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Non-trivial consensus control on directed signed networks

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Abstract This paper tackles the challenge of achieving non-trivial consensus control on directed signed networks. Although (interval) bipartite consensus and trivial consensus are convergence states on signed networks that have been extensively studied, non-trivial consensus represents a novel convergence state that has not been fully explored yet. In order to drive group states on a directed signed network to reach non-trivial consensus, a control technique based on linear system theory and our graph decomposition framework is proposed. Our technique operates under mild connectivity conditions and does not impose restrictions on whether the network is structurally balanced or unbalanced. Moreover, the consensus value can be preset arbitrarily as required. Based on matrix perturbation theory, the impact of the antagonistic interactions on the non-trivial consensus speed in signed networks with external control is further studied. Numerical simulations validate the effectiveness of our approach and theoretical framework. The work in this paper demonstrates that a group with coopetitive (cooperative-competitive) interactions under control can achieve consensus, which was previously considered exclusive to fully cooperative groups.

Keywords non-trivial consensus, signed networks, linear system, graph decomposition, matrix perturbation theory

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1 Introduction

With the rapid development of automatic control and wireless communication, multi-agent systems have emerged as a focal point in the area of networked control and optimization. This field contains various research directions including consensus [1, 2], synchronization [3, 4], and topology identification [5, 6]. The study of consensus problems holds significant practical importance in real-world scenarios, such as enabling distributed situation awareness in unmanned aerial vehicle (UAV) swarm systems [7], optimizing control strategies for battery storage system in DC shipboard microgrids [8], facilitating economic power dispatch in smart grids [9], and modeling opinion dynamics in social networks [10, 11]. Consensus control for multi-agent systems consisting of cooperative agents has attracted much attention and achieved fruitful results [12–14]. These systems are typically characterized by unsigned graphs with positive edge weights.

However, it is noteworthy that antagonistic interactions are prevalent in natural environment, unmanned aerial vehicle systems, and social networks. Therefore, the study of consensus control for signed networks, where edge weights can be negative, is of great significance. Previous studies have extensively explored the bipartite consensus control for structurally balanced signed networks [15–18], as well as the stability (trivial consensus [19]) control for structurally unbalanced signed networks [18]. Nevertheless, to the best of our knowledge, the challenge of non-trivial consensus [19] control, specifically, to drive a group of coopetitive (cooperative-competitive) agents connected by a directed signed graph to one desired non-zero consensus state, remains unsolved.

The structural balance property indicates that the agents in the system can be partitioned into two subgroups, where interactions within each subgroup are cooperative, while interactions between the two subgroups are antagonistic. A classic example of structural balance is observed in two-party political systems [20]. However, not all relationships are necessarily balanced. In a complete, multi-relational social network of a society consisting of the 300000 odd players of a massive multiplayer online game [21], different types of "signed triads" relationships—including "$++-$ triads" and "$---$ triads" coexist. This makes it impossible to partition all the agents into two opposing subgroups according to the principles of structural balance. Moreover, in insecure network environments, unbalanced communication topologies frequently emerge in bidirectional formation of unmanned vehicles

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and robots [22]. As studied in previous studies [10, 18, 23], the typical outcome for a structurally balanced signed network is bipartite consensus, namely all agents converge to a consensus value that is the same for all agents except for the sign. Conversely, for a structurally unbalanced signed network, the entire system eventually reaches trivial consensus, i.e., all agents' states converge to zero. From the perspective of opinion dynamics, the above outcomes suggest that the entire group splits into two camps with completely opposite opinions on a structurally balanced network, while on a structurally unbalanced network, all agents become indifferent to the topic in the end. Consequently, an intriguing question arises from the aforementioned discussion. Can a group of agents within a directed signed network reach a non-zero consensus state that is not only identical in value but also in sign, through the application of external controls, regardless of whether this signed network is structurally balanced or not? Furthermore, how can a controller be designed to facilitate the convergence of these agents towards a shared non-zero state, which can be preset arbitrarily as needed?

The practical significance of realizing non-trivial consensus on signed networks lies in both areas of unmanned aerial vehicle networks and social networks. To be specific, consider the deliberate attack on the communication channels of a group of cooperative unmanned aerial vehicles [24, 25]. Such an attack may bring new antagonistic relationships, resulting in a coopetitive UAV network, whose interaction topology can be characterized by a signed network. In this case, the original consensus states before the attack will be disrupted, and the states of all the vehicles may converge to different values. By realizing non-trivial consensus control of signed networks, we propose a theoretical scheme that brings such attacked UAV groups back to consensus, in other words, the disruption caused by the attack is offset. Another research background comes from opinion manipulation problems in the area of social networks [26–28]. Trust and mistrust relationships coexist in social networks, which commonly leads to opinion separation among countries, political parties and opinionated individuals [10]. Studying non-trivial consensus on signed networks gives rise to an opinion manipulation strategy, through which a group of agents that contains mistrust relationships can be driven to follow one shared opinion in a unanimous way, rather than polarize or become indifferent to the topics in the end.

Non-trivial consensus control on signed networks can be seen as an extension of traditional consensus control in the field of multi-agent systems, moving from unsigned networks to the more complex case of signed networks. Traditional consensus control [12–14] typically requires the network to be unsigned, a fundamental precondition. However, the presence of negative edges in signed networks introduces significant challenges for achieving consensus. In practical scenarios, negative edges represent antagonistic interactions in UAV groups or mistrust relationships in social networks, both of which are clearly detrimental to consensus formation.

In [19], where the concept of non-trivial consensus for signed networks was proposed, non-trivial consensus can only be established in the case of (essentially) cooperative network. In [16], which concerns undirected signed networks, necessary and sufficient conditions for the non-trivial consensus control problem to be solvable were derived. Nevertheless, this theoretical result does not provide specific controller design formulas, and requires the network under investigation to be connected and structurally balanced.

Motivated by the preceding discussion, this article primarily focuses on the non-trivial consensus control problem for directed signed networks, without restriction on the network property of structural balance or unbalance. Our work fills the gap in the related research on consensus control of directed signed networks. The theoretical results derived here can be applied to the practice of unmanned aerial vehicle network formation control in automatic control and opinion manipulation in social networks.

Antagonistic interactions, once introduced to a cooperative network with external control, shall make it scarcely possible for the agents to exactly reach consensus with the control signal. One can say that, in terms of convergence state, these antagonistic interactions hindered the trend of the system reaching consensus. It is our second objective in this article to analyse the effect of these antagonistic interactions from another perspective: convergence speed, which is related to spectral properties of the grounded Laplacian matrix [29, 30]. Our work in this part essentially draws a universal framework to analyse convergence speed variation trends when regulating edge weights in cooperative networks with external control, particularly, the case of introducing new antagonistic interactions is our major innovation.

Formally, the main contributions of this article, along with comparisons with some existing related studies, are summarized as follows.

(1) Non-trivial consensus control technique for directed signed networks is developed. This result demonstrates that a group with coopetitive (cooperative-competitive) interactions under control can achieve consensus, which was previously considered exclusive to fully cooperative groups. Specifically, the appropriate informed agents set is selected, the specific and formulaic external control signal and the coupling weights are designed, to facilitate the convergence of the agents on a directed signed network towards a shared non-zero state, which can be preset arbitrarily as needed. In contrast, in the general case of interval bipartite consensus [31] on directed signed networks,

the precise convergence states are usually diverse and unavailable. To achieve this, a novel graph decomposition framework is proposed, which includes the idea to decompose a leader-follower signed network into its positive subgraph and followers subgraph. The topology assumption in our work originates from a practical scenario in unmanned aerial vehicles, ensuring real-world practicality of our non-trivial consensus control technique, which holds potential for extensions in UAV coordination in automatic control and opinion manipulation in social networks. The connectivity conditions in our technique are more relaxed, compared with most existing results on consensus control, which often require strong connectivity [16–18] or spanning tree conditions [1, 2]. Notably, our technique imposes no restriction on structural balance or unbalance, broadening its applicability.

(2) A universal framework is first proposed to analyse how convergence speed varies when adjusting edge weights in a directed unsigned network with external control. This framework holds its generality in applicability for all types of edge weights regulation, with particular emphasis on the introduction of negative edges. In contrast, prior work in [29] is restricted on regulating edges weights within the scope of positive weights, and the distributed neighbor selection technique for faster convergence in [32] only considers edge disconnection operation, excluding the case of adding negative edge(s). Based on this framework, the effect on non-trivial consensus speed of all the antagonistic interactions in signed networks is further studied. Specifically, it is proved that, in a signed network with strongly-connected agents and external control, the antagonistic interactions within a certain range lead to faster non-trivial consensus. The analysis method can be applied to derive the theoretical results analogous to those in [29, 32], demonstrating the universality of our framework.

