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A Quantized Order Estimator

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Appendix A Proof of Lemma 1

From Assumptions 1, 2, Lemma B.3.3. of [1] (Page 486) and law of large numbers, we know that there exists a positive constant $\hat{c}$ such that $\lim_{n\to\infty}\frac{\sum_{i=0}^{n}y_i^2}{n+1}\leqslant\hat{c}$.

So, from (3), (7) and Assumption 1, we know that

$$\begin{aligned}\sum_{i=0}^{n}\left|\left|\psi_i(p_0,q^*)\right|\right|^2=&\sum_{i=0}^{n}\left(s_i^2+s_{i-1}^2+\cdots+s_{i-p_0+1}^2+u_i^2+u_{i-1}^2+\cdots+u_{i-q^*+1}^2\right)\\ \leqslant&\sum_{i=0}^{n}2\left[y_i^2+\left(\frac{\varepsilon}{2}\right)^2+y_{i-1}^2+\left(\frac{\varepsilon}{2}\right)^2+\cdots+y_{i-p_0+1}^2+\left(\frac{\varepsilon}{2}\right)^2\right]+\sum_{i=0}^{n}\left(u_i^2+u_{i-1}^2+\cdots+u_{i-q^*+1}^2\right)\\ \leqslant&2\sum_{i=0}^{n}\left(y_i^2+y_{i-1}^2+\cdots+y_{i-p_0+1}^2\right)+\left(\frac{p_0\varepsilon^2}{2}+q^*\delta^2\right)(n+1)\\ \leqslant&\left(2p_0\hat{c}+\frac{p_0\varepsilon^2}{2}+q^*\delta^2\right)(n+1)\,.\end{aligned}$$

So, as $n\to\infty$, there exists a constant $c_3>0$ such that

$$\begin{aligned}\lambda_{\max}^{(p_0,q^*)}(n)=&\left|\left|\sum_{i=0}^{n}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right|\right|\\ \leqslant&\left|\left|\sum_{i=0}^{n}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)\right|\right|+||I||\\ \leqslant&\sum_{i=0}^{n}\left|\left|\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)\right|\right|+1\\ =&\sum_{i=0}^{n}\left|\left|\psi_i(p_0,q^*)\right|\right|^2+1\\ \leqslant&\left(2p_0\hat{c}+\frac{p_0\varepsilon^2}{2}+q^*\delta^2+1\right)(n+1)\,,a.s.\end{aligned}\tag{A1}$$

Let $c_3=2p_0\hat{c}+\frac{p_0\varepsilon^2}{2}+q^*\delta^2+1$. This completes the proof.

Appendix B Proof of Lemma 2

From (13) and (14) we know that

$$\bar{\theta}(p_0,q^*)=\left[-a_1,\cdots,-a_{p_0},b_1,\cdots,b_{q_0},0,\cdots,0\right]^\top,\tag{B1}$$

and from (4), (5), (7), (13) and (B1) it can be seen that

$$s_{n+1}=\theta^\top(p_0,q_0)\psi_n(p_0,q_0)+w_{n+1}+\epsilon_{n+1}=\bar{\theta}^\top(p_0,q^*)\psi_n(p_0,q^*)+w_{n+1}+\epsilon_{n+1}.\tag{B2}$$

From (15) we know that

$$\theta_n(p_0,q^*)=\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right)^{-1}\sum_{i=0}^{n-1}\psi_i(p_0,q^*)s_{i+1}=P_n(p_0,q^*)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)s_{i+1}.\tag{B3}$$

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From (15) and (B2) the estimated parameter error can be written as

$$\begin{aligned}\tilde{\theta}_n(p_0,q^*) =&\bar{\theta}(p_0,q^*)-\theta_n(p_0,q^*)\\ =&\bar{\theta}(p_0,q^*)-P_n(p_0,q^*)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(\psi_i^\top(p_0,q^*)\bar{\theta}(p_0,q^*)+w_{i+1}+\epsilon_{i+1}\right)\\ =&\bar{\theta}(p_0,q^*)-P_n(p_0,q^*)\left(P_n^{-1}(p_0,q^*)-I\right)\bar{\theta}(p_0,q^*)-P_n(p_0,q^*)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)w_{i+1}-P_n(p_0,q^*)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\epsilon_{i+1}.\end{aligned}\tag{B4}$$

So,

$$\tilde{\theta}_n(p_0,q^*)=P_n(p_0,q^*)\bar{\theta}(p_0,q^*)-P_n(p_0,q^*)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)w_{i+1}-P_n(p_0,q^*)\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\epsilon_{i+1}.\tag{B5}$$

