind of criterion and a new system order estimation algorithm are proposed for ARX systems with quantized data. When the upper bounds of the system orders are known and the persistent excitation condition is satisfied, the system order estimates given by this algorithm are shown to be consistent for a small quantization step. Furthermore, a concrete method is given for choosing quantization parameters to ensure that the system order estimates are consistent. A numerical example is given to demonstrate the effectiveness of the theoretical results of the paper.

Keywords discrete-time linear time-invariant systems, quantized output, order estimate, system identification, estimator

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1 Introduction

System identification with quantized data is a challenging research topic [1, 2]. In many cases, using quantized data during the system identification process will bring quantization error, which increases the difficulty of analysis. Up to now, a large number of identification methods with quantized data have been developed, including [1, 3–11], to name a few. In particular, Ref. [1] proposed two different frameworks, namely, stochastic and deterministic frameworks so as to identify systems. Ref. [3] researched the identification of multi-agent systems with quantized observations. Ref. [4] gave some motivating examples of quantized measurements and introduced the methods and algorithms of system identification for set-valued linear systems. Ref. [5] used a projection algorithm to estimate parameters of quantized deterministic autoregressive moving average (DARMA) systems, and proved the boundedness of parameter estimation error by designing system inputs. Refs. [6, 7] solved the parameter estimation problem of quantized DARMA systems and quantized stochastic autoregressive exogenous input (ARX) systems with the help of the least squares, respectively. Ref. [8] concerned the system identification for FIR systems with set-valued and precise data received from multiple sensors.

The system identification task for ARX systems consists of estimating (i) the orders, (ii) the parameters, and (iii) the covariance matrix of system noise. However, the contributions listed above are all for parameter estimation with quantized data. As for order estimation by using quantized data, it is a novel problem. Actually, order estimation with quantized data plays a significant role in areas such as signal processing, control systems, and communication systems, where measurements are often quantized for practical reasons (e.g., limited sensor resolution or data storage constraints). And in sensor networks, estimating model orders from sensor data that have been quantized can reduce transmission costs or storage space. Order estimation is a statistical method used to estimate the values of orders within a statistical model based on observed data. This concept is important in numerous fields, including statistics and system identification. Here are some key characteristics of order estimation. (i) Consistency. The estimator converges to the true order value in some sense as the sample size increases. In other words, larger samples lead to more accurate estimates. (ii) Robustness. Due to the positive integer order of the linear system model, the algorithm should ensure convergence to the correct order even when the data are quantitative and imprecise. (iii) Method of estimation. Order estimation algorithms are generally based on parameter estimation algorithms, so there are high requirements for the structural design of the algorithm and the accuracy of parameter estimation. Obviously, selecting the right model order is the first step for the goal of estimating system parameters.

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A number of classic order estimation techniques such as [12–20] have been made since about the 1970s. Specifically, Akaike proposed a well-known criterion, Akaike's information criterion (AIC) [12]. Ref. [13] proved that the final prediction-error (FPE) criterion and AIC are asymptotically equivalent. Ref. [14] made some consistent studies on the order estimation. Ref. [16] proposed an approach for model order determination based on the minimum description length (MDL) criterion which is shown to depend on the minimum eigenvalues of a covariance matrix derived from the observed data. Ref. [17] proved that a strongly consistent estimation of the order can be based on the law of iterated logarithm for the partial autocorrelations. Ref. [18] established the asymptotic properties under very general conditions. Ref. [19] got a consistent estimate of the order of feedback control systems with system parameters estimated by the least squares method. Ref. [20] introduced a new criterion, control systems information criterion (CIC), so as to estimate orders of the linear stochastic feedback control system. In recent years, order estimation problems have received further attention in signal processing and other fields [21–23].

Considering the wide use of quantized data and the important value of order estimation, it is of significance to study order estimation based on quantized data. The introduction of quantized data will produce quantization error, which brings difficulties to order estimation. Using some conclusions of [7], an order estimation method of ARX models with uniform quantized data is proposed. The order estimation algorithm in the paper is utilized in the following process. First of all, the range of ARX system orders is selected (i.e., $0 \leqslant p \leqslant p_{\max}$ and $0 \leqslant q \leqslant q_{\max}$, where p is the order of the AR part and q is the order of the exogenous part). Then for each (p, q) pair the parameters of the model are estimated by the least squares under the assumption that p and q are the right model orders. Finally, a prediction error variance for the model is calculated by the proposed criterion and the (p, q) pair yielding the lowest value is chosen as the best estimate of the model order. So, the key step of estimation lies in two aspects: the design of a criterion for the order estimate algorithm as well as the choice of a quantization step. In fact, they are complementary. During the order estimation algorithm designing progress, new difficulties arise in dealing with the quantized data. For example, the robustness of the quantized parameter estimation algorithm generated estimated errors, which affect the accuracies of traditional order estimation algorithms. By analyzing the upper bound of the quantized parameter estimation, we gave the structure of the criterion in the algorithm design phase, and this difficulty has been solved.

In contrast to the previous studies [3,5–8,14–20], the main contributions of this paper are summarized as follows.

• As mentioned earlier, order estimation is one component of system identification problems. However, to the best of my knowledge, the existing papers on quantized system identification mainly focus on quantized parameter estimation. The discussion about quantized order estimation is pretty rare. Actually, studies like [3,5–8] considered quantized parameter estimation based on known system orders. Different from them, in this paper, we study the quantized order estimation problem when the system orders and parameters are both unknown.

• Compared with classic papers [14–20] on order estimation based on accurate data, we study the order estimate problem under uniform quantized observations. To be more concrete, one of the difficulties in designing an order estimate algorithm is how to make full use of the roughness of quantized observations. Quantized data make the structure of classic estimation algorithms more complex and the estimated parameter cannot converge to the real value in many cases. By designing the criterion and using some hypotheses of system parameters and orders, the quantized order estimation can converge to the real value in some sense.

• Different from [3,5,6], the model researched in this paper contains stochastic noise. So, the algorithm analysis methods in the parameter estimation part of this note are quite different.