2 Notations and preliminaries

Throughout the paper, we denote $\mathbf{1}_N = [1, 1, \dots, 1]^\top \in \mathbb{R}^N$ and $\mathbf{0}_N = [0, 0, \dots, 0]^\top \in \mathbb{R}^N$. For a vector $\boldsymbol{B} = [b_1, b_2, \dots, b_N]^\top \in \mathbb{R}^N$, the notation $\mathrm{diag}(\boldsymbol{B})$ refers to the diagonal matrix with diagonal elements $b_1, b_2, \dots, b_N$, and $|\boldsymbol{B}| = [|b_1|, |b_2|, \dots, |b_N|]^\top \in \mathbb{R}^N$ represents the absolute value of vector $\boldsymbol{B}$. The set of all $N \times N$ matrices on the complex number field is denoted by $M_N(\mathbb{C})$, or shortly, M_N. The function $\mathrm{sgn}(\cdot)$ represents the standard sign function. A signed network comprising N agents is described by a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$, where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ denotes the vertex set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} = \{(v_i, v_j) | v_i, v_j \in \mathcal{V}\}$ denotes the edge set, and $\boldsymbol{\mathcal{A}} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ such that $a_{ij} \neq 0$ if and only if $(v_j, v_i) \in \mathcal{E}$. Here, a_{ij} represents the weight of edge (v_j, v_i) and may be either positive or negative. We say vertex v_i is reachable from vertex v_j if there exists a directed path starting at v_j and ending at v_i in graph $\mathcal{G}$.

Generally, in a fully autonomous network (FAN) [32] of N agents interacting with each other over a signed digraph $\mathcal{G}$, the dynamics of each agent i can be described as

$$\dot{\boldsymbol{x}}_i(t) = \sum_{j=1}^{N} |a_{ij}| \left(\mathrm{sgn}\left(a_{ij}\right) \boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\right),\ i = 1, \dots, N, \tag{1}$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^n$ is the state of agent i at time t. The signed Laplacian matrix $\boldsymbol{L}(\mathcal{G}) = [l_{ij}] \in \mathbb{R}^{N\times N}$ corresponding to a signed digraph $\mathcal{G}$, or $\boldsymbol{L}$ for simplicity, is defined as [23]

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i, \\ \displaystyle\sum_{k=1, k\neq i}^{N} |a_{ik}|, & j = i. \end{cases}$$

3 Problem statement

Let $\boldsymbol{L} \in \mathbb{R}^{N\times N}$ represent the signed Laplacian matrix of the signed digraph $\mathcal{G}$ of the original FAN system in our study. The original FAN system is represented as

$$\dot{\boldsymbol{x}}(t) = -(\boldsymbol{L} \otimes \boldsymbol{I}_n)\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x}(t) = \left[\boldsymbol{x}_1^\top(t), \dots, \boldsymbol{x}_N^\top(t)\right]^\top \in \mathbb{R}^{nN}$ is the state of the N agents. In fact, Eq. (2) describes the entire system under dynamics (1).

It is a common observation that the state $\boldsymbol{x}(t)$ in the signed system (2) reaches (interval) bipartite consensus or converges to zero, when $\mathcal{G}$ is strongly-connected or contains a spanning tree [10, 23, 31]. The primary focus in this paper is to realize non-trivial consensus control for system (2), i.e., driving these N agents towards a non-zero consensus state, where the consensus state value can be arbitrarily configured as needed. The standard definition of non-trivial consensus control shall be given afterward.

To achieve this, the external control input will be exerted to some specified agents in the original FAN system (2)

$$\dot{\boldsymbol{x}}_i(t)=\sum_{j=1}^{N}|a_{ij}|\left(\operatorname{sgn}\left(a_{ij}\right)\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t)\right)+\sum_{l=1}^{m}|b_{il}|(\operatorname{sgn}(b_{il})\boldsymbol{u}_l-\boldsymbol{x}_i(t)),\ i=1,\ldots,N, \tag{3}$$

where $\boldsymbol{u}_l\in\mathbb{R}^n$ is the lth external control signal, $b_{il}\in\mathbb{R}$ is the coefficient indicating the coupling weight from $\boldsymbol{u}_l$ to $\boldsymbol{x}_i$, which can be positive, negative or zero. Dynamics (3) establishes a semiautonomous network (SAN) [32], in which a subset of agents (referred to as informed agents [33]) are selected to receive external control signals to steer the entire network toward the desired non-zero consensus state. The vertex set of informed agents is denoted as $\mathcal{V}_{\mathcal{I}}$, where for each agent i, $v_i\in\mathcal{V}_{\mathcal{I}}$ if and only if $b_{il}\neq 0$ for some $l\in\{1,\ldots,m\}$. The naive vertex [33] set is $\mathcal{V}_{\mathcal{N}}=\mathcal{V}\setminus\mathcal{V}_{\mathcal{I}}$. It is time now to give the specific definition of non-trivial consensus control in our framework.

Definition 1 (Non-trivial consensus control)**.** Under the control dynamics (3), if there exists $b_{il}\in\mathbb{R},\boldsymbol{u}_l\in\mathbb{R}^n$ for $i=1,\ldots,N$ and $l=1,\ldots,m$, such that

$$\lim_{t\to+\infty}\boldsymbol{x}_i(t)=\boldsymbol{\theta}\in\mathbb{R}^n,\ i=1,\ldots,N,$$

where $\boldsymbol{\theta}\neq\mathbf{0}_n$ is the preset consensus state, then we say that the non-trivial consensus control for the signed SAN system (3) is realized.

In our study, one time-invariant external control signal is sufficient to realize non-trivial consensus for the original system (2). In the following part of this paper, the control protocol under investigation is thus

$$\dot{\boldsymbol{x}}_i(t)=\sum_{j=1}^{N}|a_{ij}|\left(\operatorname{sgn}\left(a_{ij}\right)\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t)\right)+|b_i|\left(\operatorname{sgn}(b_i)\boldsymbol{x}_0-\boldsymbol{x}_i(t)\right),\ i=1,\ldots,N, \tag{4}$$

where $\boldsymbol{x}_0\in\mathbb{R}^n$ represents the state of the sole external control signal, which is time-invariant. $b_i\in\mathbb{R},\ i=1,\ldots,N$ indicates the coupling weight from $\boldsymbol{x}_0$ to $\boldsymbol{x}_i$. Correspondingly, the vertex set of informed agents is denoted as $\mathcal{V}_{\mathcal{I}}$, where for each agent i, $v_i\in\mathcal{V}_{\mathcal{I}}$ if and only if $b_i\neq 0$. The naive vertex set is $\mathcal{V}_{\mathcal{N}}=\mathcal{V}\setminus\mathcal{V}_{\mathcal{I}}$.

Given a desired non-trivial consensus state $\boldsymbol{\theta}\neq\mathbf{0}_n$ for agents $\boldsymbol{x}_i,\ i=1,\ldots,N$, the main task of this paper is to realize non-trivial consensus control by appropriately designing the external control signal $\boldsymbol{x}_0$ and the coupling weights $b_i,\ i=1,\ldots,N$, including the basic problem of selecting the informed agents set $\mathcal{V}_{\mathcal{I}}\subseteq\mathcal{V}$ to receive external control signal.

The time-invariant external control signal $\boldsymbol{x}_0$ can be viewed as an individual agent from an augmented system perspective. In this way, the SAN (4) can be equivalently written in the form of FAN:

$$\dot{\boldsymbol{z}}_i(t)=\sum_{j=1}^{\widehat{N}}|\widehat{a}_{ij}|\left(\operatorname{sgn}\left(\widehat{a}_{ij}\right)\boldsymbol{z}_j(t)-\boldsymbol{z}_i(t)\right),\ i=1,\ldots,\widehat{N} \tag{5}$$

with the compact form being

$$\dot{\boldsymbol{z}}(t)=-\left(\widehat{\boldsymbol{L}}\otimes\boldsymbol{I}_n\right)\boldsymbol{z}(t), \tag{6}$$

where $\boldsymbol{z}_i(t)=\boldsymbol{x}_i(t)$ for $i=1,\ldots,N$, $\boldsymbol{z}_{\widehat{N}}(t)\equiv\boldsymbol{x}_0$, and $\boldsymbol{z}(t)=[\boldsymbol{z}_1^\top(t),\ldots,\boldsymbol{z}_N^\top(t),\boldsymbol{z}_{\widehat{N}}^\top(t)]^\top$. The augmented system (5) (or (6)) is endowed with the augmented signed graph $\widehat{\mathcal{G}}=(\widehat{\mathcal{V}},\widehat{\mathcal{E}},\widehat{\boldsymbol{\mathcal{A}}})$ and the augmented signed Laplacian matrix $\widehat{\boldsymbol{L}}\in\mathbb{R}^{\widehat{N}\times\widehat{N}}$, in which $\widehat{\boldsymbol{\mathcal{A}}}=[\widehat{a}_{ij}]\in\mathbb{R}^{\widehat{N}\times\widehat{N}}$, for clarity. In this paper, for the augmented graph $\widehat{\mathcal{G}}=(\widehat{\mathcal{V}},\widehat{\mathcal{E}},\widehat{\boldsymbol{\mathcal{A}}})$, we denote $\widehat{\mathcal{V}}=\{\widehat{v}_1,\ldots,\widehat{v}_N,\widehat{v}_0\}$, in which $\widehat{v}_1,\ldots,\widehat{v}_N$ and $\widehat{v}_0$ correspond to the N agents $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N$ and the external control signal $\boldsymbol{x}_0$ in (4), respectively. Clearly, $\widehat{N}=N+1$, and

$$\widehat{\boldsymbol{L}}=\begin{bmatrix}\boldsymbol{L}+\operatorname{diag}(|\boldsymbol{B}|) & -\boldsymbol{B}\\ \mathbf{0}_{1\times N} & 0\end{bmatrix}, \tag{7}$$

where $\boldsymbol{B}=[b_1,\ldots,b_N]$, and the part $\boldsymbol{L}_{\boldsymbol{B}}=\boldsymbol{L}+\operatorname{diag}(|\boldsymbol{B}|)$ is referred to as the grounded Laplacian matrix [29] or the perturbed Laplacian [32], whose spectral properties play a central role in determining the convergence speed of the augmented system (6).

