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Data-driven control handling noisy input-state data and noisy input-output data: a survey of trends and techniques

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Abstract Designing controllers directly from measurement data has attracted growing attention in recent years, as it avoids the need for accurate system modeling or explicit system identification. This paper focuses on recent advances in data-driven control for linear discrete-time systems with unknown system matrices. For noisy input-state data, an in-depth analysis is provided on several representative approaches, including data-driven control based on Willems et al.'s fundamental lemma, quadratic matrix inequalities, linear fractional transformations for combining prior knowledge with data, and integral quadratic constraints. For noisy input-output data, a concise review is presented on control methods based on quadratic matrix inequalities, along with key insights into their structure and implications. The paper concludes by outlining several challenging problems that merit further investigation in future research.

Keywords data-driven control, noisy input-state data, noisy input-output data, quadratic matrix inequality, matrix S-lemma, integral quadratic constraint

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1 Introduction

Data-driven control (DDC) has witnessed remarkable development over the past few decades, fueled by advancements in computational capabilities, sensor technologies, and machine learning [1–7]. In contrast to traditional model-based control, which depends on an explicitly derived mathematical model of a system, DDC utilizes online or offline data to design controllers without requiring detailed system identification. The foundations of DDC can be traced to early techniques, such as iterative feedback tuning (IFT) [5, 6] and virtual reference feedback tuning (VRFT) [7], which aimed to optimize controller parameters directly from experimental data. In the early 2000s, subspace identification methods gained prominence by enabling the estimation of state-space models from noisy measurements for control design purposes. A pivotal development came with Willems et al.'s fundamental lemma [8], which demonstrated that sufficiently rich input and output data could implicitly capture linear system dynamics, paving the way for direct DDC approaches such as data-enabled predictive control (DeePC) [9].

The advantages of DDC are manifold, making it especially attractive for systems where obtaining accurate models is difficult or impractical [10–13]. First, DDC bypasses the often labor-intensive and error-prone process of system identification, thereby reducing development time and engineering effort. For instance, in industrial processes with high level of uncertainty or unmodeled dynamics, methods like VRFT can effectively tune controllers using only operational data. Second, DDC is inherently adaptive, capable of updating controllers in real time as new data are available, which is critical for time-varying systems such as renewable energy grids. Third, DDC methods are well-suited for handling noise and disturbances through robust optimization and regularization techniques, ensuring reliable performance even in sensor-heavy or uncertain environments. Finally, DDC's natural synergy with machine learning techniques allows it to scale efficiently to high-dimensional and nonlinear systems, offering greater flexibility and performance potential than many traditional model-based methods.

Despite its advantages, DDC faces several key challenges [14, 15]. First, the quality of control design is highly dependent on the richness and quality of the available data. Insufficient or noisy data can lead to inaccurate controller synthesis and degraded performance. Second, ensuring robustness to model uncertainties and external

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disturbances remains a difficult task, especially in systems with limited excitation or operating in changing environments. Moreover, many data-driven methods lack interpretability compared to classical model-based approaches, making analysis and validation more complex. Computational efficiency and scalability also pose concerns, particularly for high-dimensional systems or real-time applications. Finally, integrating prior knowledge, such as physical constraints or known dynamics, with data remains an ongoing research focus, aiming to bridge the gap between data-driven and model-based control frameworks.

Facing these challenges, research on DDC has never ceased; instead, it has continued to grow with increasing enthusiasm [16–20]. Since 2020, DDC has once again emerged as a hot topic in the field of control, spurred by several influential results [21–23]. In these studies, it is found that, for a linear feedback system with unknown system matrices, a data-dependent representation of either the open-loop or closed-loop dynamics can be established using Willems et al.'s fundamental lemma. As a result, under certain rank conditions on the collected data, all stabilizing controllers can be synthesized by solving a set of data-dependent linear matrix inequalities (LMIs), provided that the experimental data are noise-free. This groundbreaking development reveals a fundamental insight: explicit system matrix matrices' identification is not required to verify whether a proposed controller stabilizes the original system or not. The method thus offers a paradigm shift in control design by enabling direct controller synthesis from data. Since its introduction, this approach has been successfully extended to more complex settings, including linear time-delay systems, linear switched systems, and certain classes of nonlinear systems, further broadening its applicability and impact.

However, experimental system measurements are typically contaminated by noise. As a result, increasing attention in recent years has been devoted to designing controllers directly from noisy data, particularly for linear systems with unknown system matrices. When the method proposed in [21] is applied to such data, it reveals that not only the magnitude but also the 'direction' of the noise significantly influences system stability, which is a striking insight that has sparked widespread interest among researchers. To address a broader range of noise types, a general assumption is introduced in [24], positing that noise samples satisfy a certain quadratic matrix inequality (QMI). By appropriately selecting a matrix multiplier in the QMI, this assumption can encompass various noise characteristics, including energy-bounded noise, noise with bounded sample covariance, and noise bounded within a subspace [25]. Building on this framework, several notable methods have been developed, such as QMI-based approaches, integral quadratic constraint (IQC) techniques, and linear fractional transformation (LFT) representations combining prior knowledge with data.

This paper provides a comprehensive overview of recent advances in DDC for linear discrete-time systems with unknown system matrices, with particular emphasis on handling noisy input-state and input-output data. For noisy input-state data, insightful understanding is presented on the foundational framework based on Willems et al.'s fundamental lemma, highlighting how data-driven representations of system dynamics can be leveraged for controller synthesis via data-dependent LMIs. Then, several robust methods for controller design directly from noisy data are analyzed in depth, including QMI-based approaches, IQC techniques for systems with non-uniform sampling, and LFT frameworks that combine prior knowledge with data. For noisy input-output data, a concise review is provided covering key DDC strategies applicable to both single-input-single-output (SISO) and multiple-input-multiple-output (MIMO) systems. Finally, several challenges and promising directions are provided for future research.

The remainder of the paper is organized as follows. Section 2 discusses data-driven control based on Willems et al.'s fundamental lemma. QMI approaches are examined in Section 3. Section 4 reviews LFT representations that enable the structured integration of prior knowledge with collected data, while Section 5 presents the IQC approach for DDC. Data-driven control using noisy input-output data is addressed in Section 6. Finally, Section 7 concludes the paper by highlighting several challenges and directions for future research.

Notations: The notations throughout this paper are standard. $\mathrm{diag}\{\cdots\}$ and $\mathrm{col}\{\cdots\}$ denote a block-diagonal matrix and a block-column matrix (vector), respectively. Without confusion, 'I' and '0' in a matrix, respectively, denote an identity matrix and a zero matrix with compatible dimensions. Specifically, I_n (0_n) stands for an n-dimensional identity (zero) matrix, and $0_{m\times n}$ means an $m \times n$ zero matrix. The symbol '$\star$' in a symmetric block matrix stands for a term induced by symmetry. $\mathbb{Z}_{\geqslant}0$ means the set of non-negative integers. $\mathrm{im}A$ and $\ker A$, repsectively, represent the image space and the kernel space of the matrix A. $\mathrm{He}\{A\} = A + A^{\mathrm{T}}$, where X^{T} is the transpose of X. $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) means the minimum (maximum) eigenvalue of the symmetric matrix A. $A^{\dagger}$ denotes the Moore-Penrose inverse of A.

2 Data-driven control based on Willems et al.'s lemma

2.1 The case of noise-free data

Consider a controllable discrete-time system described by

$$x(k+1) = A_{\rm un}x(k) + B_{\rm un}u(k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and the compatible matrices $A_{\rm un}$ and $B_{\rm un}$ are unknown. However, the data-related input and state sequences can be collected from experiments, which are given in the following:

$$U_0 = \big[u(0)\ u(1)\ \cdots\ u(T-1)\big], \tag{2a}$$

$$X_0 = \big[x(0)\ x(1)\ \cdots\ x(T-1)\big], \tag{2b}$$

$$X_1 = \big[x(1)\ x(2)\ \cdots\ x(T)\big], \tag{2c}$$

where $T \in \mathbb{N}$ indicates how long the sequences should be for control design or performance requirement. A key assumption about these data is as follows.

Assumption 1. $\operatorname{rank}\begin{bmatrix} X_0 \\ U_0 \end{bmatrix} = n+m.$

Assumption 1 can be satisfied if the input signal u is persistently exciting of order $n+1$, leading to $T \geqslant nm+m+n$. Under Assumption 1, by utilizing Willems et al.'s lemma, a data-based representation of the closed-loop system associated with $u(k) = Fx(k)$ can be obtained as

$$x(k+1) = (A_{\rm un} + B_{\rm un}F)x(k) = X_1 G_F x(k), \tag{3}$$

where $G_F \in \mathbb{R}^{T\times n}$ satisfies

$$\begin{bmatrix} I \\ F \end{bmatrix} = \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} G_F. \tag{4}$$

The first key significance of the closed-loop representation (3) is that it provides a data-driven interpretation of a model-based approach, where the system matrices $A_{\rm un}$ and $B_{\rm un}$ are reconstructed using a collection of sampled trajectories. Notably, this representation is highly dependent on T. Different values of T may lead to variations in the data-based representation of system (1), revealing the inevitable presence of uncertainties when modeling real-world physical systems. However, the primary goal is not to obtain the exact values of $A_{\rm un}$ and $B_{\rm un}$ but rather to design a stabilizing controller for the system. Thus, the second key significance of the closed-loop representation (3) is that it paves a way to design stabilizing controllers directly from data.