There are two main novelties here. One is the method of proving the excitation condition. To be more specific, during studying the properties of the quantized criterion, we found that the excitation condition is the key to getting the lower bound of the quantized criterion. So, in this note, we prove that the system satisfies the excitation condition based on quantized data instead of assuming it. The other is the proposal of a quantized criterion. The inaccuracy of parameter estimation based on uniform quantized data brings essential difficulties to order estimation. To deal with the difficulties, we design a new kind of criterion, named the quantized criterion. And we focused on exploring the upper and lower bounds of the quantized criterion, which are not necessary in the classical situation.

In this paper, $\mathbb{R}$ denotes the real number field. For a given vector or matrix x, $x^\top$ denotes the transpose of x; $||x||$ denotes the Euclidean norm for the vector case and the corresponding induced norm for the matrix case. $\lambda_{\min}()$ denotes the smallest eigenvalue of the matrix between round brackets. The rest of the paper is as follows. In Section 2, we describe the model. Section 3 shows the specific order estimation algorithm for the quantized ARX model, and the influence of quantization error on the order estimation is analyzed. Section 4 uses a numerical example to demonstrate the main result. Section 5 concludes this work.

2 Model

Consider the following ARX system:

$$A(z)y_{n+1} = B(z)u_n + w_{n+1}, \quad n \geqslant 0, \tag{1}$$

where y_n, u_n, and w_n are the system output, system input, and system noise. Besides, $N(0,1)$ indicates a Gaussian distribution with zero mean and variance 1. The noise $\{w_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables and $w_n \sim N(0,1)$. For simplicity, suppose $y_n = u_n = w_n = 0$, $\forall n < 0$.

$$A(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_{p_0} z^{p_0}, \quad p_0 \geqslant 0,$$

$$B(z) = b_1 + b_2 z + \cdots + b_{q_0} z^{q_0 - 1}, \quad q_0 \geqslant 1,$$

where a_i and b_j are unknown system parameters. z is the shift-back operator and the orders p_0, q_0 are unknown. $a_{p_0} \neq 0$, $b_{q_0} \neq 0$.

Remark 1. The system orders p_0 and q_0 are to be estimated in the note. The order of a discrete linear system is determined by the number of input variables and output variables required to fully describe the system's behavior. It determines the complexity of the system's behavior and the number of variables that should be considered when modeling or controlling the system.

For the convenience of proving, the model (1) can be rewritten as follows:

$$y_{n+1} = \theta^\top(p_0, q_0)\varphi_n(p_0, q_0) + w_{n+1}, \tag{2}$$

where

$$\theta(p_0, q_0) = [-a_1, \cdots, -a_{p_0}, b_1, \cdots, b_{q_0}]^\top,$$

$$\varphi_n(p_0, q_0) = [y_n, \cdots, y_{n-p_0+1}, u_n, \cdots, u_{n-q_0+1}]^\top.$$

This paper considers the condition that the system output y_n cannot be directly measured and only its quantized value is known. We want to design an order estimation algorithm and analyze the influence of the quantization step on order estimation.

For a given constant $\varepsilon > 0$ and any $n = 0, 1, 2, \ldots$, the quantized value of y_n is from the following uniform quantizer:

$$s_n = \varepsilon \left\lfloor \frac{y_n}{\varepsilon} + \frac{1}{2} \right\rfloor. \tag{3}$$

We can call ε the quantization step and s_n is the quantized output.

Remark 2. The more direct form of (3) is

$$s_n = \begin{cases} \vdots \\ -2\varepsilon, & y_n \in \left[-\frac{5\varepsilon}{2}, -\frac{3\varepsilon}{2}\right), \\ -\varepsilon, & y_n \in \left[-\frac{3\varepsilon}{2}, -\frac{\varepsilon}{2}\right), \\ 0, & y_n \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right), \\ \varepsilon, & y_n \in \left[\frac{\varepsilon}{2}, \frac{3\varepsilon}{2}\right), \\ 2\varepsilon, & y_n \in \left[\frac{3\varepsilon}{2}, \frac{5\varepsilon}{2}\right), \\ \vdots \end{cases}$$

From (2) and (3) we know that

$$s_{n+1} = \theta^\top(p_0, q_0)\psi_n(p_0, q_0) + w_{n+1} + \epsilon_{n+1}, \tag{4}$$

where

$$\psi_n(p_0, q_0) = \left[s_n, \cdots, s_{n-p_0+1}, u_n, \cdots, u_{n-q_0+1}\right]^\top, \tag{5}$$

and ϵ_{n+1} is the quantization noise at time $n+1$, which is produced by quantized outputs and its concrete property is as follows.

From (2) and (4) we know that

$$\begin{aligned}
|\epsilon_{n+1}| &= \left|s_{n+1} - \theta^\top(p_0, q_0)\psi_n(p_0, q_0) - w_{n+1}\right| \\
&= \left|s_{n+1} - \theta^\top(p_0, q_0)\psi_n(p_0, q_0) - \left(y_{n+1} - \theta^\top(p_0, q_0)\varphi_n(p_0, q_0)\right)\right| \\
&= \left|s_{n+1} - y_{n+1} + \theta^\top(p_0, q_0)\left(\varphi_n(p_0, q_0) - \psi_n(p_0, q_0)\right)\right| \\
&\leqslant \left|s_{n+1} - y_{n+1}\right| + \left|\theta^\top(p_0, q_0)\left(\varphi_n(p_0, q_0) - \psi_n(p_0, q_0)\right)\right| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(|a_1| + |a_2| + \cdots + |a_{p_0}|\right) \\
&= \frac{\varepsilon}{2}\left(|a_1| + |a_2| + \cdots + |a_{p_0}| + 1\right).
\end{aligned} \tag{6}$$

So, we can assume ϵ_n is the bounded noise.

3 Order estimation of quantized ARX systems

The purpose of this paper is to estimate p_0 and q_0 in (4) using system inputs and quantized outputs. In this section, we give the specific order estimate method and analyze its properties.