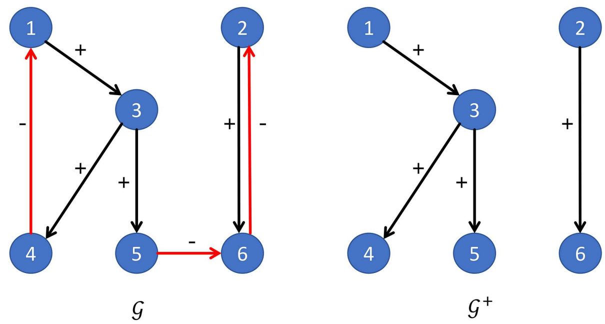

Figure 1 (Color online) An illustrative example of Assumption 1 (structurally unbalanced $\mathcal{G}$).

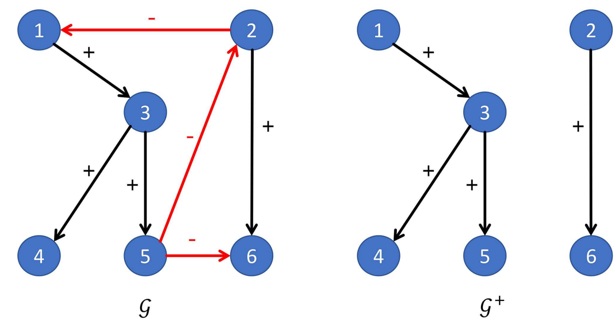

Figure 2 (Color online) An illustrative example of Assumption 1 (structurally balanced $\mathcal{G}$).

For simplicity, in the following analysis, we take the dimension of each agent $\boldsymbol{z}_i \in \mathbb{R}^n$ (as well as agent $\boldsymbol{x}_i \in \mathbb{R}^n$) as $n = 1$. The conclusions for the case of $n \geqslant 2$ can be obtained accordingly by taking a micro view of each individual element of $\boldsymbol{z}_i \in \mathbb{R}^n$ (and $\boldsymbol{x}_i \in \mathbb{R}^n$), and is thus omitted.

Before ending this section, an indispensable Lemma is proposed.

Lemma 1 ([31])**.** By the decomposition of $\widehat{\boldsymbol{L}}$ in (7), if the signed digraph $\widehat{\mathcal{G}}$ has a spanning tree, then all the eigenvalues of $\boldsymbol{L} + \mathrm{diag}(|\boldsymbol{B}|)$ have positive real parts.

4 Main results

4.1 Graph decomposition and positive subgraph

To begin with, we introduce our graph decomposition framework, which commences with the definition of positive subgraph.

Definition 2 (Positive subgraph)**.** For a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$ with Laplacian matrix $\boldsymbol{L}$, the positive subgraph associated with $\mathcal{G}$ is $\mathcal{G}^+ = \left(\mathcal{V}^+, \mathcal{E}^+, \boldsymbol{\mathcal{A}}^+\right)$, where $\mathcal{V}^+ = \mathcal{V}$, $\mathcal{E}^+ \subseteq \mathcal{E}$ is a subset of $\mathcal{E}$ that consists of all the positive edges in $\mathcal{E}$. The Laplacian matrix of $\mathcal{G}^+$ is denoted as $\boldsymbol{L}^+$.

Notice that according to the above definition, signed digraph $\mathcal{G}$ and its positive subgraph $\mathcal{G}^+$ have the same vertex set $\mathcal{V}$, and $\mathcal{G}^+ = \mathcal{G}$ if and only if $\mathcal{G}$ has no negative edge. To achieve non-trivial consensus control of the N agents in (4), Assumption 1 is necessary.

Assumption 1. The vertex set $\mathcal{V}$ of signed digraph $\mathcal{G}$ and its positive subgraph $\mathcal{G}^+$ can be partitioned into two nonoverlapping subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, such that

(1) In $\mathcal{G}$, one of the following holds:

- for any vertex v_i in $\mathcal{V}_1$, there exists at least one negative edge pointing to v_i;
- there exists no negative edge in $\mathcal{G}$;

(2) In $\mathcal{G}^+$, for any vertex v_j in $\mathcal{V}_2$, there exists at least one vertex v_k in $\mathcal{V}_1$ such that v_j is reachable from v_k.

To explain Assumption 1, consider graph $\mathcal{G}$ and its positive subgraph $\mathcal{G}^+$ in Figure 1 or 2, in which the 6 vertices can be partitioned into two subsets $\mathcal{V}_1 = \{v_1, v_2\}$ and $\mathcal{V}_2 = \{v_3, v_4, v_5, v_6\}$. Clearly, such partition satisfies (1) and (2) in Assumption 1.

From the perspective of practical significance, the proposal of Assumption 1 originates from the unmanned aircraft systems [24] in reality. Consider the practical scenario when the communication channels of a group of cooperative unmanned aerial vehicles organized by a spanning tree topology (correspondingly, we require that "in $\mathcal{G}^+$, for any vertex v_j in $\mathcal{V}_2$, there exists at least one vertex v_k in $\mathcal{V}_1$ such that v_j is reachable from v_k" in (2) of Assumption 1 come under attack. To break the consensus states, such deliberate attack may be especially targeted at those leaders vehicles and brings new antagonistic interactions (correspondingly, we require that "for any vertex v_i in $\mathcal{V}_1$, there exists at least one negative edge pointing to v_i" in (1) of Assumption 1), as shown in Figure 3. Such attack belongs to the category of interference and forgery of signals in availability attack [25], since according to the paradigm of dynamics on signed network (1), introducing new antagonistic interaction from agent j to agent i can be viewed as forging a control signal that is opposite of the real state of agent j, and delivering this fake signal to agent i.

After the attack, this UAV group is no longer a completely collaborative unit. Accordingly, their states may tend to polarize or reach trivial consensus, which is meaningless since all agents' states converge to zero. Our work in this

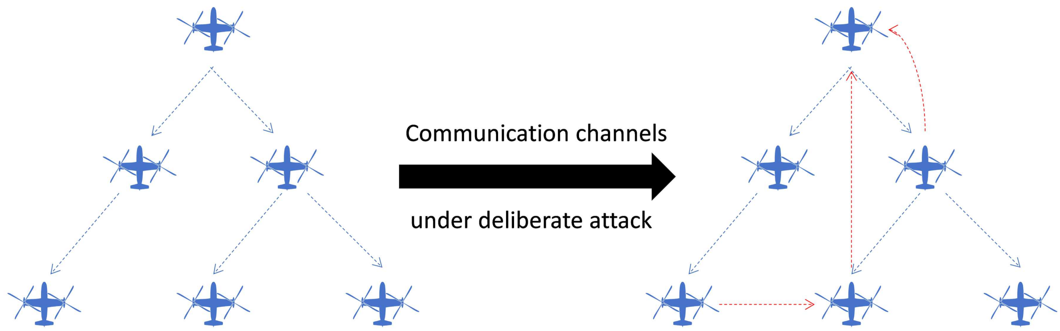

Figure 3 (Color online) A practical interaction topology satisfying Assumption 1. Blue dashed lines represent cooperative interactions, and red dashed lines represent antagonistic interactions.

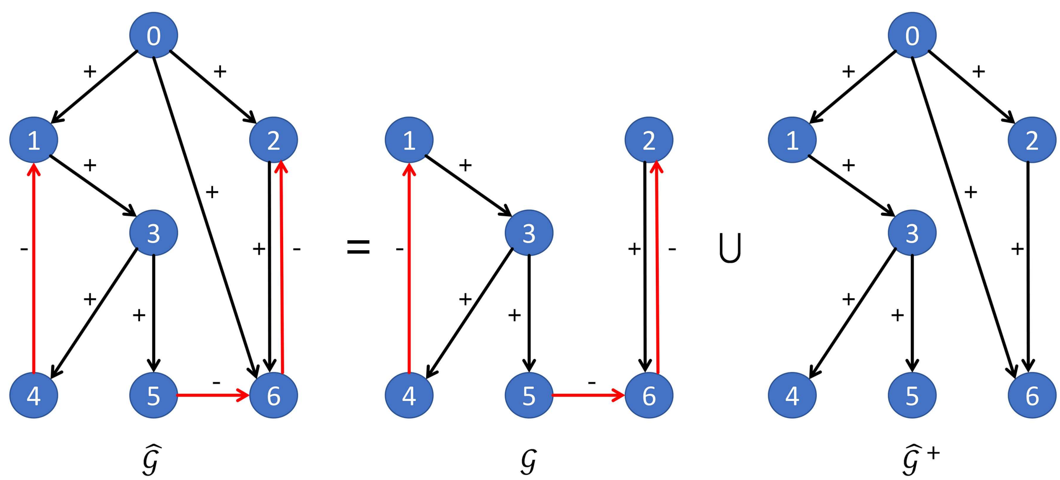

Figure 4 (Color online) An illustrative example of graph decomposition.

paper can be applied to deal with such corrupted UAV networks, to be specific, our control technique drives these agents in the new coopetitve (cooperative-competitive) network to reach a non-trivial consensus, in which all agents tend to one shared non-zero consensus state, just like the original status of the network before being attacked.