With the data-based representation (3), the closed-loop system is asymptotically stable if and only if there exists a real matrix $\bar{P} \in \mathbb{S}_+^n$ such that

$$X_1 G_F \bar{P} (X_1 G_F)^{\rm T} - \bar{P} < 0. \tag{5}$$

From (4), since F is to be designed, G_F can be regarded as a decision variable. Let $P = G_F\bar{P}$. Then a stabilization criterion follows.

Theorem 1 ([21, Theorem 3])**.** Under Assumption 1, the state feedback control $u(k) = Fx(k)$ with $F = U_0P(X_0P)^{-1}$ stabilizes the system (1) if there exists a real matrix $P \in \mathbb{R}^{T\times n}$ such that

$$\begin{bmatrix} -X_0P & X_1P \\ P^{\rm T}X_1^{\rm T} & -X_0P \end{bmatrix} < 0. \tag{6}$$

It is clear that the matrix inequality (6) is data-based and linear with respect to the matrix variable P. A key feature of Theorem 1 lies in that it characterizes the whole set of stabilizing state feedback gains, given by $F = U_0P(X_0P)^{-1}$, for all P satisfying (6). However, the matrix variable P depends on the time horizon of experiments due to $P \in \mathbb{R}^{T\times n}$. Thus, the computational complexity of the LMI condition increases significantly as the dataset size grows.

Building on the idea above, several results on optimal control and linear quadratic regulation are obtained in [21]. Moreover, this approach is also extended to time-delay systems [26] and switched systems [27], among others.

2.2 The case of noisy data

The data collected from experiments are usually corrupted by external disturbances. Thus, in recent years, control design directly from noisy data has been a hot topic. In the following, we present several ways to deal with noisy data.

2.2.1 *Method I from [21]*

Suppose that the measured signal is $\eta(k) = x(k) + w(k)$, where w is an unknown measurement noise. Let

$$W_0 = [w(0)\ w(1)\ \cdots\ w(T-1)], \qquad (7)$$
$$W_1 = [w(1)\ w(2)\ \cdots\ w(T)]$$

and let $Z_0 = X_0 + W_0$ and $Z_1 = X_1 + W_1$. Since the measured signals are corrupted by noise, the exact values of X_0 and X_1 are unavailable for control design. Consequently, the objective is to design a stabilizing controller using the noisy data U_0, Z_0 and Z_1. For this goal, a conservative method is proposed in [21], based on the idea that the LMI (6) possesses an intrinsic degree robustness with respect to perturbations in X_0 and X_1. That is, if X_0 and X_1 are replaced with their noisy counterparts Z_0 and Z_1, there may still exist a matrix $P > 0$ such that the modified LMI remains feasible. In this situation, the control law $u(k) = Fx(k)$ with $F = U_0P(Z_0P)^{-1}$ continues to serve as a stabilizing controller. To formalize this idea, a robust version of Theorem 1 is presented as follows.

Theorem 2. Assume that rank col$\{U_0, Z_0\} = n + m$. The state feedback control $u(k) = Fx(k)$ with $F = U_0P(Z_0P)^{-1}$ stabilizes the system (1) if there exists a matrix $P \in \mathbb{R}^{T\times n}$ and a scalar $\alpha > 0$ such that

$$\begin{bmatrix} -Z_0P + \alpha Z_1Z_1^{\mathrm{T}} & Z_1P \\ P^{\mathrm{T}}Z_1^{\mathrm{T}} & -Z_0P \end{bmatrix} < 0, \quad \begin{bmatrix} I & P \\ P^{\mathrm{T}} & Z_0P \end{bmatrix} > 0. \qquad (8)$$

Based on Theorem 2, a couple of results are derived to quantify the level of measurement noise under which the obtained solution still yields a stabilizing controller. These results provide explicit conditions on the admissible noise bounds, ensuring that the controller designed from noisy data retains its stabilizing properties.

To show that, we need the following assumptions.

Assumption 2. For some scalar $\gamma > 0$, $\mathcal{E}\mathcal{E}^{\mathrm{T}} \leqslant \gamma Z_1Z_1^{\mathrm{T}}$ where $\mathcal{E} \triangleq A_{\mathrm{un}}W_0 - W_1$.

Assumption 3. For some scalars $\gamma_1 \in (0, 0.5)$ and $\gamma_2 > 0$, the following hold:

$$\begin{bmatrix} 0 \\ W_0 \end{bmatrix}\begin{bmatrix} 0 \\ W_0 \end{bmatrix}^{\mathrm{T}} \leqslant \gamma_1 \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}^{\mathrm{T}}, \quad W_1W_1^{\mathrm{T}} \leqslant \gamma_2 Z_1Z_1^{\mathrm{T}}.$$

Theorem 3. Under Assumption 2, rank col$\{U_0, Z_0\} = n + m$, and rank $Z_1 = n$, the state feedback control $u(k) = Fx(k)$ with $F = U_0P(Z_0P)^{-1}$ stabilizes the system (1) if (P, α) is a solution to the LMIs in (8) such that $\gamma < \alpha^2/(4 + 2\alpha)$.

Theorem 4. Under Assumption 3, rank col$\{U_0, Z_0\} = n + m$, and rank $Z_1 = n$, the state feedback control $u(k) = Fx(k)$ with $F = U_0P(Z_0P)^{-1}$ stabilizes the system (1) if (P, α) is a solution to the LMIs in (8) such that $\frac{6\gamma_1+3\gamma_2}{1-2\gamma_1} < \frac{\alpha^2}{4+2\alpha}$.

Assumption 2 involves the unknown system matrix A_{un}, making it difficult to verify directly from data. However, an interesting insight can be drawn from this assumption. That is, if the disturbance evolves according to $w(k+1) = A_{\mathrm{un}}w(k)$, then Assumption 2 is always satisfied. In this case, Theorem 3 ensures that any solution to (8) yields a stabilizing controller for the system (1). This observation suggests that the effect of noise on control design does not depend only on the magnitude of the disturbance w, but also on its direction in the state space.

2.2.2 *Method II from [28]*

From the analysis above, it is clear that Method I relies heavily on the robustness of the LMI condition (6) with respect to the perturbations in X_0 and X_1. In the following, we introduce a robust method based on Petersen's Lemma, as proposed in [28].

Consider a disturbed linear system described by

$$x(k+1) = A_{\mathrm{un}}x(k) + B_{\mathrm{un}}u(k) + w(k), \qquad (9)$$

where the meanings of $x, u, A_{\rm un}$ and $B_{\rm un}$ are the same as those in (1), and $w \in \mathbb{R}^n$ is an unknown disturbance. To maintain consistency throughout this paper, let the same notations U_0, X_0 and X_1 as in (2) denote the noisy data collected from an experiment on the unknown system (9). For the unknown disturbance $w(k)$, we define $W_0 \in \mathbb{R}^{n\times T}$ as in (7). From (9), one has

$$W_0 = X_1 - A_{\rm un}X_0 - B_{\rm un}U_0. \tag{10}$$

Assumption 4. There exists a known matrix $D_0 \in \mathbb{R}^{n\times r}$ such that $W_0W_0^{\rm T} \leqslant D_0D_0^{\rm T}$.

Assumption 4 implies that the disturbance matrix W_0 has bounded energy. It can capture a variety of noise scenarios, including signal-to-noise ratio (SNR) conditions [21], over-approximated instantaneous bounds [29], and probabilistic bounds for Gaussian noise [22]. Under Assumption 4, define a set as

$$\mathscr{S}_1 = \left\{[A\ B]:\ X_1 = AX_0 + BU_0 + W,\ WW^{\rm T} \leqslant D_0D_0^{\rm T}\right\}. \tag{11}$$

It is clear that $[A_{\rm un}\ B_{\rm un}] \in \mathscr{S}_1$. Then the stabilization problem of the system (9) can be stated as follows.

Problem I: Design a state-feedback control $u(k) = Fx(k)$ such that the closed-loop system $x(k+1) = (A+BF)x(k)$ is asymptotically stable for $\forall [A\ B] \in \mathscr{S}_1$.