From (7) we have

$$\begin{aligned}&\left|\left|\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right)^{-\frac{1}{2}}\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\right|\right|^2\\ =&\left|\left|\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\right)^\top P_n(p_0,q^*)\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\right)\right|\right|.\end{aligned}\tag{B6}$$

From Assumption 5 we know that

$$\lambda_{\min}^{(p_0,q^*)}(n)\geqslant c_1 n,a.s.\tag{B7}$$

So, from (B5)-(B7) and Theorem 1 of [2] it can be seen that

$$\begin{aligned}&\left|\left|\left(\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\psi_i^\top(p_0,q^*)+I\right)^{-\frac{1}{2}}\sum_{i=0}^{n-1}\psi_i(p_0,q^*)\left(w_{i+1}+\epsilon_{i+1}\right)\right|\right|^2\\ =&\left|\left|\left(\bar{\theta}(p_0,q^*)-P_n^{-1}(p_0,q^*)\tilde{\theta}_n(p_0,q^*)\right)^\top P_n(p_0,q^*)\left(\bar{\theta}(p_0,q^*)-P_n^{-1}(p_0,q^*)\tilde{\theta}_n(p_0,q^*)\right)\right|\right|\\ \leqslant&2\tilde{\theta}_n^\top(p_0,q^*)P_n^{-1}(p_0,q^*)\tilde{\theta}_n(p_0,q^*)+2\bar{\theta}^\top(p_0,q^*)P_n(p_0,q^*)\bar{\theta}(p_0,q^*)\\ \leqslant&c'+(1+p_0c)\,\varepsilon n+O\left(\log\lambda_{\max}^{(p_0,q^*)}(n-1)\right)+O\left(\left|\left|\bar{\theta}(p_0,q^*)\right|\right|^2\frac{1}{c_1 n}\right)\\ =&c'+(1+p_0c)\,\varepsilon n+O\left(\log n\right)+o\left(1\right)\\ =&\left(1+p_0c\right)\varepsilon n+o\left(n\right),a.s.,\end{aligned}\tag{B8}$$

where c' is a constant, and its definition can be found in Theorem 1 of [2]. This completes the proof.

Remark 1. $2\tilde{\theta}_n^\top(p_0,q^*)P_n^{-1}(p_0,q^*)\tilde{\theta}_n(p_0,q^*)\leqslant c'+(1+p_0c)\,\varepsilon n+O\left(\log\lambda_{\max}^{(p_0,q^*)}(n-1)\right)$ in (B8) is similar with that in Theorem 1 of [2]. To be more concrete, from (13) and (14), we just need to treat 0 in (13) as parameters to be estimated.

Appendix C Proof of Lemma 3

From (15) we get

$$\theta_n(p,q^*)=\left(\sum_{i=0}^{n-1}\psi_i(p,q^*)\psi_i^\top(p,q^*)+I\right)^{-1}\sum_{i=0}^{n-1}\psi_i(p,q^*)s_{i+1}.\tag{C1}$$

So,

$$\left|\left|\theta_n(p,q^*)\right|\right|\leqslant\left|\left|\left(\sum_{i=0}^{n-1}\psi_i(p,q^*)\psi_i^\top(p,q^*)+I\right)^{-1}\right|\right|\left|\left|\sum_{i=0}^{n-1}\psi_i(p,q^*)s_{i+1}\right|\right|.\tag{C2}$$

From Assumption 5 it can be seen that

$$\left|\left|\left(\sum_{i=0}^{n-1}\psi_i(p,q^*)\psi_i^\top(p,q^*)+I\right)^{-1}\right|\right|\leqslant\frac{1}{c_1 n},a.s.,\tag{C3}$$

By (C2), (C3) and Assumptions 1, 2 it can be seen that $||\theta_n(p,q^*)||$ is bounded (a.s.).

From (13), (14) and Assumption 3 we know that $\left|\left|\bar{\theta}(p_0,q^*)\right|\right|$ is bounded.

So, there is a constant γ such that

$$\left|\left|\tilde{\theta}_n(p)\right|\right|=\left|\left|\bar{\theta}(p_0,q^*)-\hat{\theta}_n(p)\right|\right|\leqslant||\bar{\theta}(p_0,q^*)||+\left|\left|\hat{\theta}_n(p)\right|\right|=||\bar{\theta}(p_0,q^*)||+||\theta_n(p,q^*)||\leqslant\gamma,a.s.\tag{C4}$$

This completes the proof.

References

1 Goodwin G C, Sin K S. Adaptive Filtering Prediction and Control. NJ: Prentice Hall, 1984
2 Jing L D. Quantized-output-based least squares of ARX systems. Asian Journal of Control, 2023, 25: 1971-1980