Remark 1. The signed digraph $\mathcal{G}$ satisfying Assumption 1 is not necessarily structurally unbalanced, as shown in Figure 2. In other words, Assumption 1 holds generality in its applicability for both structurally balanced and unbalanced signed digraphs.

Remark 2. The above assumption on network topology indicates milder connectivity conditions in our control techique, compared with most of the existing studies on consensus control, where strong connectivity (connectivity for undirected graph) [17, 18] or spanning tree condition [1] is often required.

Definition 3 (Graph decomposition)**.** Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$ and a sequence of graphs $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i, \boldsymbol{\mathcal{A}_i})$, $i = 1, \ldots, m,\ m \geqslant 2$. If $\mathcal{G}$ is the union of $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$, meaning that $\mathcal{V} = \bigcup_{i=1}^{m} \mathcal{V}_i$ and $\mathcal{E} = \bigcup_{i=1}^{m} \mathcal{E}_i$, then we say that $\mathcal{G}$ can be decomposed into m subgraphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$, which is represented as $\mathcal{G} = \bigcup_{i=1}^{m} \mathcal{G}_i$.

Based on the above definitions, we decompose the graph $\widehat{\mathcal{G}}$ of the augmented system (6) into the graph $\mathcal{G}$ of the original system (2) and the positive subgraph $\widehat{\mathcal{G}}^{+}$, denoted as $\widehat{\mathcal{G}} = \mathcal{G} \bigcup \widehat{\mathcal{G}}^{+}$. Figure 4 provides an illustrative example. The detailed topology of augmented graph $\widehat{\mathcal{G}}$ will be explicated in Subsection 4.2. Based on our graph decomposition framework, we are now in a position to establish our main result of non-trivial consensus control on directed signed networks.

4.2 Non-trivial consensus control on directed signed networks

Recall the control protocol (4) under investigation with the basic underlying topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$, and recall the non-trivial consensus control task illustrated in Definition 1. Suppose that Assumption 1 holds for $\mathcal{G}$, then the vertex set $\mathcal{V}$ can be partitioned into $\mathcal{V}_1 = \{v_1, \ldots, v_{N_1}\}$ and $\mathcal{V}_2 = \{v_{N_1+1}, \ldots, v_N\}$ after appropriately reordering, and the vertex set consisting of every vertex that has incoming negative edge(s) in $\mathcal{G}$ can be denoted as $\mathcal{U} = \{v_1, \ldots, v_{N_1}, v_{N_1+1}, \ldots, v_{N_1+P}\}$, if there exists negative edge(s) in $\mathcal{E}$. For each vertex v_i, denote Ω_i as the vertex set consisting of all the neighbor vertices of v_i that has outgoing negative edge pointing to v_i in $\mathcal{G}$. Then one has the following theorem.

Theorem 1. Consider the signed SAN (4), or equivalently in the FAN form (5), with the basic underlying topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$ in (1) satisfying Assumption 1. Let the coupling weights $b_i,\ i \in \mathcal{V}_\mathcal{I}$ have the same sign. If there exists negative edge(s) in $\mathcal{E}$, the informed agents set is chosen as $\mathcal{V}_\mathcal{I} = \mathcal{U}$; otherwise, $\mathcal{V}_\mathcal{I} = \mathcal{V}_1$. If for every $i \in \mathcal{V}_\mathcal{I}$,

$$\frac{b_i x_0}{|b_i| + 2\sum_{k\in\Omega_i} |a_{ik}|} = \theta, \tag{8}$$

in which $\theta \neq 0$ is the desired consensus state, then

$$\lim_{t\to+\infty} x_i(t) = \theta,\ i = 1, \ldots, N, \tag{9}$$

which indicates that the non-trivial consensus control for the N agents in signed SAN system (4) is accomplished.

Proof. Since x_0 is the sole time-invariant control signal and $b_i, i \in \mathcal{V}_\mathcal{I}$ have the same sign, one can assume without loss of generality that

$$b_i > 0,\ i \in \mathcal{V}_\mathcal{I}. \tag{10}$$

For the special case where $\mathcal{G}$ is unsigned, non-trivial consensus (9) can be established by setting $x_0 = \theta$. In the following proof, we assume that $\mathcal{G}$ has at least one negative edge. Analogous to (7), for the two unsigned Laplacian matrices $\widehat{\boldsymbol{L}}^+$ and $\boldsymbol{L}^+$, one has

$$\widehat{\boldsymbol{L}}^+ = \begin{bmatrix} \boldsymbol{L}^+ + \mathrm{diag}(\boldsymbol{B}) & -\boldsymbol{B} \\ \boldsymbol{0}_{1\times N} & 0 \end{bmatrix}.$$

Write (5) in the compact form

$$\dot{\boldsymbol{z}}(t) = -\widehat{\boldsymbol{L}}\boldsymbol{z}(t) = -\begin{bmatrix} \boldsymbol{L} + \mathrm{diag}(|\boldsymbol{B}|) & -\boldsymbol{B} \\ \boldsymbol{0}_{1\times N} & 0 \end{bmatrix} \boldsymbol{z}(t) \tag{11}$$

with $\boldsymbol{z}(t) = [x_1(t), \ldots, x_n(t), x_0(t)]^\top$.

Under Assumption 1, $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}^+$ both have a spanning tree with leader $\widehat{v}_0$ being the root. By Lemma 1, the eigenvalues of $\boldsymbol{L} + \mathrm{diag}(|\boldsymbol{B}|)$ and $\boldsymbol{L}^+ + \mathrm{diag}(\boldsymbol{B})$ all have positive real parts. As a result, $\boldsymbol{L} + \mathrm{diag}(|\boldsymbol{B}|)$ and $\boldsymbol{L}^+ + \mathrm{diag}(\boldsymbol{B})$ are two invertible matrices. Therefore, $\widehat{\boldsymbol{L}} \in \mathbb{R}^{\widehat{N}\times\widehat{N}}$ has rank $N = \widehat{N} - 1$, rendering the right and left eigenvectors of $\widehat{\boldsymbol{L}}$ corresponding to zero eigenvalue falling into a pair of one-dimensional subspaces, each spanned by a non-zero vector. According to [31], Eq. (11) yields

$$\lim_{t\to+\infty} \boldsymbol{z}(t) = \boldsymbol{w}_r\boldsymbol{w}_l^\top \boldsymbol{z}(0) = \left(\boldsymbol{w}_l^\top \boldsymbol{z}(0)\right)\boldsymbol{w}_r,$$

where $\boldsymbol{w}_l$ and $\boldsymbol{w}_r$ are the normalized left and right eigenvectors of $\widehat{\boldsymbol{L}}$ associated with its zero eigenvalue, such that $\boldsymbol{w}_l^\top\widehat{\boldsymbol{L}} = \boldsymbol{0}_{\widehat{N}}^\top$, $\widehat{\boldsymbol{L}}\boldsymbol{w}_r = \boldsymbol{0}_{\widehat{N}}$ and $\boldsymbol{w}_l^\top\boldsymbol{w}_r = 1$. Specifically, since

$$\boldsymbol{z}(0) = \left(z_1(0), \ldots, z_N(0), z_{\widehat{N}}(0)\right)^\top = \left(z_1(0), \ldots, z_N(0), x_0\right)^\top,$$

take $\boldsymbol{w}_l^\top = (0, \ldots, 0, \frac{1}{x_0})$ and $\boldsymbol{w}_r = (\theta, \ldots, \theta, x_0)^\top$, then

$$\lim_{t\to+\infty} \boldsymbol{z}(t) = \boldsymbol{w}_r = (\theta, \ldots, \theta, x_0)^\top,$$

which includes the desired outcome (9). It is clear that $\boldsymbol{w}_l^\top\boldsymbol{w}_r = 1$, and $\boldsymbol{w}_l^\top\widehat{\boldsymbol{L}} = \boldsymbol{0}_{\widehat{N}}^\top$ since the last row of $\widehat{\boldsymbol{L}}$ is $\boldsymbol{0}_{\widehat{N}}^\top$. In the following, we prove that

$$\begin{aligned} \boldsymbol{0}_{\widehat{N}} &= \widehat{\boldsymbol{L}}\boldsymbol{w}_r \\ &= \begin{bmatrix} \boldsymbol{L} + \mathrm{diag}\left(|\boldsymbol{B}|\right) & -\boldsymbol{B} \\ \boldsymbol{0}_{1\times N} & 0 \end{bmatrix} \begin{bmatrix} \theta\boldsymbol{1}_N \\ x_0 \end{bmatrix} \\ &= \begin{bmatrix} |b_1| + \sum_{k\neq 1}^N |a_{1k}| & \cdots & -a_{1N} & -b_1 \\ \vdots & \ddots & \vdots & \vdots \\ -a_{N1} & \cdots & |b_N| + \sum_{k\neq N}^N |a_{Nk}| & -b_N \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \vdots \\ \theta \\ x_0 \end{bmatrix}. \end{aligned} \tag{12}$$