A key contribution in [28] is that the set $\mathscr{S}_1$ is equivalently described by an uncertain set, where the element is expressed as a nominal matrix plus a norm-bounded uncertain matrix under Assumption 1. First, substituting $W = X_1 - AX_0 - BU_0$ into $WW^{\rm T} \leqslant D_0D_0^{\rm T}$ yields

$$\mathscr{S}_1 = \left\{[A\ B] = G^{\rm T}:\ (G - \Pi_2)^{\rm T}\Pi_1(G - \Pi_2) \leqslant \Pi_3\right\}, \tag{12}$$

where

$$\Pi_1 = \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\rm T},\ \Pi_2 = \Pi_1^{-1}\begin{bmatrix} X_0 \\ U_0 \end{bmatrix} X_1^{\rm T},\ \Pi_3 = \Pi_2^{\rm T}\Pi_1\Pi_2 + D_0D_0^{\rm T} - X_1X_1^{\rm T}.$$

It is worth noting that the set $\mathscr{S}_1$ of form (12) can be interpreted as a 'matrix ellipsoid'. In the absence of disturbances, i.e., when $W = W_0 = 0$, then the matrix G reduces to Π_2, which represents the 'centre' of the ellipsoid. Moreover, with the form (12), an insightful fact is revealed in [28]: the set $\mathscr{S}_1$ is equivalent to an uncertainty set whose elements are norm-bounded, as formally stated below.

Fact 1. Under Assumption 1, $\Pi_3 \geqslant 0$, and the set $\mathscr{S}_1$ is equal to $\mathscr{S}_2$, where $\mathscr{S}_2$ is defined as

$$\mathscr{S}_2 = \left\{(\Pi_2 + \Pi_1^{-\frac{1}{2}}\triangle\Pi_3^{\frac{1}{2}})^{\rm T} : \triangle^{\rm T}\triangle \leqslant I\right\}. \tag{13}$$

From Fact 1, the matrix $[A\ B]$ can be expressed as

$$[A\ B] = \Pi_2^{\rm T} + \Pi_3^{\frac{1}{2}}\triangle^{\rm T}\Pi_1^{-\frac{1}{2}},\quad \triangle^{\rm T}\triangle \leqslant I. \tag{14}$$

Thus, by the Lyapunov stability theorem, the solution to Problem I can be given by $(A + BF)^{\rm T}P^{-1}(A + BF) - P^{-1} < 0$ with $P > 0$, which can be equivalently transformed into

$$\Upsilon := \begin{bmatrix} -P & PA^{\rm T} + PF^{\rm T}B^{\rm T} \\ \star & -P \end{bmatrix} < 0. \tag{15}$$

Let $Y = PF^{\rm T}$ and substitute $[A\ B]$ in (14) into (15). Then one obtains

$$\Upsilon = \begin{bmatrix} -P & [P\ Y]\Pi_2 \\ \star & -P \end{bmatrix} + \begin{bmatrix} [P\ Y]\Pi_1^{-\frac{1}{2}} \\ 0 \end{bmatrix}\triangle[0\ \Pi_3^{\frac{1}{2}}] + \begin{bmatrix} 0 \\ \Pi_3^{\frac{1}{2}} \end{bmatrix}\triangle^{\rm T}\left[\Pi_1^{-\frac{1}{2}}\begin{pmatrix} P \\ Y^{\rm T} \end{pmatrix}\ 0\right] < 0,\ \triangle^{\rm T}\triangle \leqslant I. \tag{16}$$

Employ Petersen's lemma to obtain a necessary and sufficient condition for the solution to Problem I.

Theorem 5. Problem I is solvable with $F = Y^{\rm T}P^{-1}$ if and only if there exist $P \in \mathbb{S}_+^n$ and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} -P & [P\ Y]\Pi_2 & [P\ Y]\Pi_1^{-\frac{1}{2}} \\ \star & -P + \Pi_3 & 0 \\ \star & \star & -I \end{bmatrix} < 0. \tag{17}$$

As a special case, in the absence of disturbances, it follows from (16) that the LMI (17) reduces to

$$\begin{bmatrix} -P & [P\ Y]\Pi_2 \\ \star & -P \end{bmatrix} < 0. \tag{18}$$

2.2.3 *Comparison between Methods I and II*

In this subsection, we make a comparison between Methods I and II. First, it is clear to see that several common points of these two methods can be given as follows.

- A rank condition from Assumption 1 on data sequences is required.
- Controllers are designed directly from data-based LMIs (Theorems 2 and 5).
- In the absence of disturbances, both LMIs in (6) and (18) are equivalent. In fact, from the definition of Moore-Penrose inverse, one has $\Pi_2^{\mathrm{T}} = X_1 \left[\begin{smallmatrix} X_0 \\ U_0 \end{smallmatrix}\right]^{\dagger}$, and from (4), $G_F = \left[\begin{smallmatrix} X_0 \\ U_0 \end{smallmatrix}\right]^{\dagger}\left[\begin{smallmatrix} I \\ F \end{smallmatrix}\right]$. Then, by the Schur complement, together with some matrix manipulations, the LMI (18) is equivalent to

$$-P^{-1} + \left(X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} I \\ F \end{bmatrix}\right)^{\mathrm{T}} P^{-1} X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} I \\ F \end{bmatrix} = (X_1 G_F)^{\mathrm{T}} P^{-1} (X_1 G_F) - P^{-1} < 0 \tag{19}$$

which is equivalent to (5), as well as (6).

Next, we discuss the differences between Methods I and II, corresponding to Theorems 2 and 5, respectively.

(1) The dimension of the matrix variable P in Theorem 2 depends on the time horizon T of the experiment, whereas the matrix variables P and Y in Theorem 5 are independent of T.

(2) Theorem 2 is directly derived from the noise-free result (Theorem 1), which tends to be conservative in the presence of noise. Although Theorem 1 possesses some inherent robustness to measurement noise, explicitly quantifying this robustness is challenging. Moreover, Theorem 2 provides only a sufficient condition, not an equivalent one to Theorem 1, which further contributes to its conservativeness. While Theorems 3 and 4 are introduced to characterize noise levels that allow for the existence of stabilizing controllers, they remain overly conservative because they are based on Theorem 2 rather than directly on Theorem 1.

Compared to Method I, Method II offers a more effective approach for designing stabilizing controllers from noisy data. The core concept involves representing the set of all unknown matrices $[A_{\mathrm{un}}\ B_{\mathrm{un}}]$ as a matrix ellipsoid $\mathscr{S}_1$, derived from noisy data under the assumption of a disturbance sequence with bounded energy (Assumption 4). This matrix ellipsoid is then equivalently reformulated as an uncertain matrix set $\mathscr{S}_2$ with norm-bounded elements. Consequently, the problem of designing a stabilizing controller directly from noisy data is transformed into a classical robust control problem that is solved using Petersen's lemma.

2.2.4 *Discussion on Method II*

From Fact 1, under Assumption 1, the set $\mathscr{S}_1$ is equivalent to $\mathscr{S}_2$. In the proof of Fact 1 [28], two cases of the matrix Π_3, i.e., positive definite and positive semi-definite, are considered. In what follows, it is shown that when Π_3 is positive semi-definite, the set $\mathscr{S}_2$ can be replaced with an alternative form.

Suppose that $\Pi_3^{\frac{1}{2}}$ has $r(\leqslant n)$ positive eigenvalues $\lambda_1, \cdots, \lambda_r$, and let $\Sigma_r = \mathrm{diag}\{\lambda_1, \cdots, \lambda_r\}$. By the singular value decomposition, there exists an orthogonal matrix $\mathcal{U} = [\mathcal{U}_1\ \mathcal{U}_2]$ such that

$$\Pi_3^{\frac{1}{2}} = \left[\mathcal{U}_1 \middle| \mathcal{U}_2\right] \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \mathcal{U}_1^{\mathrm{T}} \\ \mathcal{U}_2^{\mathrm{T}} \end{bmatrix} = \mathcal{U}_1 [\Sigma_r\ 0] \mathcal{U}^{\mathrm{T}} = \mathcal{U}_1 \Sigma_r \mathcal{U}_1^{\mathrm{T}}. \tag{20}$$

Then it is easy to verify that

$$\Pi_3 = \mathcal{U}_1 \Sigma_r^2 \mathcal{U}_1^{\mathrm{T}},\ \mathcal{U}_1\mathcal{U}_1^{\mathrm{T}} + \mathcal{U}_2\mathcal{U}_2^{\mathrm{T}} = I,\ \mathcal{U}_1^{\mathrm{T}}\mathcal{U}_1 = I,\ \mathcal{U}_2^{\mathrm{T}}\mathcal{U}_2 = I,\ \mathcal{U}_1^{\mathrm{T}}\mathcal{U}_2 = 0. \tag{21}$$

Then we have the following conclusion.

Lemma 1. Under Assumption 1, the set $\mathscr{S}_1$ is equal to $\mathscr{S}_3$, where $\mathscr{S}_3$ is defined as

$$\mathscr{S}_3 = \left\{(\Pi_2 + \Pi_1^{-\frac{1}{2}} \triangle \Pi_3^{\frac{1}{2}})^{\mathrm{T}} : \triangle^{\mathrm{T}}\triangle \leqslant \mathcal{U}_1\mathcal{U}_1^{\mathrm{T}}\right\}. \tag{22}$$

The proof of Lemma 1 can be completed by following the same line as that in [28]. Its simplified version can be referred to [30] in detail.