Let

$$\begin{cases}
\psi_i(p, q) := \left[s_i, \cdots, s_{i-p+1}, u_i, \cdots, u_{i-q+1}\right]^\top, \\
P_{n+1}(p, q) := \left(I + \sum_{i=0}^{n} \psi_i(p, q)\psi_i^\top(p, q)\right)^{-1},
\end{cases} \tag{7}$$

where $s_i = u_i = 0$, when $i \leqslant 0$. And let $\lambda_{\min}^{(p,q)}(n)$ denote the smallest eigenvalue of $P_{n+1}^{-1}(p, q)$.

3.1 Assumptions

In order to proceed the analysis, we introduce the following assumptions.

Assumption 1. $\{u_i\}$ is a sequence of independent and identically distributed (i.i.d.) random variables and u_i satisfies uniform distribution in $[-\delta, \delta]$, $\delta > 0$.

Assumption 2. $A(z)$ is stable, i.e., $A(z) \neq 0$, $\forall |z| \leqslant 1$.

Assumption 3. There exists a constant $c > 0$ such that $|a_i| \leqslant c$, $|b_j| \leqslant c$, $i = 1, \ldots, p_0$, $j = 1, \ldots, q_0$, and $\varepsilon < \frac{1}{2(1+p_0 c)}$.

Assumption 4. $\{p_0, q_0\}$ belongs to a known finite set M:

$$M \triangleq \{(p, q) : 0 \leqslant p \leqslant p^*, 0 \leqslant q \leqslant q^*\},$$

where $p^* > 0$, $q^* > 0$.

Assumption 5. There exists a constant $c_1 > 0$ such that

$$\lambda_{\min}^{(p,q^*)}(n) \geqslant c_1(n+1), \text{a.s.}, n \to \infty,$$

for all $0 \leqslant p \leqslant p^*$.

Assumption 6. There exists a constant $c_2 > 0$ such that

$$\lambda_{\min}^{(p^*,q)}(n) \geqslant c_2(n+1), \text{a.s.}, n \to \infty,$$

for all $0 \leqslant q \leqslant q^*$.

Remark 3. Assumption 1 means system inputs $\{u_i\}$ are bounded and satisfy a uniform distribution. Assumptions 2 and 4 are common in classic system identification literature. Assumption 3 is always used in quantized identification. Assumptions 5 and 6 mean persistent excitation condition can be satisfied and they are pretty important to the proof of the theorem in the paper.

3.2 Examples about the assumptions

We use one example to explain the existence of Assumptions 5 and 6 in a specific situation.

Consider the system: $A(z)y_{n+1} = B(z)u_n + w_{n+1}$, where $A(z) = 1 + 0.7z + 0.1z^2$, $B(z) = 1$, $p_0 = 2$, $q_0 = 1$. We assume that $p^* = 3$, $q^* = 2$. $\{y_n\}$ is quantized by (5).

For any $x \in \mathbb{R}^{p+q}$, $||x|| = 1$, write x in the form $x = [x_1, x_2, \cdots, x_{p+q}]^\top$ and let

$$\phi_i(p,q) = A(z)\psi_i(p,q). \tag{8}$$

From (8) it can be seen that

$$\begin{aligned}
x^\top \left(\sum_{i=0}^{n} \phi_i(p,q)\phi_i^\top(p,q)\right) x &= \sum_{i=0}^{n} \left(x^\top \phi_i(p,q)\right)^2 \\
&= \sum_{i=0}^{n} \left(\sum_{j=0}^{2} a_j x^\top \psi_{i-j}(p,q)\right)^2 \\
&\leqslant \sum_{j=0}^{2} a_j^2 \sum_{i=0}^{n}\sum_{j=0}^{2} \left(x^\top \psi_{i-j}(p,q)\right)^2 \\
&\leqslant 3\sum_{j=0}^{2} a_j^2 x^\top \left(\sum_{i=0}^{n} \psi_i(p,q)\psi_i^\top(p,q)\right) x \\
&= 4.5 x^\top \left(\sum_{i=0}^{n} \psi_i(p,q)\psi_i^\top(p,q)\right) x,
\end{aligned} \tag{9}$$

where $a_0 = 1, a_1 = 0.7, a_2 = 0.1$. So we have

$$\lambda_{\min}^{(p,q)}(n) \geqslant \lambda_{\min}\left(\sum_{i=0}^{n} \psi_i(p,q)\psi_i^\top(p,q)\right) \geqslant \frac{2}{9}\lambda_{\min}\left(\sum_{i=0}^{n} \phi_i(p,q)\phi_i^\top(p,q)\right). \tag{10}$$

Example 1. We give the analyses of $\lambda_{\min}^{(p,2)}(n), 0 \leqslant p \leqslant 3$ and $\lambda_{\min}^{(3,q)}(n), 0 \leqslant q \leqslant 2$.

When $p = 0$, $q = 2$,

$$\begin{aligned}
&\lambda_{\min}\left(\sum_{i=0}^{n} \phi_i(0,2)\phi_i^\top(0,2)\right) \\
&= \inf_{||x||=1} x^\top \left(\sum_{i=0}^{n} \phi_i(0,2)\phi_i^\top(0,2)\right) x \\
&= \inf_{||x||=1} \sum_{i=0}^{n} \left[x_1u_i + (0.7x_1 + x_2)\, u_{i-1} + (0.1x_1 + 0.7x_2)\, u_{i-2} + 0.1x_2u_{i-3}\right]^2.
\end{aligned}$$

When $p = 1$, $q = 2$,

$$\begin{aligned}
&\lambda_{\min}\left(\sum_{i=0}^{n} \phi_i(1,2)\phi_i^\top(1,2)\right) \\
&= \inf_{||x||=1} x^\top \left(\sum_{i=0}^{n} \phi_i(1,2)\phi_i^\top(1,2)\right) x \\
&= \inf_{||x||=1} \sum_{i=0}^{n} \left[x_2u_i + (x_1 + 0.7x_2 + x_3)\, u_{i-1} + (0.1x_2 + 0.7x_3)\, u_{i-2} + 0.1x_3u_{i-3} + x_1w_i + x_1\epsilon_i\right]^2.
\end{aligned}$$