Denote $\widehat{\boldsymbol{L}} = \left[\widehat{\boldsymbol{\alpha}}_1^\top; \ldots; \widehat{\boldsymbol{\alpha}}_{\widehat{N}}^\top\right]$, where $\widehat{\boldsymbol{\alpha}}_i^\top$ represents the ith row of $\widehat{\boldsymbol{L}}$. By the selection of informed agents set $\mathcal{V}_{\mathcal{I}} = \mathcal{U} = \{v_1, \ldots, v_{N_1}, v_{N_1+1}, \ldots, v_{N_1+P}\}$ and the designed condition of external control signal x_0 and coupling weight b_i in (8),

$$\begin{aligned}\widehat{\boldsymbol{\alpha}}_i^\top \boldsymbol{w}_r &= \left(|b_i| + \sum_{k\neq i}^{N} |a_{ik}| - \sum_{k\neq i}^{N} a_{ik}\right)\theta - b_i x_0 \\ &= \left(b_i + 2\sum_{k\in\Omega_i} |a_{ik}|\right)\theta - b_i x_0 \\ &= 0, \quad i = 1, \ldots, N_1 + P.\end{aligned} \tag{13}$$

Denote $\widehat{\boldsymbol{L}}^{+} = \left[\widehat{\boldsymbol{\alpha}}_1^{+\top}; \ldots; \widehat{\boldsymbol{\alpha}}_{\widehat{N}}^{+\top}\right]$. Since $\widehat{\mathcal{G}}^{+}$ is an unsigned digraph that contains a spanning tree, according to [1], the system $\dot{\boldsymbol{x}}(t) = -\widehat{\boldsymbol{L}}^{+}\boldsymbol{x}(t)$ reaches consensus. Equivalently, each row $\widehat{\boldsymbol{\alpha}}_i^{+\top}$ of submatrix $\widehat{\boldsymbol{L}}^{+}$ admits the eigenvector $\theta\mathbf{1}_{\widehat{N}} \in \mathbb{R}^{\widehat{N}}$. Notice that for $i = N_1 + P + 1, \ldots, \widehat{N}$, the $\widehat{N}$th elements of $\widehat{\boldsymbol{\alpha}}_i^{+\top}$ are all zero, one has

$$\widehat{\boldsymbol{\alpha}}_i^{+\top} \boldsymbol{w}_r = 0, \quad i = N_1 + P + 1, \ldots, \widehat{N}.$$

Recall that in our setting, the informed agents set $\mathcal{V}_{\mathcal{I}} = \mathcal{U} = \{v_1, \ldots, v_{N_1+P}\}$ includes all the vertices in $\widehat{\mathcal{G}}$ that has incoming negative edges, therefore, $\widehat{\boldsymbol{\alpha}}_i^\top = \widehat{\boldsymbol{\alpha}}_i^{+\top}$ for $i = N_1 + P + 1, \ldots, \widehat{N}$, and further

$$\widehat{\boldsymbol{\alpha}}_i^\top \boldsymbol{w}_r = 0, \quad i = N_1 + P + 1, \ldots, \widehat{N}. \tag{14}$$

Combining (13) with (14), one has $\widehat{\boldsymbol{L}}\boldsymbol{w}_r = \mathbf{0}_{\widehat{N}}$, which fulfills the proof of (12).

Remark 3. Given a certain original FAN system (2) with determinate topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$, given the determinate external control signal x_0, the consensus state θ can be viewed as an outcome varying with coupling weights $b_i,\ i \in \mathcal{V}_{\mathcal{I}}$. From (8) one has

$$\left|\frac{\theta(b_i)}{x_0}\right| = \left|\frac{b_i}{|b_i| + 2\sum_{k\in\Omega_i}|a_{ik}|}\right| < 1,\ i \in \mathcal{V}_{\mathcal{I}}. \tag{15}$$

Furthermore

$$\lim_{b_i\to+\infty} \frac{\theta(b_i)}{x_0} = 1,\ i \in \mathcal{V}_{\mathcal{I}},$$

and

$$\lim_{b_i\to-\infty} \frac{\theta(b_i)}{x_0} = -1,\ i \in \mathcal{V}_{\mathcal{I}}.$$

Remark 3 highlights that, as the positive coupling strength $b_i, i \in \mathcal{V}_{\mathcal{I}}$ getting strong, the consensus state of the N agents in (4) gradually approaches the external control signal. Meanwhile, Eq. (15) is consistent with the theoretical results in [31]. The augmented signed graph $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{\boldsymbol{\mathcal{A}}})$ contains a spanning tree, where $\widehat{v}_0$ serves as the sole root vertex. According to [31], the states of the N followers $x_1, \ldots, x_N$ shall converge and lie in the interval $[-x_0, x_0]$, which is determined by the state of the sole leader.

Remark 4. Eq. (8) integrally provides the design of both the external control signal x_0 and coupling weights $b_i,\ i \in \mathcal{V}_{\mathcal{I}}$. Through condition (10), one can equivalently design x_0 and $b_i,\ i \in \mathcal{V}_{\mathcal{I}}$ in a separate manner:

$$x_0 = \rho\theta,\ b_i = \frac{2}{\rho - 1}\sum_{k\in\Omega_i}|a_{ik}|,\ i \in \mathcal{V}_{\mathcal{I}},$$

in which ρ is a constant that can be arbitrarily chosen from interval $(1, +\infty)$.

Remark 5. Recall the conditions for signed digraph $\mathcal{G}$ and its positive subgraph $\mathcal{G}^{+}$ in Assumption 1. From Theorem 1 and its proof one can see that, the key idea of our non-trivial consensus control technique is to "balance" the negative impact of the antagonistic interactions on consensus by exerting "positive" external control, and meanwhile ensuring accessibility of the external control signal for all the agents, which explains significance of Assumption 1 from the perspective of technical realization.

Theorem 1 develops a universal framework that enables agents on a directed signed network to reach non-trivial consensus under mild connectivity conditions, regardless of whether this network is structurally balanced or not, whereas the existing work in the field of consensus control on signed networks mainly focus on bipartite consensus control [16–18] and stability control [19]. In [19], where the concept of non-trivial consensus for signed networks was proposed, non-trivial consensus can only be established in the case of (essentially) cooperative network. In [16], which concerns undirected signed networks, necessary and sufficient conditions for non-trivial consensus control problem to be solvable were derived. Nevertheless, this theoretical result does not contain specific and formulaic controller design method, and requires the network under investigation to be connected and structurally balanced.

As mentioned in Subsection 4.1, the topology condition in Assumption 1 of Theorem 1 originates from the practical scenario of deliberate attacks. These attacks introduce new antagonistic interactions into the communication channels of a cooperative UAV group, causing the group states to polarize or reach a trivial consensus. However, the destructive effect of the attack on consensus can be counteracted by the control technique proposed in Theorem 1. In the context of opinion dynamics, Theorem 1 indicates that, when a dominant outside leader comes to issue proper commands, the agents that are coupled by a signed network satisfying Assumption 1 tend to unanimously follow the opinion of the dominant leader.

4.3 Antagonistic interactions lead to faster consensus

Based on Theorem 1, the impact of negative edges on consensus in a signed network is counteracted. Following the network topology and coupling weights setting in Theorem 1 and its proof, let us take another view of our graph decomposition framework: $\widehat{\mathcal{G}} = \mathcal{G} \bigcup \widehat{\mathcal{G}}^{+}$, then we can see $\widehat{\mathcal{G}}$ as the signed rooted graph obtained by adding negative edges to the unsigned rooted graph $\widehat{\mathcal{G}}^{+}$. Compare the two systems on $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}^{+}$:

$$\dot{\boldsymbol{z}}(t) = -\widehat{\boldsymbol{L}}\boldsymbol{z}(t) = -\begin{bmatrix} \boldsymbol{L} + \mathrm{diag}(|\boldsymbol{B}|) & -\boldsymbol{B} \\ \boldsymbol{0}_{1\times N} & 0 \end{bmatrix} \boldsymbol{z}(t), \tag{16}$$

and

$$\dot{\boldsymbol{z}}^{+}(t) = -\widehat{\boldsymbol{L}}^{+}\boldsymbol{z}^{+}(t) = -\begin{bmatrix} \boldsymbol{L}^{+} + \mathrm{diag}(\boldsymbol{B}) & -\boldsymbol{B} \\ \boldsymbol{0}_{1\times N} & 0 \end{bmatrix} \boldsymbol{z}^{+}(t), \tag{17}$$

both of which realized non-trivial consensus control for the N agents, according to Theorem 1 and Definition 1. In terms of convergence state, these negative edges weakened the consensus state of the related system, i.e., from complete consensus [1] in (17) to the final state in (16) where the N agents can only approach but cannot exactly follow the external control signal as the control coupling strength increasing, which is indicated by Theorem 1 and Remark 3. In the following, we shall analyse the effects of these negative edges from the perspective of consensus speed. To begin with, one essential Lemma is introduced.