When $r = n$, $\Pi_3 > 0$. In this case, the set $\mathscr{S}_3$ is of the same form as $\mathscr{S}_2$ due to $\mathcal{U}_1 = \mathcal{U}$ and $\mathcal{U}\mathcal{U}^{\mathrm{T}} = I$. When $0 < r < n$, compared to $\mathscr{S}_2$, a tight bound on the uncertain matrix $\triangle$ is presented in $\mathscr{S}_3$ due to $\triangle^{\mathrm{T}}\triangle \leqslant \mathcal{U}_1\mathcal{U}_1^{\mathrm{T}} \neq \mathcal{U}_1\mathcal{U}_1^{\mathrm{T}} + \mathcal{U}_2\mathcal{U}_2^{\mathrm{T}} = I$. However, based on the set $\mathscr{S}_3$, the feasibility of Problem I from Theorem 5 remains unchangeable even though Π_3 is positive semi-definite. In fact, based on the set $\mathscr{S}_3$, Problem I can be solved if

$$\begin{bmatrix} -P & [P\ Y]\Pi_2 \\ \star & -P \end{bmatrix} + \begin{bmatrix} [P\ Y]\Pi_1^{-\frac{1}{2}} \\ 0 \end{bmatrix} \triangle [0\ \Pi_3^{\frac{1}{2}}] + \begin{bmatrix} 0 \\ \Pi_3^{\frac{1}{2}} \end{bmatrix} \triangle^{\mathrm{T}} \left[\Pi_1^{-\frac{1}{2}} \left(\begin{smallmatrix} P \\ Y^{\mathrm{T}} \end{smallmatrix}\right)\ 0\right] < 0,\ \triangle^{\mathrm{T}}\triangle \leqslant \mathcal{U}_1\mathcal{U}_1^{\mathrm{T}}. \tag{23}$$

Using Petersen's lemma, the inequality (23) holds if and only if there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} -P & [P\ Y]\Pi_2 \\ \star & -P \end{bmatrix} + \varepsilon \begin{bmatrix} [P\ Y]\Pi_1^{-\frac{1}{2}} \\ 0 \end{bmatrix} \begin{bmatrix} [P\ Y]\Pi_1^{-\frac{1}{2}} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1} \begin{bmatrix} 0 \\ \Pi_3^{\frac{1}{2}} \end{bmatrix} \mathcal{U}_1 \mathcal{U}_1^{\mathrm{T}} [0\ \Pi_3^{\frac{1}{2}}] < 0 \tag{24}$$

which is equivalent to (17) due to

$$\begin{bmatrix} 0 \\ \Pi_3^{\frac{1}{2}} \end{bmatrix} \mathcal{U}_1 \mathcal{U}_1^{\mathrm{T}} [0\ \Pi_3^{\frac{1}{2}}] = \begin{bmatrix} 0 & 0 \\ 0 & \Pi_3 \end{bmatrix}.$$

Note that the set $\mathscr{S}_3$ depends on the singular value decomposition (SVD) or orthogonal decomposition. After an insightful observation of the set $\mathscr{S}_1$, we present below an alternative, yet equivalent, representation of $\mathscr{S}_1$ without matrix decompositions.

Lemma 2. Under Assumption 1, the set $\mathscr{S}_1$ is equal to $\mathscr{S}_4$, where $\mathscr{S}_4$ is defined as

$$\mathscr{S}_4 = \left\{ (\Pi_2 + \Pi_1^{-\frac{1}{2}} \triangle)^{\mathrm{T}} : \triangle^{\mathrm{T}} \triangle \leqslant \Pi_3 \right\}. \tag{25}$$

Proof. For $\forall G^{\mathrm{T}} \in \mathscr{S}_4$, $G = \Pi_2 + \Pi_1^{-\frac{1}{2}} \triangle$ with $\triangle^{\mathrm{T}} \triangle \leqslant \Pi_3$. Then

$$(G - \Pi_2)^{\mathrm{T}} \Pi_1 (G - \Pi_2) = \triangle^{\mathrm{T}} \Pi_1^{-\frac{1}{2}} \Pi_1 \Pi_1^{-\frac{1}{2}} \triangle = \triangle^{\mathrm{T}} \triangle \leqslant \Pi_3. \tag{26}$$

Thus, $G^{\mathrm{T}} \in \mathscr{S}_1$, leading to $\mathscr{S}_4 \subseteq \mathscr{S}_1$. On the other hand, for $\forall G^{\mathrm{T}} \in \mathscr{S}_1$, one has

$$(G - \Pi_2)^{\mathrm{T}} \Pi_1 (G - \Pi_2) \leqslant \Pi_3. \tag{27}$$

Set $\triangle = \Pi_1^{\frac{1}{2}} (G - \Pi_2)$. Then $G = \Pi_2 + \Pi_1^{-\frac{1}{2}} \triangle$, and it follows directly from (27) that $\triangle^{\mathrm{T}} \triangle \leqslant \Pi_3$. Thus, $G^{\mathrm{T}} = (\Pi_2 + \Pi_1^{-\frac{1}{2}} \triangle)^{\mathrm{T}} \in \mathscr{S}_4$, leading to $\mathscr{S}_1 \subseteq \mathscr{S}_4$.

Lemma 2 offers an equivalent set $\mathscr{S}_4$ of $\mathscr{S}_1$, which takes a different form from $\mathscr{S}_2$ and $\mathscr{S}_3$. However, when using the set $\mathscr{S}_4$ to solve Problem I, the resulting solution remains the same as Theorem 5. To clarify this, based on the set $\mathscr{S}_4$, the matrix Υ given in (15) can be written as

$$\Upsilon = \begin{bmatrix} -P & [P\ Y]\Pi_2 \\ \star & -P \end{bmatrix} + \begin{bmatrix} [P\ Y]\Pi_1^{-\frac{1}{2}} \\ 0 \end{bmatrix} \triangle [0\ I] + \begin{bmatrix} 0 \\ I \end{bmatrix} \triangle^{\mathrm{T}} \left[\Pi_1^{-\frac{1}{2}} \left(\begin{smallmatrix} P \\ Y^{\mathrm{T}} \end{smallmatrix} \right)\ 0 \right],\ \triangle^{\mathrm{T}} \triangle \leqslant \Pi_3. \tag{28}$$

Applying Petersen's lemma, it is straightforward to conclude that $\Upsilon < 0$ for $\triangle^{\mathrm{T}} \triangle \leqslant \Pi_3$ if and only if the LMI (17) is satisfied.

3 Data-driven control using QMI approaches

Recent research such as [31] has shown that the rank condition in Assumption 1 is not necessary for achieving several system-theoretic properties. For instance, it is possible to design suitable stabilizing controllers based on collected data even when Assumption 1 is not satisfied. On the other hand, although Method II introduced in the previous section establishes a useful framework for data-driven control, it relies not only on the rank condition but also on a specific noise model described in Assumption 4, which cannot be readily extended to other noise settings, such as those characterized by cross-covariance bounds [32]. Motivated by this important insight, a QMI approach has been developed for data-driven control in [24, 25].

Consider the unknown system (9). To keep the consistency of the paper, the data collected from an experiment are still defined as U_0, X_0 and X_1, which are given in (2), and the unknown disturbance matrix W_0 is defined in (7). The objective is to design a data-based state feedback controller $u(k) = Fx(k)$ such that the closed-loop system below is asymptotically stable:

$$x(k+1) = (A_{\mathrm{un}} + B_{\mathrm{un}} F) x(k). \tag{29}$$

The initial idea of the QMI approach is to assume the unknown disturbance matrix W_0 to satisfy a QMI, as stated in the following assumption.

Assumption 5. The unknown matrix W_0 satisfies

$$\begin{bmatrix} I \\ W_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \Phi \begin{bmatrix} I \\ W_0^{\mathrm{T}} \end{bmatrix} \geqslant 0, \quad \Phi \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix}, \tag{30}$$

where $\Phi_{11} \in \mathbb{S}^n, \Phi_{12} \in \mathbb{R}^{n\times T}, \Phi_{22} \in \mathbb{S}^T$.

The noise model described in Assumption 5 encompasses several types of disturbances, including those characterized by energy bounds, individual noise sample bounds, sample covariance bounds, and bounded noise within a subspace [25]. Under Assumption 5, the unknown system matrix $[A_{\mathrm{un}}\ B_{\mathrm{un}}]$ belongs to a matrix set, defined as follows:

$$\mathscr{S}_5 = \left\{[A\ B] \in \mathbb{R}^{n\times(n+m)} \mid X_1 = AX_0 + BU_0 + W_0 \text{ for } W_0 \text{ satisfying (30)}\right\}. \tag{31}$$

If one sets $\Phi_{11} = D_0 D_0^{\mathrm{T}}$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$, then Assumption 5 reduces to Assumption 4. Thus, $\mathscr{S}_1$ in (12) is just a subset of $\mathscr{S}_5$. Under Assumption 5, the quadratic stabilization problem of the system (9) can be stated as follows.