When $p = 2$, $q = 2$,

$$\lambda_{\min}\left(\sum_{i=0}^{n} \phi_i(2,2)\phi_i^\top(2,2)\right)$$

$$
\begin{aligned}
&= \inf_{||x||=1} x^\top \left(\sum_{i=0}^{n} \phi_i(2,2)\phi_i^\top(2,2) \right) x \\
&= \inf_{||x||=1} \sum_{i=0}^{n} \left[x_3 u_i + (x_1 + 0.7x_3 + x_4)\, u_{i-1} + (x_2 + 0.1x_3 + 0.7x_4)\, u_{i-2} + 0.1x_4 u_{i-3} \right. \\
&\quad \left. + x_1 w_i + x_2 w_{i-1} + x_1 \epsilon_i + x_2 \epsilon_{i-1} \right]^2 .
\end{aligned}
$$

When $p = 3$, $q = 2$,

$$
\begin{aligned}
&\lambda_{\min} \left(\sum_{i=0}^{n} \phi_i(3,2)\phi_i^\top(3,2) \right) \\
&= \inf_{||x||=1} x^\top \left(\sum_{i=0}^{n} \phi_i(3,2)\phi_i^\top(3,2) \right) x \\
&= \inf_{||x||=1} \sum_{i=0}^{n} \left[x_4 u_i + (x_1 + 0.7x_4 + x_5)\, u_{i-1} + (x_2 + 0.1x_4 + 0.7x_5)\, u_{i-2} + (x_3 + 0.1x_5)\, u_{i-3} \right. \\
&\quad \left. + x_1 w_i + x_2 w_{i-1} + x_3 w_{i-2} + x_1 \epsilon_i + x_2 \epsilon_{i-1} + x_3 \epsilon_{i-2} \right]^2 .
\end{aligned}
$$

When $p = 3$, $q = 0$,

$$
\begin{aligned}
&\lambda_{\min} \left(\sum_{i=0}^{n} \phi_i(3,0)\phi_i^\top(3,0) \right) \\
&= \inf_{||x||=1} x^\top \left(\sum_{i=0}^{n} \phi_i(3,0)\phi_i^\top(3,0) \right) x \\
&= \inf_{||x||=1} \sum_{i=0}^{n} \left[x_1 u_{i-1} + x_2 u_{i-2} + x_3 u_{i-3} + x_1 w_i + x_1 \epsilon_i + x_2 w_{i-1} + x_2 \epsilon_{i-1} + x_3 w_{i-2} + x_3 \epsilon_{i-2} \right]^2 .
\end{aligned}
$$

When $p = 3$, $q = 1$,

$$
\begin{aligned}
&\lambda_{\min} \left(\sum_{i=0}^{n} \phi_i(3,1)\phi_i^\top(3,1) \right) \\
&= \inf_{||x||=1} x^\top \left(\sum_{i=0}^{n} \phi_i(3,1)\phi_i^\top(3,1) \right) x \\
&= \inf_{||x||=1} \sum_{i=0}^{n} \left[x_4 u_i + (x_1 + 0.7x_4)\, u_{i-1} + (x_2 + 0.1x_4)\, u_{i-2} + x_3 u_{i-3} + x_1 w_i + x_1 \epsilon_i + x_2 w_{i-1} \right. \\
&\quad \left. + x_2 \epsilon_{i-1} + x_3 w_{i-2} + x_3 \epsilon_{i-2} \right]^2 .
\end{aligned}
$$

It can be seen that in each case, the coefficients of u_i, u_{i-1}, u_{i-2}, u_{i-3}, w_i, w_{i-1}, w_{i-2} are not all 0. So, from (10) and [7] (confer (16), Lemma 1), we know that Assumptions 5 and 6 can be satisfied.

3.3 The estimation of p_0

In this section, we will prove the convergence of the estimate of p_0.

First, we give the analyses of the matrix composed of quantized regressor vectors.

Lemma 1. Suppose Assumptions 1 and 2 are satisfied. Then, as $n \to \infty$, there is a constant $c_3 > 0$ such that

$$
\lambda_{\max}^{(p_0,q^*)}(n) \leqslant c_3\,(n+1)\,, \text{a.s.}, \tag{11}
$$

where $\lambda_{\max}^{(p_0,q^*)}(n)$ denotes the largest eigenvalue of $\sum_{i=0}^{n} \psi_i(p_0,q^*)\psi_i^\top(p_0,q^*) + I$.

Proof. The proof can be seen in Appendix A.

For $(p,q) \in M$, define the cost function

$$J_n(\theta) := \sum_{i=0}^{n} \left(s_{i+1} - \bar{\theta}^\top(p,q)\psi_i(p,q)\right)^2, \tag{12}$$

where

$$\bar{\theta}(p,q) = [-a_1, \cdots, -a_p, b_1, \cdots, b_q]^\top \tag{13}$$

and

$$a_i = 0, b_j = 0, \quad i > p_0, j > q_0. \tag{14}$$

Let (12) be minimized by the least squares with respect to the parameters $\bar{\theta}(p,q)$.

Based on the regularized method, the estimation of $\bar{\theta}(p,q)$ can be defined as

$$\theta_n(p,q) := \left(\sum_{i=0}^{n-1} \psi_i(p,q)\psi_i^\top(p,q) + I\right)^{-1} \sum_{i=0}^{n-1} \psi_i(p,q)s_{i+1} = P_n(p,q)\sum_{i=0}^{n-1} \psi_i(p,q)s_{i+1}, \tag{15}$$

where

$$\theta_n(p,q) = [-a_{1n}, \cdots, -a_{pn}, b_{1n}, \cdots, b_{qn}]^\top. \tag{16}$$

Lemma 2. Suppose Assumptions 1–5 are satisfied. Then as $n \to \infty$,

$$\begin{aligned} &\left\| \left(\sum_{i=0}^{n-1} \psi_i(p_0,q^*)\psi_i^\top(p_0,q^*) + I\right)^{-\frac{1}{2}} \sum_{i=0}^{n-1} \psi_i(p_0,q^*)\left(w_{i+1} + \epsilon_{i+1}\right) \right\|^2 \\ &\leqslant (1 + p_0 c)\,\varepsilon n + o\left(n\right), \text{a.s.} \end{aligned} \tag{17}$$

Proof. The proof can be seen in Appendix B.