Lemma 2 ([34])**.** Let $\boldsymbol{A}, \boldsymbol{E} \in M_N$ and suppose that λ is a simple eigenvalue of $\boldsymbol{A}$. Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be, respectively, right and left eigenvectors of $\boldsymbol{A}$ corresponding to λ. Then

(a) for each given $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $t \in \mathbb{C}$ such that $|t| < \delta$, there is a unique eigenvalue $\lambda(t)$ of $\boldsymbol{A} + t\boldsymbol{E}$ such that $\left|\lambda(t) - \lambda - t\frac{\boldsymbol{\eta}^{\top}\boldsymbol{E}\boldsymbol{\xi}}{\boldsymbol{\eta}^{\top}\boldsymbol{\xi}}\right| \leqslant |t|\varepsilon$;

(b) $\lambda(t)$ is continuous at $t = 0$, and $\lim_{t\to 0}\lambda(t) = \lambda$;

(c) $\lambda(t)$ is differentiable at $t = 0$, and

$$\left.\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\right|_{t=0} = \frac{\boldsymbol{\eta}^{\top}\boldsymbol{E}\boldsymbol{\xi}}{\boldsymbol{\eta}^{\top}\boldsymbol{\xi}}.$$

Consider a general directed unsigned network with external control:

$$\dot{\boldsymbol{z}}(t) = -\widehat{\boldsymbol{\mathcal{L}}}\boldsymbol{z}(t), \tag{18}$$

where

$$\widehat{\boldsymbol{\mathcal{L}}} = \begin{bmatrix} \boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}} & \boldsymbol{\mathcal{L}}_{12} \\ \boldsymbol{\mathcal{L}}_{21} & \boldsymbol{\mathcal{L}}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{L}} + \mathrm{diag}(\boldsymbol{\mathcal{B}}) & -\boldsymbol{\mathcal{B}} \\ \boldsymbol{0}_{1\times N} & 0 \end{bmatrix},$$

in which $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}} = \boldsymbol{\mathcal{L}} + \mathrm{diag}(\boldsymbol{\mathcal{B}})$ is called the grounded Laplacian matrix of $\widehat{\boldsymbol{\mathcal{L}}}$. Denote $\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}})$ as the eigenvalue of $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}$ that has the smallest real part, which is a measure of the network (18)'s convergence speed [29, 30].

Regarding this critical indicator, various researches have been made by studying the spectral properties of the grounded Laplacian matrix. The bounds on the smallest eigenvalue of grounded Laplacian matrices were provided for unsigned networks with bidirectional interactions, utilizing Rayleigh quotient inequality and the interlacing theorem for symmetric Laplacian matrices of undirected graphs [35, 36]. Spectral analysis has been performed for a directed unsigned network to reveal the variation trend of $\lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}})$ when weakening or strengthening some specific edges' weights [29], based on tools from nonnegative matrix theory. For structurally balanced signed networks, the distributed neighbor selection technique was derived to ensure faster convergence [32]. Nevertheless, this distributed neighbor selection technique only considers the edge disconnection operation, excluding the case of adding negative edge(s), which is of our particular interest.

The work in this subsection essentially establishes a universal framework to analyse convergence speed variation trend when regulating edge weights in directed unsigned networks with external control. We primarily focus on the case of adding new negative edge(s). The unidirectional interactions render tools like the Rayleigh quotient inequality and the interlacing theorem for symmetric Laplacian matrices of undirected graph not applicable here, and the nonnegative matrix theory becomes insufficient due to the emergence of new negative edge(s).

Let us first review an existing result induced from Theorem 2 in [29], this result is restricted on unsigned networks and positive edges.

Lemma 3. For the unsigned network (18), assume the grounded Laplacian matrix $\boldsymbol{\mathcal{L}_{\mathcal{B}}}$ is irreducible. Take a positive right eigenvector $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_N)^\top$ corresponding to $\lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}})$, for any edge (v_j, v_i) in the network, denote $\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}}$ as the grounded Laplacian matrix of the new network after changing the edge weight a_{ij} into a^{δ}_{ij}. Assume $\xi_i < \xi_j$, one has

(i) if $a^{\delta}_{ij} > a_{ij} > 0$, then $\lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}}) > \lambda(\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}}) > 0$;

(ii) if $a_{ij} > a^{\delta}_{ij} > 0$, then $\lambda(\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}}) > \lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}}) > 0$.

Lemma 3 indicates that, for an unsigned network, the positive right eigenvector of $\lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}})$ encodes the vital information to regulate convergence speed by adjusting edge weight. It will be shown that this eigenvector also plays an important role in our theoretical analysis.

Theorem 2. For unsigned network with external control (18), assume that $\boldsymbol{\mathcal{L}_{\mathcal{B}}}$ is irreducible. The new grounded Laplacian matrix $\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}}$ is obtained by adding a negative edge from vertex v_j to vertex v_i in $\mathcal{G}(\widehat{\boldsymbol{\mathcal{L}}})$, $i, j \in \{1, \dots, N\}$, and the newly introduced negative edge's weight is denoted as $a^{\delta}_{ij} = -\delta < 0$, where δ is a positive real number. Then

$$\left.\frac{\mathrm{d}\lambda(\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}})}{\mathrm{d}\delta}\right|_{\delta=0} > 0. \tag{19}$$

Proof. Denote the new signed Laplacian matrix $\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}} = \boldsymbol{\mathcal{L}_{\mathcal{B}}} + \delta \boldsymbol{E}$, in which $\boldsymbol{E}$ is the perturbation matrix that declares the location of newly introduced negative edge(s), then

$$\boldsymbol{E} = \begin{array}{cc} & \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & 1 & & \\ & & & \ddots & \vdots & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \end{array}.$$

Assume that α is a sufficiently large number such that $\boldsymbol{P} = -\boldsymbol{\mathcal{L}_{\mathcal{B}}} + \alpha \boldsymbol{I_N}$ is a nonnegative matrix. Since $\boldsymbol{\mathcal{L}_{\mathcal{B}}}$ is irreducible, From Perron-Frobenius theorem [34] we know that $\rho(\boldsymbol{P})$ is a simple eigenvalue and there exists positive vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ such that $\boldsymbol{P}\boldsymbol{\xi} = \rho(\boldsymbol{P})\boldsymbol{\xi}$ and $\boldsymbol{\eta}^\top \boldsymbol{P} = \rho(\boldsymbol{P})\boldsymbol{\eta}^\top$. Since $\boldsymbol{\mathcal{L}_{\mathcal{B}}} = \alpha \boldsymbol{I}_n - \boldsymbol{P}$, $\lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}}) = \alpha - \rho(\boldsymbol{P})$ is a simple eigenvalue of $\boldsymbol{\mathcal{L}_{\mathcal{B}}}$, and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are the corresponding right and left eigenvectors, respectively.

Based on the above results, according to Lemma 2, one has

$$\lim_{\delta \to 0} \lambda\left(\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}}\right) = \lambda(\boldsymbol{\mathcal{L}_{\mathcal{B}}}),$$

and

$$\left.\frac{\mathrm{d}\lambda(\boldsymbol{\mathcal{L}}^{\delta}_{\boldsymbol{\mathcal{B}}})}{\mathrm{d}\delta}\right|_{\delta=0} = \frac{\boldsymbol{\eta}^\top \boldsymbol{E} \boldsymbol{\xi}}{\boldsymbol{\eta}^\top \boldsymbol{\xi}}.$$

Normalize eigenvectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, such that $\boldsymbol{\eta}^\top \boldsymbol{\xi} = 1$, then

$$\left.\frac{\mathrm{d}\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}^{\delta})}{\mathrm{d}\delta}\right|_{\delta=0} = \eta_i(\xi_i + \xi_j) > 0,$$

since $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are both positive eigenvectors. This completes the proof.

Remark 6. If there is more than one negative edge added, the corresponding perturbation matrix $\boldsymbol{E}$ would remain a nonnegative matrix with elements 0 and 1, and thus Eq. (19) still holds.

The eigenvalue with the smallest real part $\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}})$ of the grounded Laplacian matrix $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}$ measures the convergence speed of network (18). Therefore, Theorem 2 and Remark 6 indicate that introducing antagonistic interactions in a cooperative network with external control and strongly-connected agents leads to faster convergence.

Remark 7. The above result relies on the assumption that $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}$ is irreducible. From the proof of Theorem 2, one can find that generally, as long as $\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}})$ is a simple eigenvalue of $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}$ and the corresponding left and right eigenvectors are positive, then Eq. (19) still holds.

Remark 8. Technically, Eq. (19) indicates that, under mild perturbation, i.e., when the modulus δ of the added negative edge's weight is sufficiently small, the convergence speed would be improved. Nevertheless, our simulation results in the next section show that, as the perturbation getting strong, the convergence speed in general maintains a growing trend.