Problem II: Design a state-feedback controller $u(k) = Fx(k)$ such that the closed-loop system $x(k+1) = (A+BF)x(k)$ is asymptotically stable for $\forall [A\ B] \in \mathscr{S}_5$.

It is known that the system $x(k+1) = (A+BF)x(k)$ is asymptotically stable if and only if there exists $P \in \mathbb{S}_+^n$ such that

$$(A+BF)P(A+BF)^{\mathrm{T}} - P < 0. \tag{32}$$

The crucial idea of the QMI approach is to describe both the constraint $X_1 = AX_0 + BU_0 + W_0$ for W_0 satisfying (30) and the inequality (32) as two different QMIs but with a similar structural form. First, substituting $W_0 = X_1 - AX_0 - BU_0$ into (30) yields

$$\mathcal{M}_1 := \begin{bmatrix} I \\ A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I \\ A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} \geqslant 0. \tag{33}$$

Second, rewrite (32) as

$$\mathcal{M}_2 := \begin{bmatrix} I \\ A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -P & 0 & 0 \\ 0 & P & PF^{\mathrm{T}} \\ 0 & FP & FPF^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I \\ A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} < 0. \tag{34}$$

Clearly, two matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ share the same 'edges', i.e., $\mathcal{Z} \triangleq \mathrm{col}\{I, A^{\mathrm{T}}, B^{\mathrm{T}}\}$ and its transpose, on both sides. In this sense, they possess a similar structural form. Then Problem II is transformed into a QMI problem:

$$\text{To determine } F \text{ such that } \mathcal{M}_2 < 0 \text{ subject to } \mathcal{M}_1 \geqslant 0. \tag{35}$$

If the 'edge' $\mathcal{Z}$ is a vector, Problem II can be efficiently addressed using the well-known S-procedure. To solve the QMI (35), several matrix S-lemmas, regarded as extensions of the classical S-procedure, have been developed in [24, 25]. To move on, let $\Psi, \Xi \in \mathbb{S}^{q+r}$ with

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \star & \Psi_{22} \end{bmatrix}, \quad \Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \star & \Xi_{22} \end{bmatrix}. \tag{36}$$

Lemma 3 (Matrix S-lemma with α [25, Theorem 4.10])**.** Let Ψ and Ξ given in (36). Suppose that $\Psi_{22} < 0$, $\Psi_{11} - \Psi_{12}\Psi_{22}^{-1}\Psi_{12}^{\mathrm{T}} \geqslant 0$ and $\ker \Psi_{22} \subseteq \ker \Psi_{12}$. Then for $\forall Z \in \mathbb{R}^{r\times q}$,

$$\begin{bmatrix} I \\ Z \end{bmatrix}^{\mathrm{T}} \Xi \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \text{ subject to } \begin{bmatrix} I \\ Z \end{bmatrix}^{\mathrm{T}} \Psi \begin{bmatrix} I \\ Z \end{bmatrix} \geqslant 0, \tag{37}$$

if and only if there exists a scalar $\alpha \geqslant 0$ such that $\Xi - \alpha\Psi > 0$.

Lemma 4 (Matrix S-lemma with α and β [25, Corollary 4.13]). Let Ψ and Ξ given in (36). Suppose that $\Xi_{22} \leqslant 0$, $\Psi_{22} \leqslant 0$, $\Psi_{11} - \Psi_{12}\Psi_{22}^{\dagger}\Psi_{12}^{\mathrm{T}} \geqslant 0$ and $\ker \Psi_{22} \subseteq \ker \Psi_{12}$. Then for $\forall Z \in \mathbb{R}^{r\times q}$,

$$\begin{bmatrix} I \\ Z \end{bmatrix}^{\mathrm{T}} \Xi \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \text{ subject to } \begin{bmatrix} I \\ Z \end{bmatrix}^{\mathrm{T}} \Psi \begin{bmatrix} I \\ Z \end{bmatrix} \geqslant 0, \tag{38}$$

if and only if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ such that $\Xi - \alpha\Psi \geqslant \operatorname{diag}\{\beta I, 0\}$.

The key difference between Lemmas 3 and 4 lies in that Ψ_{22} should be negative definite in Lemma 3 while it is relaxed to be negative semi-definite in Lemma 4. Applying Lemmas 3 and 4 to the QMI (35) yields the following results.

Theorem 6 ([25, Theorem 5.1(a)]). Under Assumption 5, suppose that $\Phi_{22} \leqslant 0$, $\Phi_{11} - \Phi_{12}\Phi_{22}^{\dagger}\Phi_{12}^{\mathrm{T}} \geqslant 0$ and $\ker \Phi_{22} \subseteq \ker \Phi_{12}$. Problem II is solvable with $F = YP^{-1}$ if and only if there exist $P \in \mathbb{S}_{+}^{n}$, $Y \in \mathbb{R}^{m\times n}$ and a scalar $\varepsilon_0 > 0$ such that

$$\begin{bmatrix} I & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} P - \varepsilon_0 I & 0 & 0 & 0 \\ \star & -P & -Y^{\mathrm{T}} & 0 \\ \star & \star & 0 & Y \\ \star & \star & \star & P \end{bmatrix} \leqslant 0. \tag{39}$$

Theorem 7 ([25, Theorem 5.1(b)]). Under Assumptions 1 and 5, suppose that $\Phi_{22} < 0$, $\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^{\mathrm{T}} \geqslant 0$ and $\ker \Phi_{22} \subseteq \ker \Phi_{12}$. Problem II is solvable with $F = YP^{-1}$ if and only if there exist $P \in \mathbb{S}_{+}^{n}$ and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} I & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} P & 0 & 0 & 0 \\ \star & -P & -Y^{\mathrm{T}} & 0 \\ \star & \star & 0 & Y \\ \star & \star & \star & P \end{bmatrix} < 0. \tag{40}$$

Theorem 6 does not depend on the rank condition in Assumption 1. Specifically, under Assumption 5, and with certain matrix constraints on Φ, Problem II is solvable as long as the data-based LMI in (39) is feasible. Furthermore, as shown in Theorem 7, if the rank condition in Assumption 1 is also satisfied, then suitable stabilizing controllers can be directly designed from noisy data that are not merely energy-bounded.

If we set $\Phi_{11} = D_0 D_0^{\mathrm{T}}$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$, then Assumption 5 reduces to Assumption 4. In this case, $\Phi_{22} < 0$, $\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^{\mathrm{T}} = D_0 D_0^{\mathrm{T}} \geqslant 0$ and $\ker \Phi_{22} \subset \ker \Phi_{12}$. Applying Theorem 7 yields Corollary 1.

Corollary 1. Problem I is solvable with $F = YP^{-1}$ if and only if there exist $P \in \mathbb{S}_{+}^{n}$ and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} D_0 D_0^{\mathrm{T}} - X_1 X_1^{\mathrm{T}} - P & X_1 X_0^{\mathrm{T}} & X_1 U_0^{\mathrm{T}} & 0 \\ \star & P - X_0 X_0^{\mathrm{T}} & -X_0 U_0^{\mathrm{T}} + Y^{\mathrm{T}} & 0 \\ \star & \star & -U_0 U_0^{\mathrm{T}} & -Y \\ \star & \star & \star & -P \end{bmatrix} < 0. \tag{41}$$

Corollary 1 and Theorem 5 establish two necessary and sufficient conditions for the solution to Problem I, formulated as data-based LMIs in different forms, (41) and (17), respectively. Their equivalence is not immediately clear and warrants further investigation.

At the end of this section, we consider a more general case of system (9), where the coefficient matrix of the disturbance $w(k)$ is no longer the identity. Specifically, consider the system

$$x(k+1) = A_{\mathrm{un}} x(k) + B_{\mathrm{un}} u(k) + B_w w(k). \tag{42}$$

The key difference between systems (42) and (9) is that the disturbance vector $w(k)$ in (42) has a dimension $n_w \leqslant n$, and B_w is a known matrix with full column rank. One can equivalently define $\tilde{w}(k) = B_w w(k)$, in which case system (42) reduces to the form of (9), with $\tilde{w}(k)$ regarded as an unknown disturbance. Consequently, the previously introduced QMI approach can be applicable. In what follows, we generalize the QMI approach to directly accommodate this more general case.

Due to $w(k) \in \mathbb{R}^{n_w}$, we first modify Assumption 5 slightly as follows.

Assumption 5′. The unknown matrix W_0 satisfies

$$\Upsilon_0 := \begin{bmatrix} I_{n_w} \\ W_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} I_{n_w} \\ W_0^{\mathrm{T}} \end{bmatrix} \geqslant 0, \tag{43}$$

where $\Phi_{11} \in \mathbb{S}^{n_w}, \Phi_{12} \in \mathbb{R}^{n_w \times T}, \Phi_{22} \in \mathbb{S}^T$.