Next, we show the properties of parameter estimation error.

Lemma 3. Suppose Assumptions 1–5 are satisfied under the condition $p \leqslant p_0$, and define

$$\hat{\theta}_n(p) := [-a_{1n}(p), \cdots, -a_{pn}(p), \underbrace{0, \cdots, 0}_{p_0-p}, b_{1n}(p), \cdots, b_{q^*n}(p)]^\top, \tag{18}$$

where $a_{in}(p)$, $b_{in}(p)$ are of $\theta_n(p,q^*)$.

Let

$$\tilde{\theta}_n(p) = \bar{\theta}(p_0,q^*) - \hat{\theta}_n(p). \tag{19}$$

Then as $n \to \infty$, there is a constant γ such that

$$\left\|\tilde{\theta}_n(p)\right\| \leqslant \gamma, \text{a.s.} \tag{20}$$

Proof. The proof can be seen in Appendix C.

Then, we give the form of the quantized criterion $L_n(p,q)$ and the order estimation algorithm.

Let

$$L_n(p,q) := \sigma_n(p,q) + l_n \cdot (p+q), \tag{21}$$

where

$$\sigma_n(p,q) = \sum_{i=0}^{n-1} \left(s_{i+1} - \theta_n^\top(p,q)\psi_i(p,q)\right)^2, \tag{22}$$

and the restrictions of l_n will be given later.

The order estimation $\hat{p}_n$ of p_0 is defined as

$$\hat{p}_n := \operatorname{argmin}_{0\leqslant p\leqslant p^*} L_n(p,q^*). \tag{23}$$

Now, we give the upper bound of $\sigma_n(p_0,q^*)$ in the following lemma.

Lemma 4. Suppose Assumptions 1–5 are satisfied, then as $n \to \infty$,

$$\sigma_n(p_0, q^*) \leqslant 3\left(1 + p_0 c\right)\varepsilon n + \sum_{i=0}^{n-1}\left(w_{i+1} + \epsilon_{i+1}\right)^2 + o\left(n\right), \text{a.s.} \tag{24}$$

Proof. From (4), (5), (7), (13), (14), (22) we have

$$\sigma_n(p_0, q^*) = \sum_{i=0}^{n-1}\left(\bar{\theta}^{\top}(p_0, q^*)\psi_i(p_0, q^*) + w_{i+1} + \epsilon_{i+1} - \theta_n^{\top}(p_0, q^*)\psi_i(p_0, q^*)\right)^2. \tag{25}$$

So,

$$\begin{aligned}\sigma_n(p_0, q^*) =& \sum_{i=0}^{n-1}\left(\tilde{\theta}_n^{\top}(p_0, q^*)\psi_i(p_0, q^*) + w_{i+1} + \epsilon_{i+1}\right)^2\\ =&\tilde{\theta}_n^{\top}(p_0, q^*)\sum_{i=0}^{n-1}\psi_i(p_0, q^*)\psi_i^{\top}(p_0, q^*)\tilde{\theta}_n(p_0, q^*) + 2\tilde{\theta}_n^{\top}(p_0, q^*)\sum_{i=0}^{n-1}\psi_i(p_0, q^*)\left(w_{i+1} + \epsilon_{i+1}\right)\\ &+ \sum_{i=0}^{n-1}\left(w_{i+1} + \epsilon_{i+1}\right)^2.\end{aligned} \tag{26}$$

From Theorem 1 of [7] we get

$$\tilde{\theta}_n^{\top}(p_0, q^*)\sum_{i=0}^{n-1}\psi_i(p_0, q^*)\psi_i^{\top}(p_0, q^*)\tilde{\theta}_n(p_0, q^*) \leqslant \left(1 + p_0 c\right)\varepsilon n + o\left(n\right), \text{a.s.}, \tag{27}$$

and

$$\begin{aligned}&2\left|\tilde{\theta}_n^{\top}(p_0, q^*)\sum_{i=0}^{n-1}\psi_i(p_0, q^*)\left(w_{i+1} + \epsilon_{i+1}\right)\right|\\ &= 2\left|\tilde{\theta}_n^{\top}(p_0, q^*)\left(\bar{\theta}(p_0, q^*) - P_n^{-1}(p_0, q^*)\tilde{\theta}_n(p_0, q^*)\right)\right|\\ &\leqslant 2\left|\tilde{\theta}_n^{\top}(p_0, q^*)\bar{\theta}(p_0, q^*)\right| + 2\tilde{\theta}_n^{\top}(p_0, q^*)P_n^{-1}(p_0, q^*)\tilde{\theta}_n(p_0, q^*)\\ &\leqslant 2\left(1 + p_0 c\right)\varepsilon n + o\left(n\right), \text{a.s.}\end{aligned} \tag{28}$$

From (26)–(28) we obtain

$$\sigma_n(p_0, q^*) \leqslant 3\left(1 + p_0 c\right)\varepsilon n + \sum_{i=0}^{n-1}\left(w_{i+1} + \epsilon_{i+1}\right)^2 + o\left(n\right), \text{a.s.} \tag{29}$$

This completes the proof.

Based on the above lemmas, we can get the main theoretical result of the paper.