Essentially, the proof of Theorem 2 establishes a universal framework to analyse convergence speed variation trends for all types of edge weights regulation in directed unsigned networks with external control. Theoretical results analogous to Lemma 3 above, and Theorem 9 in [32] can be derived by applying the same method in this framework, using different selections of the perturbation matrix. This demonstrates the universality of our framework. Detailed explanation is omitted for brevity.

Back to the original objective of this subsection, which is to determine the effect on non-trivial consensus speed of the antagonistic interactions in network (16). Based on Theorem 2, the conclusion is summarized as follows.

Corollary 1. Assume that the network (16) on $\widehat{\mathcal{G}}$ has at least one negative edge, and the modulus of these negative edges' weights is upper bounded by $\delta > 0$. Assume that $\mathcal{G}^+$ is strongly-connected. Then there exists a positive constant C such that if $\delta < C$, then

$$\mathrm{Re}\left[\lambda\left(\boldsymbol{L}_{\boldsymbol{B}}^{+}\right)\right] < \mathrm{Re}\left[\lambda\left(\boldsymbol{L}_{\boldsymbol{B}}\right)\right].$$

Remark 9. Based on Theorem 2 and Remark 7, the assumption "$\mathcal{G}^+$ is strongly-connected" in Corollary 1 can be theoretically relaxed to: $\lambda(\boldsymbol{L}^+)$ is a simple eigenvalue of $\boldsymbol{L}^+$, and the corresponding left and right eigenvectors are positive. However, in practice, the relaxed assumption is difficult to verify, which involves matrix operations with high complexity, complex number processing and the challenge of numerical stability. Therefore, we retain the original "$\mathcal{G}^+$ is strongly-connected" assumption in Corollary 1 for conciseness and practicality.

Theorem 1 and Corollary 1 together reveal the role of the antagonistic interactions in our non-trivial consensus control framework. It is proven that the antagonistic interactions within a certain range lead to faster non-trivial consensus, at the cost of losing the ability for the agents to fully align with the external control signal.

5 Simulation

In this section, two numerical simulations are carried out first to illustrate the effectiveness of our control technique proposed in Theorem 1, with the underlying topology shown in Figures 1 and 2. For simplicity, the time-invariant external control signal is set as $x_0 = 1$.

5.1 Non-trivial consensus control on structurally unbalanced signed networks

Consider original FAN system (1) and the corresponding SAN (4) under control coefficients setting (8) in Theorem 1, with the basic underlying structurally unbalanced network topology $\mathcal{G}$ satisfying Assumption 1 and the corresponding augmented system topology $\widehat{\mathcal{G}}$ depicted in Figure 4. The augmented Laplacian matrix, which includes

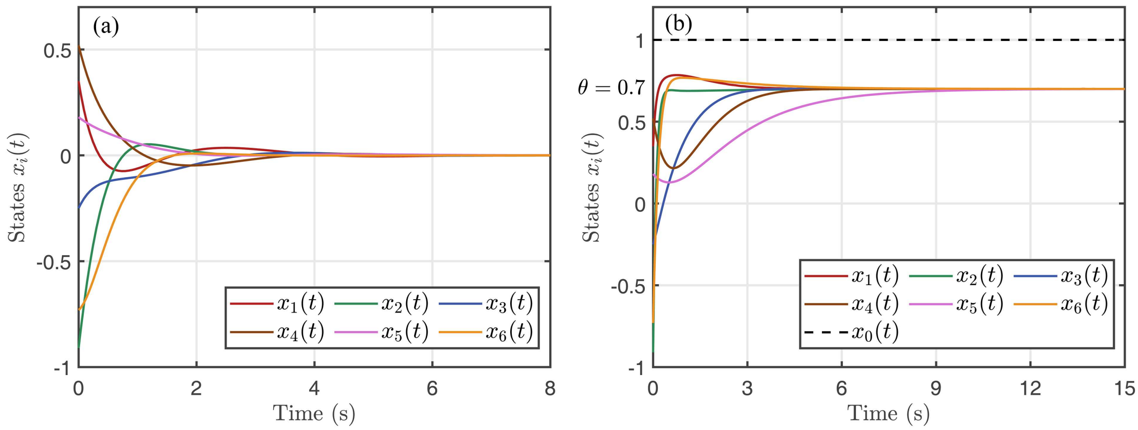

Figure 5 (Color online) (a) States evolution of the original FAN system (1) on a structurally unbalanced signed network; (b) states evolution of the corresponding SAN system (4) on a structurally unbalanced signed network.

both topology information for $\mathcal{G}$ and $\widehat{\mathcal{G}}$, is

$$\widehat{\boldsymbol{L}} = \begin{bmatrix} 2+2\left|\dfrac{2\theta}{1-\theta}\right| & 0 & 0 & 2 & 0 & 0 & -2\dfrac{2\theta}{1-\theta} \\ 0 & 1.5+1.5\left|\dfrac{2\theta}{1-\theta}\right| & 0 & 0 & 0 & 1.5 & -1.5\dfrac{2\theta}{1-\theta} \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.5 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 3+\left|\dfrac{2\theta}{1-\theta}\right| & -\dfrac{2\theta}{1-\theta} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Take the desired consensus state $\theta = 0.7$, the evolution of both FAN and SAN is shown in Figure 5.

5.2 Non-trivial consensus control on structurally balanced signed networks

Consider original FAN system (1) and the corresponding SAN (4) under control coefficients setting (8) in Theorem 1, with the basic underlying structurally balanced network topology $\mathcal{G}$ satisfying Assumption 1 and the corresponding augmented system topology $\widehat{\mathcal{G}}$ depicted in Figure 6. The augmented Laplacian matrix is

$$\widehat{\boldsymbol{L}} = \begin{bmatrix} 2+2\left|\dfrac{2\theta}{1-\theta}\right| & 2 & 0 & 0 & 0 & 0 & -2\dfrac{2\theta}{1-\theta} \\ 0 & 1.5+1.5\left|\dfrac{2\theta}{1-\theta}\right| & 0 & 0 & 1.5 & 0 & -1.5\dfrac{2\theta}{1-\theta} \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.5 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 3+\left|\dfrac{2\theta}{1-\theta}\right| & -\dfrac{2\theta}{1-\theta} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Take the desired consensus state $\theta = 0.7$, the evolution of both FAN and SAN is shown in Figure 7. Simulation results shown in Figure 7, together with that in Figure 5, verified the applicability of our control technique for both structurally unbalanced and balanced signed networks.

5.3 Antagonistic interactions lead to faster non-trivial consensus

This subsection visualizes the conclusion illustrated in Subsection 4.3. Take the topology in Figure 8 for network (18).

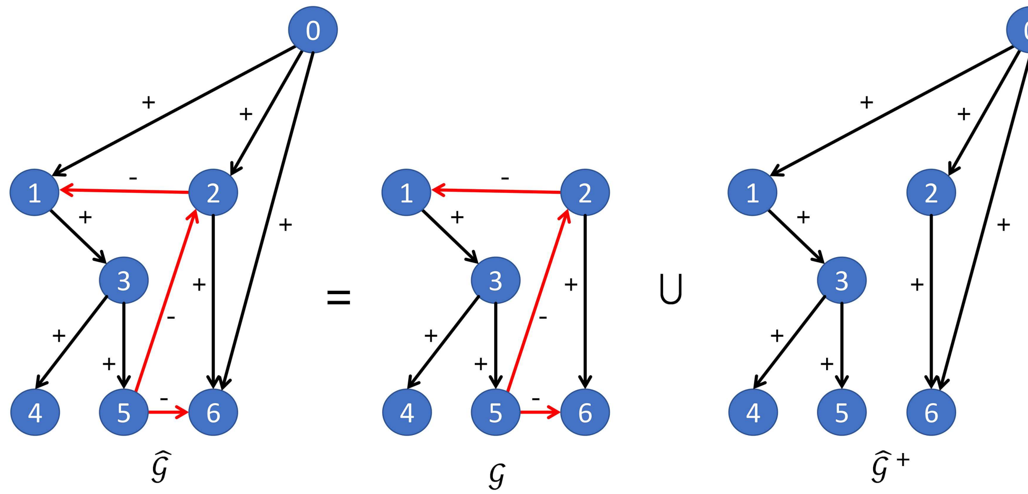

Figure 6 (Color online) Graph decomposition framework for structurally balanced basic topology $\mathcal{G}$.