Under Assumption 5′, the set $\mathscr{S}_5$ for the system (42) should be modified accordingly as

$$\mathscr{S}_6 = \big\{[A\ B] \in \mathbb{R}^{n\times(n+m)} \mid X_1 = AX_0 + BU_0 + B_w W_0 \text{ for } W_0 \text{ satisfying (43)}\big\}. \tag{44}$$

Lemma 5. Suppose that B_w has full column rank. Then $\mathscr{S}_6 = \mathscr{S}_7$, where $\mathscr{S}_7$ is defined as

$$\mathscr{S}_7 = \left\{[A\ B] \in \mathbb{R}^{n\times(n+m)} \;\middle|\; \begin{bmatrix} I_n \\ [A\ B]^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \mathcal{M}_3 \begin{bmatrix} I_n \\ [A\ B]^{\mathrm{T}} \end{bmatrix} \geqslant 0 \right\}, \tag{45}$$

where

$$\mathcal{M}_3 := \begin{bmatrix} B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix}^{\mathrm{T}}. \tag{46}$$

Proof. Let $G = [A\ B] \in \mathscr{S}_6$. Then $B_w W_0 = X_1 - AX_0 - BU_0$ and W_0 satisfies (43), which follows $B_w \Upsilon_0 B_w^{\mathrm{T}} \geqslant 0$. Note that

$$[B_w\ B_w W_0] = [B_w\ X_1 - AX_0 - BU_0] = [I_{n_w}\ G] \begin{bmatrix} B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix}. \tag{47}$$

After simple matrix manipulations, the inequality $B_w \Upsilon_0 B_w^{\mathrm{T}} \geqslant 0$ becomes $\left[\begin{smallmatrix} I_{n_w} \\ G^{\mathrm{T}} \end{smallmatrix}\right]^{\mathrm{T}} \mathcal{M}_3 \left[\begin{smallmatrix} I_{n_w} \\ G^{\mathrm{T}} \end{smallmatrix}\right] \geqslant 0$. Thus $G = [A\ B] \in \mathscr{S}_7$.

On the other hand, let $G = [A\ B] \in \mathscr{S}_7$. Then we have $B_w \Upsilon_0 B_w^{\mathrm{T}} \geqslant 0$, where $B_w W_0 = X_1 - AX_0 - BU_0$. Since B_w has full column rank, the inequality $B_w \Upsilon_0 B_w^{\mathrm{T}} \geqslant 0$ implies $\Upsilon_0 \geqslant 0$. In fact, B_w having full column rank allows to choose $B_0 \in \mathbb{R}^{n\times(n-n_w)}$ such that $[B_w\ B_0]$ is nonsingular. Then

$$B_w \Upsilon_0 B_w^{\mathrm{T}} = [B_w\ B_0] \begin{bmatrix} \Upsilon_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_w^{\mathrm{T}} \\ B_0^{\mathrm{T}} \end{bmatrix} \geqslant 0 \Longrightarrow \begin{bmatrix} \Upsilon_0 & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0 \Longrightarrow \Upsilon_0 \geqslant 0,$$

which leads to $G = [A\ B] \in \mathscr{S}_6$. Thus, one concludes that $\mathscr{S}_6 = \mathscr{S}_7$.

For the system (42), Problem II is then described equivalently as

$$\text{To determine } F \text{ such that } \mathcal{M}_2 < 0 \text{ subject to } \mathcal{M}_3 \geqslant 0. \tag{48}$$

Under Assumption 5′, by employing Lemma 4, similar to the proof of [25, Theorem 5.1], solutions to Problem II for the system (42) can be obtained from the following result.

Theorem 8. Under Assumption 5′ with $\Phi_{22} \leqslant 0$, $\Phi_{11} - \Phi_{12}\Phi_{22}^{\dagger}\Phi_{12}^{\mathrm{T}} \geqslant 0$ and $\ker \Phi_{22} \subseteq \ker \Phi_{12}$. Problem II is solvable with $F = YP^{-1}$ if and only if there exist $P \in \mathbb{S}_+^n$, $Y \in \mathbb{R}^{m\times n}$ and a scalar $\varepsilon_0 > 0$ such that

$$\begin{bmatrix} B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} P - \varepsilon_0 I & 0 & 0 & 0 \\ \star & -P & -Y^{\mathrm{T}} & 0 \\ \star & \star & 0 & Y \\ \star & \star & \star & P \end{bmatrix} \leqslant 0. \tag{49}$$

The QMI approach is a powerful framework in data-driven control that enables the design of stabilizing controllers directly from data, without the need for explicit identification of the system matrices. Compared to other methods, such as those based on Willems' fundamental lemma, the QMI approach does not require strict rank conditions on the data and can accommodate a broader class of noise models (e.g., bounded energy, covariance bounds, bounded noise within a subspace). This makes it particularly well-suited for practical scenarios involving uncertainty or measurement noise.

Moreover, the matrix S-lemmas (Lemmas 3 and 4) employed in this approach are notably general. They encompass matrix versions of classical results, such as the S-procedure [33, 34], Finslers lemma [35], and Petersens lemma [36], as special cases. This broader applicability is made possible by the removal of the generalized Slater condition [24], which is typically required in traditional formulations.

4 Control design combining prior knowledge and noisy data

In [37], a novel framework is presented to design suitable controllers by combining data of an unknown linear time-invariant (LTI) system with prior knowledge either on the system matrices or on the uncertainty. The key idea is to model the partially known system as a LFT that can separate known and unknown components effectively.

Consider an uncertain LTI system as

$$\begin{bmatrix} x(k+1) \\ \hline z(k) \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & C_2 & 0 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hline u(k) \\ w(k) \\ \tau(k) \end{bmatrix}, \tag{50a}$$

$$\tau(k) = \Delta_0 z(k), \tag{50b}$$

where $\tau(k) \in \mathbb{R}^{n_\tau}$ and $z(k) \in \mathbb{R}^{n_z}$ represent an uncertain channel; the real matrices A, B_1, B_2, B_3, C_1 and C_2 are known while the real matrix Δ_0 is unknown. All matrices have compatible dimensions. The system (50) is an LFT, which consists of an LTI system interconnected with an uncertain channel from z to τ with an uncertainty Δ_0.

As is well known, an LFT provides a structured framework for representing uncertain systems, allowing a control problem to be formulated as an interconnection between a nominal plant and an uncertainty. An insightful observation from [37] is that an LFT can also be used to describe system (42) in a data-driven setting. Specifically, by setting $A = 0, B_1 = 0, B_2 = B_w, B_3 = I, C_1 = \mathrm{col}\{I\ 0\}, C_2 = \mathrm{col}\{0\ I\}$ and $\Delta_0 = [A_{\mathrm{un}}\ B_{\mathrm{un}}]$, the LFT representation (50) reduces exactly to the system (42). This observation opens up a brand new door for designing suitable robust controllers by combining data with prior knowledge of uncertainty in a partially unknown system. An equivalent form of the LFT (50) can be given as follows:

$$\begin{bmatrix} x(k+1) \\ \hline z(k) \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_1 & B_2 & I \\ \hline C_1 & C_2 & 0 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hline u(k) \\ w(k) \\ \bar{\tau}(k) \end{bmatrix}, \tag{51a}$$

$$\bar{\tau}(k) = \bar{\Delta}_0 z(k), \tag{51b}$$

where $\bar{\tau}(k) = B_3\tau(k)$ and $\bar{\Delta}_0 = B_3\Delta_0$.

Suppose that some prior knowledge on Δ_0 is available, i.e., $\Delta_0 \in \mathbf{\Delta}_{\mathrm{pr}} \subseteq \mathbb{R}^{n_\tau \times n_z}$ ($\bar{\Delta}_0 \in B_3\mathbf{\Delta}_{\mathrm{pr}}$), and the unknown disturbance matrix $W = [w(0)\ w(1)\ \cdots\ w(T-1)]$ belongs to a set $\mathcal{W} \subseteq \mathbb{R}^{n_w \times T}$ given as

$$\mathcal{W} = \left\{ W \,\middle|\, \begin{bmatrix} I_{n_w} \\ W^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \Theta \begin{bmatrix} I_{n_w} \\ W^{\mathrm{T}} \end{bmatrix} \geqslant 0,\ \forall \Theta \in \mathbf{\Theta} \right\} \tag{52}$$

with $\mathbf{\Theta}$ being a convex cone of real symmetric matrices and $[0\ I]\Theta[0\ I]^{\mathrm{T}} < 0$ for $\forall \Theta \in \mathbf{\Theta}$.