Theorem 1. Supposing Assumptions 1–5 are satisfied and l_n satisfies

$$l_n \geqslant \left[5\left(1 + p^* c\right)\varepsilon + \alpha_1\right]n, \quad \alpha_1 > 0, \tag{30}$$

and

$$l_n \leqslant \frac{\alpha_2}{p^*}\left[a_{p_0}^2 c_1 - 2\gamma\sqrt{c_3\left(1 + p^* c\right)\varepsilon} - 3\left(1 + p^* c\right)\varepsilon\right]n, 0 < \alpha_2 < 1, \tag{31}$$

then

$$\hat{p}_n \xrightarrow[n\to\infty]{} p_0, \text{a.s.} \tag{32}$$

Proof. First, we want to prove

$$\limsup_{n\to\infty} \hat{p}_n \leqslant p_0, \text{a.s.} \tag{33}$$

For $p > p_0$, similar with (28) we have

$$2\left|\tilde{\theta}_n^\top(p, q^*)\sum_{i=0}^{n-1}\psi_i(p, q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\right| \leqslant 2\left(1+p^*c\right)\varepsilon n + o\left(n\right), \text{a.s.} \tag{34}$$

Similar with (26) we have

$$\begin{aligned}\sigma_n(p, q^*) =&\tilde{\theta}_n^\top(p, q^*)\sum_{i=0}^{n-1}\psi_i(p, q^*)\psi_i^\top(p, q^*)\tilde{\theta}_n(p, q^*) + 2\tilde{\theta}_n^\top(p, q^*)\sum_{i=0}^{n-1}\psi_i(p, q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\\ &+\sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2.\end{aligned} \tag{35}$$

From (34) and (35) we have

$$\begin{aligned}\sigma_n(p, q^*) \geqslant& 2\tilde{\theta}_n^\top(p, q^*)\sum_{i=0}^{n-1}\psi_i(p, q^*)\left(w_{i+1}+\epsilon_{i+1}\right) + \sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2\\ \geqslant& -2\left(1+p^*c\right)\varepsilon n + \sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2 + o\left(n\right), \text{a.s.}\end{aligned} \tag{36}$$

From (36) and Lemma 4 we have

$$\begin{aligned}\sigma_n(p, q^*) - \sigma_n(p_0, q^*) \geqslant& -2\left(1+p^*c\right)\varepsilon n + \sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2 + o\left(n\right)\\ &-\left[3\left(1+p_0c\right)\varepsilon n + \sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2 + o\left(n\right)\right]\\ \geqslant& -5\left(1+p^*c\right)\varepsilon n + o\left(n\right), \text{a.s.}\end{aligned} \tag{37}$$

From (30) it can be seen that

$$l_n \cdot (p - p_0) \geqslant l_n \geqslant \left[5\left(1+p^*c\right)\varepsilon + \alpha_1\right]. \tag{38}$$

From (21), (37), (38) and noticing $\alpha_1 > 0$, we have

$$\begin{aligned}\min\nolimits_{p_0<p\leqslant p^*}\left[L_n(p, q^*) - L_n(p_0, q^*)\right] \geqslant& -5\left(1+p^*c\right)\varepsilon n + l_n\cdot(p-p_0) + o\left(n\right)\\ \geqslant& -5\left(1+p^*c\right)\varepsilon n + \left[5\left(1+p^*c\right)\varepsilon + \alpha_1\right]n + o\left(n\right)\\ \geqslant& \alpha_1 n + o\left(n\right) \xrightarrow[n\to\infty]{} \infty, \text{a.s.}\end{aligned} \tag{39}$$

So, Eq. (33) is proven.

Next, we want to prove

$$\liminf_{n\to\infty} \hat{p}_n \geqslant p_0, \text{a.s.} \tag{40}$$

For $p < p_0$, from (4), (5), (7), (13), (14), (16), (18), (19) we have

$$\begin{aligned}s_{i+1} - \theta_n^\top(p, q^*)\psi_i(p, q^*) =& s_{i+1} - \hat{\theta}_n^\top(p)\psi_i(p_0, q^*)\\ =& \bar{\theta}^\top(p_0, q^*)\psi_i(p_0, q^*) + w_{i+1} + \epsilon_{i+1} - \hat{\theta}_n^\top(p)\psi_i(p_0, q^*)\\ =& \tilde{\theta}_n^\top(p)\psi_i(p_0, q^*) + w_{i+1} + \epsilon_{i+1}.\end{aligned} \tag{41}$$

From (22) and (41) we have

$$\begin{aligned}\sigma_n(p,q^*) &= \sum_{i=0}^{n-1}\left(s_{i+1}-\theta_n^\top(p,q^*)\psi_i(p,q^*)\right)^2\\&= \sum_{i=0}^{n-1}\left(\tilde{\theta}_n^\top(p)\psi_i(p_0,q^*)+w_{i+1}+\epsilon_{i+1}\right)^2\\&= \tilde{\theta}_n^\top(p)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)\tilde{\theta}_n(p)+2\tilde{\theta}_n^\top(p)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\\&\quad+\sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2. \end{aligned}\tag{42}$$

From (13), (18), and (19) we get

$$\left\|\tilde{\theta}_n^\top(p)\right\|^2 \geqslant a_{p_0}^2 > 0. \tag{43}$$

From (20), (43) and Assumption 5 we have

$$\begin{aligned}\tilde{\theta}_n^\top(p)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)\tilde{\theta}_n(p) &= \tilde{\theta}_n^\top(p)\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I-I\right)\tilde{\theta}_n(p)\\&\geqslant a_{p_0}^2\lambda_{\min}^{(p_0,q^*)}(n-1)-\left\|\tilde{\theta}_n(p)\right\|^2\\&\geqslant a_{p_0}^2 c_1 n-\gamma^2,\end{aligned}\tag{44}$$

$$\begin{aligned}&2\tilde{\theta}_n^\top(p)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\\&= 2\left\|\tilde{\theta}_n^\top(p)\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right)^{-\frac{1}{2}}\right.\\&\quad\left.\cdot\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\right\|.\end{aligned}\tag{45}$$

From Lemmas 1–3 and (45) we have

$$\begin{aligned}&2\tilde{\theta}_n^\top(p)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\\&\leqslant 2\left\|\tilde{\theta}_n^\top(p)\right\|\left\|\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right)^{\frac{1}{2}}\right\|\sqrt{\left(1+p_0c\right)\varepsilon n+o\left(n\right)}\\&\leqslant 2\gamma\sqrt{\lambda_{\max}^{(p_0,q^*)}(n-1)}\sqrt{\left(1+p_0c\right)\varepsilon n+o\left(n\right)}\\&\leqslant 2\gamma\sqrt{c_3 n}\sqrt{\left(1+p_0c\right)\varepsilon n+o\left(n\right)}\\&= 2\gamma\sqrt{c_3\left(1+p_0c\right)\varepsilon+c_3 o\left(1\right)}n\\&\leqslant 2\gamma\sqrt{c_3\left(1+p_0c\right)\varepsilon}n+o\left(n\right).\end{aligned}\tag{46}$$