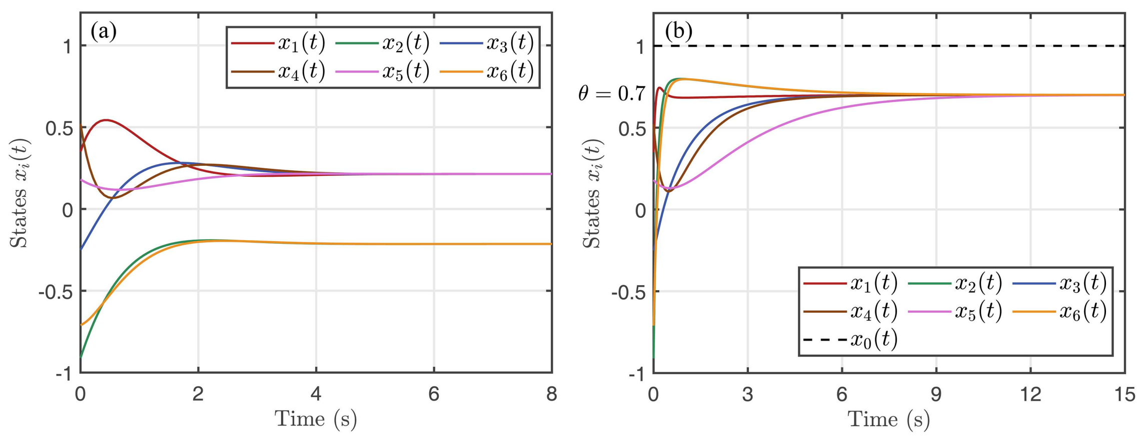

Figure 7 (Color online) (a) States evolution of the original FAN system (1) on a structurally balanced signed network; (b) states evolution of the corresponding SAN system (4) on a structurally balanced signed network.

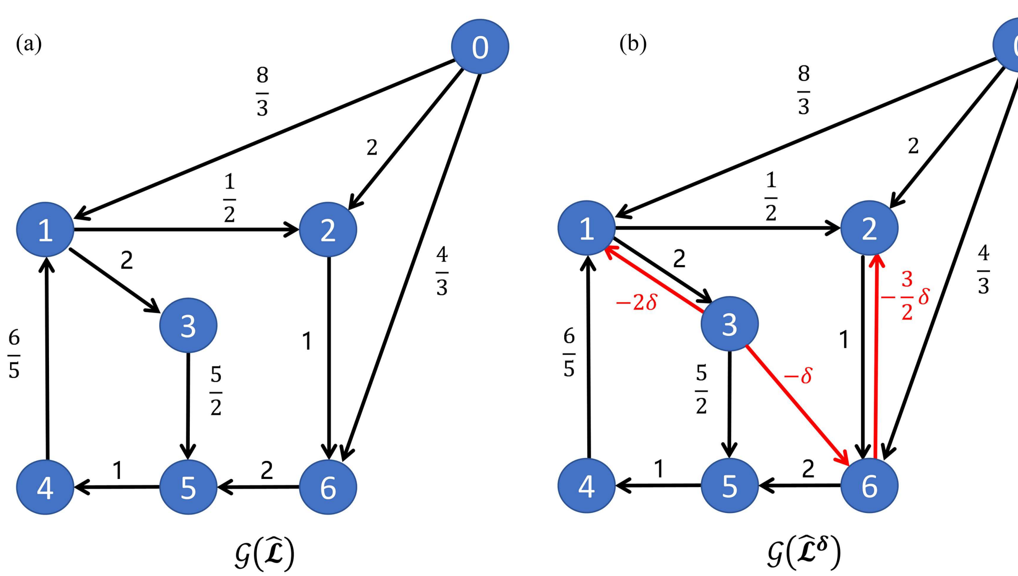

Figure 8 (Color online) (a) Topology of the original unsigned network with external control (18); (b) a case of perturbed topology by choosing the negative edges combination $2(v_3, v_1) + \frac{3}{2}(v_6, v_2) + 1(v_3, v_6)$.

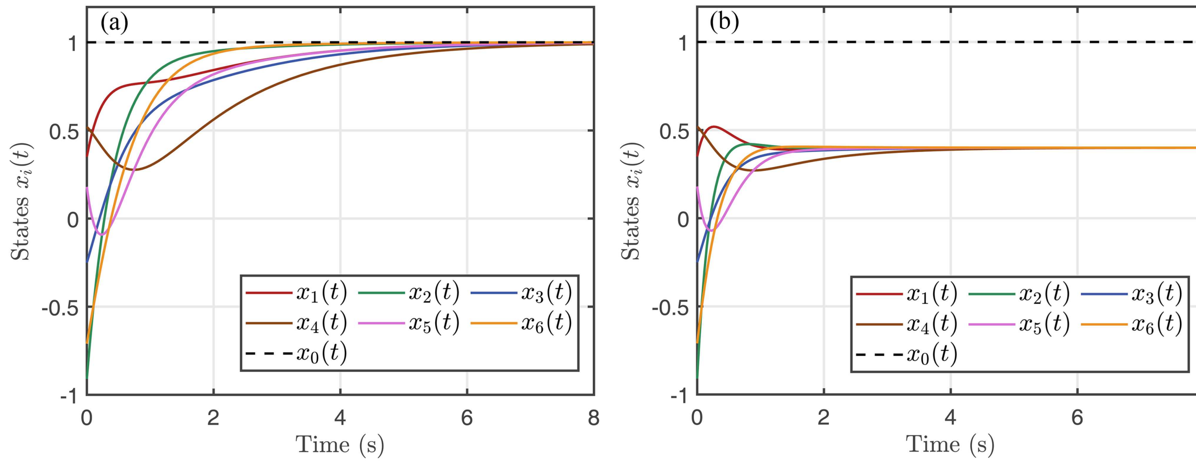

Figure 9 (Color online) (a) States evolution of the original unsigned system on $\mathcal{G}(\widehat{\boldsymbol{\mathcal{L}}})$. (b) States evolution of the perturbed system on $\mathcal{G}(\widehat{\boldsymbol{\mathcal{L}}}^{\delta})$ with $\delta = 1$. The black dotted line in each plot represents external inputs. The topology of each system is taken from Figure 8.

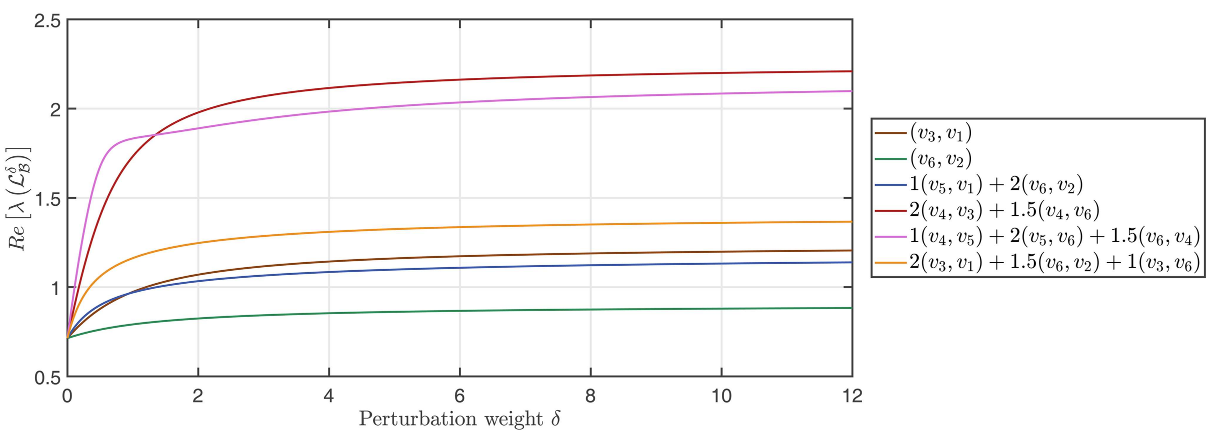

Figure 10 (Color online) Variation of $\mathrm{Re}[\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}^{\delta})]$ regarding δ under several selections of location to set the nagative edge(s) at $\mathcal{G}(\widehat{\boldsymbol{\mathcal{L}}})$ in Figure 8. The coefficient written ahead of (v_j, v_i), multiplied by δ, is exactly the modulus of the newly introduced negative edge (v_j, v_i)'s weight after perturbation.

As one can see from Figure 9, the non-trivial consensus speed is improved on the perturbed system, which includes newly introduced negative edges. Further, by selecting several different locations to set the negative edge(s), one obtains the variation trend of $\mathrm{Re}[\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}^{\delta})]$ regarding perturbation weight δ under each selection, as shown in Figure 10, in which $\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}^{\delta} = \boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}} + \delta \boldsymbol{E}$, and $\boldsymbol{E}$ is the perturbation matrix that declares the location of newly introduced negative edge(s). It is shown that as the new negative edge(s) introduced, with relatively small perturbation weight δ, $\mathrm{Re}[\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}})]$ immediately increases to $\mathrm{Re}[\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}^{\delta})] > \mathrm{Re}[\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}})]$. Moreover, as the perturbation getting stronger, the variation of $\mathrm{Re}[\lambda(\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{B}}}^{\delta})]$ in general maintains growing trend.

6 Conclusion

In this paper, a non-trivial consensus control technique for directed signed networks is established under mild connectivity conditions, without restriction on the network property of structural balance or unbalance. This paper serves as one pioneering and significant work in the field of consensus control on directed signed networks, as achieving non-trivial consensus is a rare and challenging result for signed networks, irrespective of the presence of external control. The theoretical results derived here can be applied to address deliberate attack on UAV groups in automatic control and opinion manipulation in social networks. Further, the effect on non-trivial consensus speed of the antagonistic interactions among agents is studied based on matrix perturbation theory. Numerical simulations are presented to verify our technique and theory.

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