From the LFT (50), the collected data X_0, X_1 and U_0 satisfy

$$X_1 = AX_0 + B_1U_0 + B_2W + B_3\Delta_0(C_1X_0 + C_2U_0). \tag{53}$$

Let $M = X_1 - AX_0 - B_1U_0$ and $N = C_1X_0 + C_2U_0$. Define a set of learnt uncertainty as

$$\mathbf{\Delta}_{\rm ln} = \{\Delta | M = B_3\Delta N + B_2W \text{ for } W \in \mathcal{W}\}. \tag{54}$$

Then the uncertainty Δ_0 belongs not only to $\mathbf{\Delta}_{\rm pr}$ but also to $\mathbf{\Delta}_{\rm ln}$. That is, $\Delta_0 \in \mathbf{\Delta}_{\rm com} := \mathbf{\Delta}_{\rm pr} \cap \mathbf{\Delta}_{\rm ln}$. However, the set $\mathbf{\Delta}_{\rm com}$ is not convenient to use for robust stability analysis. To address this issue, a QMI representation for the set $\mathbf{\Delta}_{\rm com}$ is proposed in [37].

4.1 QMI representations of the sets $\mathbf{\Delta}_{\rm ln}$ and $\mathbf{\Delta}_{\rm pr}$

First, with the set $\mathcal{W}$ in (52), similar to the proof of Lemma 5, if the matrix B_2 has full column rank, the set $\mathbf{\Delta}_{\rm ln}$ in (54) can be equivalently written as

$$\mathbf{\Delta}_{\rm ln} = \left\{\Delta \middle| \begin{bmatrix} I_n \\ (B_3\Delta)^{\rm T} \end{bmatrix}^{\rm T} \begin{bmatrix} B_2 & M \\ 0 & -N \end{bmatrix} \Theta \begin{bmatrix} B_2 & M \\ 0 & -N \end{bmatrix}^{\rm T} \begin{bmatrix} I_n \\ (B_3\Delta)^{\rm T} \end{bmatrix} \geqslant 0,\ \Theta \in \mathbf{\Theta} \right\}. \tag{55}$$

Let

$$\bar{\mathbf{\Delta}}_{\rm ln} = \left\{\bar{\Delta} \middle| \begin{bmatrix} I_n \\ \bar{\Delta}^{\rm T} \end{bmatrix}^{\rm T} \begin{bmatrix} B_2 & M \\ 0 & -N \end{bmatrix} \Theta \begin{bmatrix} B_2 & M \\ 0 & -N \end{bmatrix}^{\rm T} \begin{bmatrix} I_n \\ \bar{\Delta}^{\rm T} \end{bmatrix} \geqslant 0,\ \Theta \in \mathbf{\Theta} \right\}. \tag{56}$$

Then $\Delta_0 \in \mathbf{\Delta}_{\rm ln}$ implies $\bar{\Delta}_0 \in \bar{\mathbf{\Delta}}_{\rm ln}$. Clearly, each element in the set $\bar{\mathbf{\Delta}}_{\rm ln}$ satisfies a QMI.

Second, for the description of the set $\mathbf{\Delta}_{\rm pr}$, we need the following assumption.

Assumption 6. Suppose that Δ_0 is of a block-diagonal structure, i.e.,

$$\Delta_0 = {\rm diag}\{\Delta_1, \Delta_2, \cdots, \Delta_q\}, \tag{57}$$

where Δ_j is either a full-block or a repeated scalar block $\delta_j I$ with $\delta_j \in \mathbb{R}$, satisfying $\Delta_j \in \mathbf{\Delta}_j$, and

$$\mathbf{\Delta}_j = \left\{\Delta_j \in \mathbb{R}^{n_{\tau,j} \times n_{z,j}} \middle| \begin{bmatrix} I_{n_{\tau,j}} \\ \Delta_j^{\rm T} \end{bmatrix}^{\rm T} \Omega_j \begin{bmatrix} I_{n_{\tau,j}} \\ \Delta_j^{\rm T} \end{bmatrix} \geqslant 0,\ \forall \Omega_j \in \mathbf{\Omega}_j \right\} \tag{58}$$

with $\mathbf{\Omega}_j$ being a convex cone of real symmetric matrices and $[0\ I]\Omega_j[0\ I]^{\rm T} < 0$ for $\forall\, \Omega_j \in \mathbf{\Omega}_j$.

Under Assumption 6, the set $\mathbf{\Delta}_{\rm pr}$ can be described by

$$\mathbf{\Delta}_{\rm pr} = \left\{\Delta \in \mathbb{R}^{n_\tau \times n_z} \middle| \Delta = {\rm diag}\{\Delta_1, \Delta_2, \cdots, \Delta_q\},\ \Delta_j \in \mathbf{\Delta}_j \right\}. \tag{59}$$

Let R_j be the jth row-block matrix of the n_z-dimensional identity matrix and L_j be the jth column-block matrix of the n_τ-dimensional identity matrix such that

$$\Delta = {\rm diag}\{\Delta_1, \Delta_2, \cdots, \Delta_q\} = \sum_{j=1}^{q} L_j \Delta_j R_j. \tag{60}$$

Thus, it is clear that $\Delta R_j^{\rm T} = L_j\Delta_j$. Further, from (58), one has that

$$\begin{aligned} 0 \leqslant (B_3L_j) \begin{bmatrix} I_{n_{\tau,j}} \\ \Delta_j^{\rm T} \end{bmatrix}^{\rm T} \Omega_j \begin{bmatrix} I_{n_{\tau,j}} \\ \Delta_j^{\rm T} \end{bmatrix} (B_3L_j)^{\rm T} &= \begin{bmatrix} (B_3L_j)^{\rm T} \\ (B_3L_j\Delta_j)^{\rm T} \end{bmatrix}^{\rm T} \Omega_j \begin{bmatrix} (B_3L_j)^{\rm T} \\ (B_3L_j\Delta_j)^{\rm T} \end{bmatrix} \\ &= \begin{bmatrix} (B_3L_j)^{\rm T} \\ R_j(B_3\Delta)^{\rm T} \end{bmatrix}^{\rm T} \Omega_j \begin{bmatrix} (B_3L_j)^{\rm T} \\ R_j(B_3\Delta)^{\rm T} \end{bmatrix} \\ &= \begin{bmatrix} I_n \\ (B_3\Delta)^{\rm T} \end{bmatrix}^{\rm T} \begin{bmatrix} B_3L_j & 0 \\ 0 & R_j^{\rm T} \end{bmatrix} \Omega_j \begin{bmatrix} B_3L_j & 0 \\ 0 & R_j^{\rm T} \end{bmatrix}^{\rm T} \begin{bmatrix} I_n \\ (B_3\Delta)^{\rm T} \end{bmatrix}. \end{aligned} \tag{61}$$

Let

$$\bar{\mathbf{\Delta}}_{\rm pr}=\left\{\bar{\Delta}\left|\begin{bmatrix}I_n\\ \bar{\Delta}^{\rm T}\end{bmatrix}^{\rm T}\sum_{j=1}^{q}\begin{bmatrix}B_3L_j & 0\\ 0 & R_j^{\rm T}\end{bmatrix}\Omega_j\begin{bmatrix}B_3L_j & 0\\ 0 & R_j^{\rm T}\end{bmatrix}^{\rm T}\begin{bmatrix}I_n\\ \bar{\Delta}^{\rm T}\end{bmatrix}\geqslant 0,\ \Omega_j\in\mathbf{\Omega}_j\right.\right\}. \tag{62}$$

Similar to the proof of [37, Lemma 1], if each of the matrices B_3L_j $(j=1,2,\cdots,q)$ has full column rank, under Assumption 6, one has $\bar{\mathbf{\Delta}}_{\rm pr}=B_3\mathbf{\Delta}_{\rm pr}$. Hence, $\Delta_0\in\mathbf{\Delta}_{\rm pr}$ implies $\bar{\Delta}_0\in\bar{\mathbf{\Delta}}_{\rm pr}$. Clearly, each element in the set $\bar{\mathbf{\Delta}}_{\rm pr}$ also satisfies a QMI.

From the analysis above, $\Delta_0\in\mathbf{\Delta}_{\rm com}(=\mathbf{\Delta}_{\rm pr}\cap\mathbf{\Delta}_{\rm ln})$ implies $\bar{\Delta}_0\in\bar{\mathbf{\Delta}}_{\rm com}:=\bar{\mathbf{\Delta}}_{\rm pr}\cap\bar{\mathbf{\Delta}}_{\rm ln}$. Consequently, a QMI representation of $\bar{\mathbf{\Delta}}_{\rm com}$ can be given by

$$\bar{\mathbf{\Delta}}_{\rm com}=\left\{\bar{\Delta}\left|\begin{bmatrix}I_n\\ \bar{\Delta}^{\rm T}\end{bmatrix}^{\rm T}\Omega_{\rm com}\begin{bmatrix}I_n\\ \bar{\Delta}^{\rm T}\end{bmatrix}\geqslant 0\right.\right\}=\left\{\bar{\Delta}\left|\begin{bmatrix}\bar{\Delta}^{\rm T}\\ I_n\end{bmatrix}^{\rm T}\mathcal{I}_0^{\rm T}\Omega_{\rm com}\mathcal{I}_0\begin{bmatrix}\bar{\Delta}^{\rm T}\\ I_n\end{bmatrix}\geqslant 0\right.\right\}, \tag{63}$$

where $\mathcal{I}_0=\left[\begin{smallmatrix}0 & I_n\\ I_{n_z} & 0\end{smallmatrix}\right]$ and

$$\Omega_{\rm com}=\begin{bmatrix}B_2 & M\\ 0 & -N\end{bmatrix}\Theta\begin{bmatrix}B_2 & M\\ 0 & -N\end{bmatrix}^{\rm T}+\sum_{j=1}^{q}\begin{bmatrix}B_3L_j & 0\\ 0 & R_j^{\rm T}\end{bmatrix}\Omega_j\begin{bmatrix}B_3L_j & 0\\ 0 & R_j^{\rm T}\end{bmatrix}^{\rm T},\ \Theta\in\mathbf{\Theta},\ \Omega_j\in\mathbf{\Omega}_j.$$

4.2 Controller design

With the QMI representation (63), the S-lemmas, i.e., Lemmas 3 and 4, can be used for stability analysis of the LFT (50) using the QMI approach aforementioned. However, some conditions on $\Omega_{\rm com}$ in these lemmas should be checked. Alternatively, in [37], a full-block S-procedure [34] is employed to address this issue.