From (42), (44) and (46) it follows that

$$\sigma_n(p,q^*) \geqslant a_{p_0}^2 c_1 n-2\gamma\sqrt{c_3\left(1+p_0c\right)\varepsilon}n+\sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2+o\left(n\right). \tag{47}$$

From (47) and Lemma 4 we have

$$\begin{aligned}
&\sigma_n(p,q^*)-\sigma_n(p_0,q^*)\\
&\quad\geqslant a_{p_0}^2c_1n-2\gamma\sqrt{c_3\left(1+p_0c\right)\varepsilon}n+\sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2-\left[3\left(1+p_0c\right)\varepsilon n+\sum_{i=0}^{n-1}\left(w_{i+1}+\epsilon_{i+1}\right)^2+o\left(n\right)\right]\\
&\qquad+o\left(n\right)\\
&\quad=a_{p_0}^2c_1n-2\gamma\sqrt{c_3\left(1+p_0c\right)\varepsilon}n-3\left(1+p_0c\right)\varepsilon n+o\left(n\right)\\
&\quad\geqslant a_{p_0}^2c_1n-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}n-3\left(1+p^*c\right)\varepsilon n+o\left(n\right),\text{a.s.}
\end{aligned}\tag{48}$$

From (31) it can be seen that

$$\begin{aligned}
l_n\cdot(p_0-p)\leqslant&\frac{\alpha_2}{p^*}\left[a_{p_0}^2c_1-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right]np^*\\
=&\alpha_2\left[a_{p_0}^2c_1-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right]n.
\end{aligned}\tag{49}$$

From (21), (48), (49) and noticing $0<\alpha_2<1$, we have

$$\begin{aligned}
&\min\nolimits_{0\leqslant p<p_0}\left[L_n(p,q^*)-L_n(p_0,q^*)\right]\\
&\quad\geqslant a_{p_0}^2c_1n-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}n-3\left(1+p^*c\right)\varepsilon n+o\left(n\right)-l_n\cdot(p_0-p)\\
&\quad\geqslant a_{p_0}^2c_1n-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}n-3\left(1+p^*c\right)\varepsilon n+o\left(n\right)-\alpha_2\left[a_{p_0}^2c_1-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right]n\\
&\quad=\left(1-\alpha_2\right)\left[a_{p_0}^2c_1-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right]n+o\left(n\right)\xrightarrow[n\to\infty]{}\infty,\text{a.s.}
\end{aligned}$$

So, Eq. (40) is proven.

From (33) and (40) we know that

$$\hat{p}_n\xrightarrow[n\to\infty]{}p_0,\text{a.s.}\tag{50}$$

This completes the proof.

3.4 The estimation of q_0

The quantized criterion $V_n(p,q)$ can be defined as

$$V_n(p,q):=\sigma_n(p,q)+v_n\cdot\left(p+q\right),\tag{51}$$

where $\sigma_n(p,q)$ is defined in (22) and the restrictions of v_n will be given later.

The order estimation $\hat{q}_n$ of q_0 is defined as

$$\hat{q}_n:=\text{argmin}_{0\leqslant q\leqslant q^*}V_n(p^*,q).\tag{52}$$

Theorem 2. Supposing Assumptions 1–4 and 6 are satisfied and v_n satisfies

$$v_n\geqslant\left[5\left(1+p^*c\right)\varepsilon+\beta_1\right]n,\quad\beta_1>0\tag{53}$$

and

$$v_n\leqslant\frac{\beta_2}{q^*}\left[b_{q_0}^2c_2-2\gamma'\sqrt{c_4\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right]n,0<\beta_2<1,\tag{54}$$

then

$$\hat{q}_n\xrightarrow[n\to\infty]{}q_0,\quad\text{a.s.}\tag{55}$$

Proof. The proof is similar to Theorem 1.

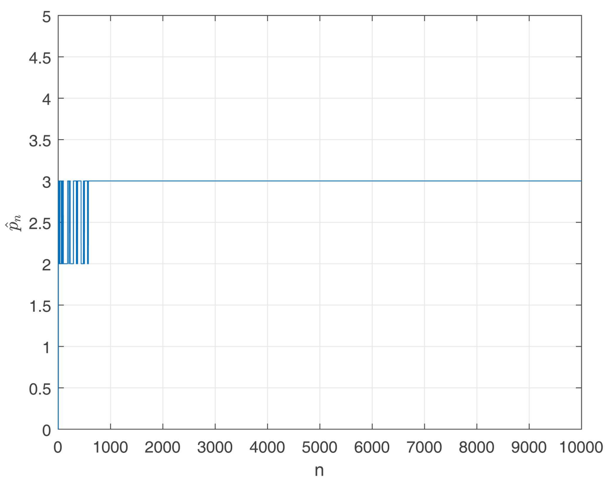

Figure 1 (Color online) The trajectory of $\hat{p}_n$ with $p^*=3$, $q^*=3$.

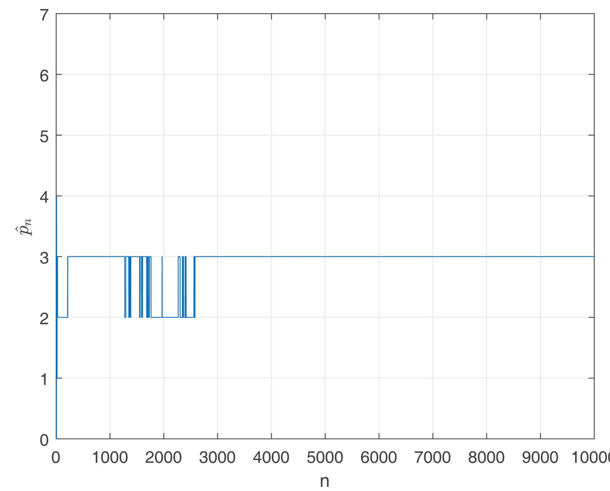

Figure 2 (Color online) The trajectory of $\hat{p}_n$ with $p^*=6$, $q^*=6$.

Remark 4. By choosing suitable ε, α_1, α_2, β_1 and β_2 it can be made sure that

$$\left[5\left(1+p^*c\right)\varepsilon+\alpha_1, \frac{\alpha_2}{p^*}\left(a_{p_0}^2c_1-2\gamma\sqrt{c_3\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right)\right]$$

and

$$\left[5\left(1+p^*c\right)\varepsilon+\beta_1, \frac{\beta_2}{q^*}\left(b_{q_0}^2c_2-2\gamma'\sqrt{c_4\left(1+p^*c\right)\varepsilon}-3\left(1+p^*c\right)\varepsilon\right)\right]$$

are not empty sets. So, Eqs. (30), (31), (53) and (54) are meaningful.

4 Numerical example

In this section, we will illustrate the theoretical result with a simulation example.

Consider the following ARX system: $y_n=a_1y_{n-1}+a_2y_{n-2}+a_3y_{n-3}+b_1u_{n-1}+b_2u_{n-2}+w_n, n=1,2,...,$ where the system noise w_n follows $N(0,1)$, $p_0=3$, $q_0=2$. $\theta=[a_1,a_2,a_3,b_1,b_2]^\top=[-1.5,-0.66,-0.08,1,1]^\top$. Let y_n be quantized by (3) under $\varepsilon=0.001$, $p^*=3$, $q^*=3$ and $p^*=6$, $q^*=6$, respectively.

With the selected p ($p\leqslant p^*$) and q ($q\leqslant q^*$), we use Algorithm 1 to estimate p_0 and q_0.

Algorithm 1 The estimate of p_0 and q_0.

Input: u_i.
Output: $\hat{p}_n$ and $\hat{q}_n$.
1: Compute $\theta_n(p,q)$ according to (15);
2: Compute $\sigma_n(p,q)$ according to (22);
3: Compute $L_n(p,q)$ according to (21);
4: Compute $V_n(p,q)$ according to (51);
5: Compute $\hat{p}_n$ according to (23);
6: Compute $\hat{q}_n$ according to (52).

From this estimate of θ_n, we use (23) to estimate p_0, where u_i satisfies uniform distribution in $[-6,6]$ ($\delta=6$). From (30) and (31), let $l_n=0.006n$. The trajectories of $\hat{p}_n$ are given by Figures 1 and 2.

From this estimate of θ_n, we use (52) to estimate q_0, where u_i satisfies uniform distribution in $[-1,1]$ ($\delta=1$). From (53) and (54), let $v_n=0.006n$. The trajectories of $\hat{q}_n$ are given by Figures 3 and 4.

From Figures 1–4, we can learn that the estimate $\hat{p}_n$ converges to the true value p_0 and the estimate $\hat{q}_n$ converges to the true value q_0. The convergence rates of $\hat{p}_n$ and $\hat{q}_n$ are affected by the bounds p^* and q^*. To be more concrete, the larger the bounds, the slower the convergence rates of $\hat{p}_n$ and $\hat{q}_n$.

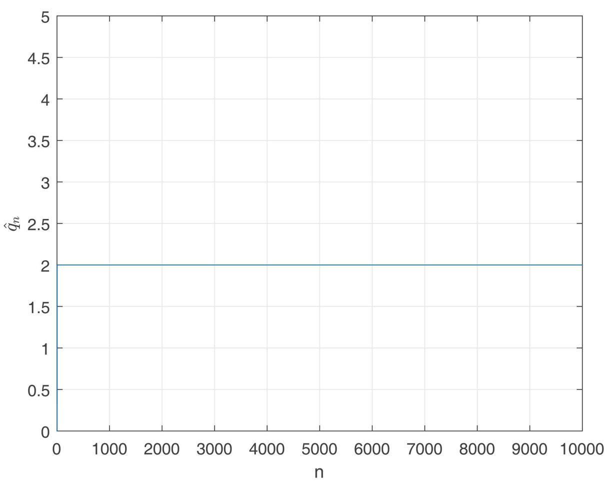

Figure 3 (Color online) The trajectory of $\hat{q}_n$ with $p^* = 3$, $q^* = 3$.

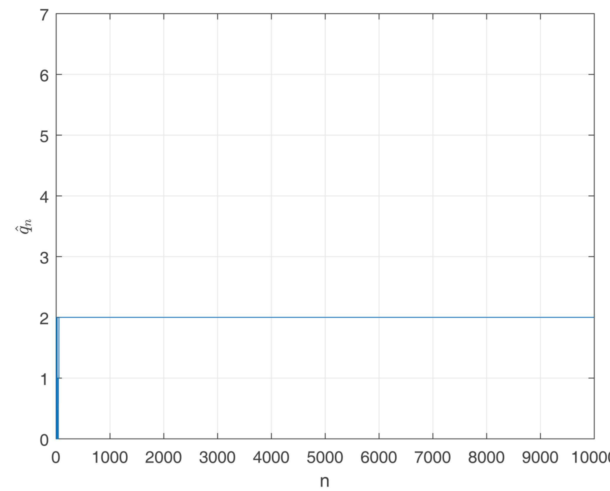

Figure 4 (Color online) The trajectory of $\hat{q}_n$ with $p^* = 6$, $q^* = 6$.

5 Conclusion

This paper has considered the order estimation of ARX systems using uniform quantized data. We design a novel criterion so as to estimate orders based on the persistent excitation condition and some assumptions. Obviously, Ref. [7] provided ideas for this paper and the least squares method is the key to the algorithm of this paper. It is shown that the estimated order is consistent. For further research, the conditions on the system itself may be relaxed by finding other suitable criteria. Another topic is how to reduce the amount of calculation. The methods proposed by [24] may be useful to solve such a problem.

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Supporting information Appendixes A–C. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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