Problem III: Design a controller $u(k)=Fx(k)$ such that the following matrix inequality

$$\begin{bmatrix}I_n\\ \mathcal{A}_{\bar{\Delta}}^{\rm T}\end{bmatrix}^{\rm T}\begin{bmatrix}-P & 0\\ 0 & P\end{bmatrix}\begin{bmatrix}I_n\\ \mathcal{A}_{\bar{\Delta}}^{\rm T}\end{bmatrix}<0 \tag{64}$$

is feasible on $P>0$ for $\forall\bar{\Delta}\in\bar{\mathbf{\Delta}}_{\rm com}$, where $\mathcal{A}_{\bar{\Delta}}=\bar{A}_F+\bar{\Delta}\bar{C}_F$ with $\bar{A}_F=A+B_1F$ and $\bar{C}_F=C_1+C_2F$.

To solve Problem III, we need the following full-block S-procedure.

Lemma 6. Let $\mathcal{S}\subseteq\mathbb{R}^t$, $\mathbf{U}\subset\mathbb{R}^{r\times l}$ and $\mathcal{T}\in\mathbb{R}^{l\times t}$, where $\mathbf{U}$ is a compact set of matrices of full row rank and $\mathcal{T}$ is a full row rank matrix. For $\mathcal{U}\in\mathbf{U}$, let $\mathcal{S}_u=\mathcal{S}\cap\ker(\mathcal{U}\mathcal{T})$. Then the following conditions are equivalent[1)]:

(i) $\forall\mathcal{U}\in\mathbf{U}$: $\mathcal{S}_u\cap\mathcal{S}_0=\{0\}$ and $\mathcal{N}<0$ on $\mathcal{S}_u$;

(ii) $\forall\mathcal{U}\in\mathbf{U}$: $\mathcal{N}+\mathcal{T}^{\rm T}\mathcal{P}\mathcal{T}<0$ on $\mathcal{S}$ and $\mathcal{P}>0$ on $\ker(\mathcal{U})$.

As an application of Lemma 6, set $\mathcal{U}=[I_{n_z}\ \ -\bar{\Delta}^{\rm T}]$, $\mathcal{P}=\mathcal{I}_0^{\rm T}\Omega_{\rm com}\mathcal{I}_0$ and

$$\mathcal{N}=\left[\begin{array}{cc|cc}-P & 0 & 0 & 0\\ 0 & P & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right],\ \mathcal{S}=\mathrm{im}\left[\begin{array}{cc}I_n & 0\\ \bar{A}_F^{\rm T} & \bar{C}_F^{\rm T}\\ \hline 0 & I_{n_z}\\ I_n & 0\end{array}\right],\ \mathcal{S}_0=\mathrm{im}\left[\begin{array}{c}0\\ \bar{C}_F^{\rm T}\\ \hline I_{n_z}\\ 0\end{array}\right],\ \mathcal{T}=\left[\begin{array}{cc|cc}0 & 0 & I_{n_z} & 0\\ 0 & 0 & 0 & I_n\end{array}\right]. \tag{65}$$

It is easy to verify that

$$\mathcal{S}_u=\mathrm{im}\begin{bmatrix}I_n\\ \mathcal{A}_{\bar{\Delta}}^{\rm T}\\ \bar{\Delta}^{\rm T}\\ I_n\end{bmatrix},\ \mathcal{S}_u\cap\mathcal{S}_0=\{0\}. \tag{66}$$

Thus, one can conclude that

$$\mathcal{N}<0\text{ on }\mathcal{S}_u\iff(64), \tag{67}$$

1) '$\mathcal{N}<0$ on $\mathcal{S}_u$' means that '$S^{\rm T}\mathcal{N}S<0$ for any basis matrix S of $\mathcal{S}_u$'.

$$\mathcal{P} > 0 \text{ on } \ker(\mathcal{U}) \iff \begin{bmatrix} \bar{\Delta}^{\mathrm{T}} \\ I_n \end{bmatrix}^{\mathrm{T}} \mathcal{I}_0^{\mathrm{T}} \Omega_{\mathrm{com}} \mathcal{I}_0 \begin{bmatrix} \bar{\Delta}^{\mathrm{T}} \\ I_n \end{bmatrix} > 0, \tag{68}$$

$$\begin{aligned} \mathcal{N} + \mathcal{T}^{\mathrm{T}}\mathcal{P}\mathcal{T} < 0 \text{ on } \mathcal{S} &\iff \begin{bmatrix} I_n & 0 \\ \bar{A}_F^{\mathrm{T}} & \bar{C}_F^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \bar{A}_F^{\mathrm{T}} & \bar{C}_F^{\mathrm{T}} \end{bmatrix} + \Omega_{\mathrm{com}} < 0 \\ &\iff \Omega_{\mathrm{com}} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{A}_F \\ \bar{C}_F \end{bmatrix} P \begin{bmatrix} \bar{A}_F^{\mathrm{T}} & \bar{C}_F^{\mathrm{T}} \end{bmatrix} < 0 \\ &\iff \begin{bmatrix} \Omega_{\mathrm{com}} - \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \bar{A}_F P \\ \bar{C}_F P \end{pmatrix} \\ \star & -P \end{bmatrix} < 0. \end{aligned} \tag{69}$$

Set $Y = FP$. Then a solution to Problem III can be given as follows.

Theorem 9. Problem III is solvable with $F = YP^{-1}$ if there exist $P > 0$ and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} \Omega_{\mathrm{com}} - \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} AP + B_1 Y \\ C_1 + C_2 Y \end{pmatrix} \\ \star & -P \end{bmatrix} < 0. \tag{70}$$

Theorem 9 presents a condition for solving Problem III. If the LMI (70) is feasible, then the LFT (50) (or (51)) is asymptotically stable. It is important to note that this LMI depends not only on the known system matrices A, B_1, B_2, B_3, but also on the collected data X_0, X_1, U_0, as well as on the multipliers Ω_j and Θ. As such, Theorem 9 offers a novel approach for stabilizing an LFT system with an unknown parameter Δ_0 without relying on system identification. Moreover, this framework can also be extended to the design of robust out-feedback controllers based on noisy data. In fact, systems of the form $y(k) = A_1 y(k-1) + \cdots + A_n y(k-n) + B_0 u(k) + \cdots + B_n u(k-n) + B_w w(k)$ can be equivalently represented as an LFT system in the form of (50); see [37] for details.

5 Data-driven control using IQC approaches

The IQC approach is a powerful and unified framework used for the analysis and design of control systems involving uncertainties and nonlinearities. At its core, the IQC method characterizes uncertain or nonlinear components by a constraint on their input-output signals, expressed as an integral (or sum, in discrete time) of a quadratic form. This allows the uncertain system to be modeled as an interconnection between a known LTI system and an uncertain operator satisfying a given IQC. Stability or performance of the overall system is then verified by checking whether the associated IQC holds over all admissible trajectories of the uncertainty.

What makes the IQC framework especially attractive is its generality and compatibility with convex optimization tools. It can capture a broad class of uncertainties, such as gain-bounded nonlinearities, time delays, unmodeled dynamics, and even dynamic uncertainties, within a single unified framework. Moreover, the resulting stability conditions can typically be formulated as LMIs, making the approach computationally tractable. IQCs thus bridge robust control theory and convex optimization, enabling the analysis and synthesis of controllers for systems where conventional techniques fall short. In [38], the IQC method is used to deal with data-driven control for discrete-time systems with non-uniform sampling.

5.1 Problem statement

Consider the following linear system with unknown system matrices:

$$x(k+1) = A_{\mathrm{un}} x(k) + B_{\mathrm{un}} u(k),\ x(k_0) = \phi_0, \tag{71a}$$

$$u(k) = B_{\mathrm{un}} F x(k_t),\ k \in \mathbb{N} \cap [k_t, k_{t+1}), \tag{71b}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and ϕ_0 is the initial condition at the initial time k_0. A_{un} and B_{un} are unknown matrices with compatible dimensions while F is the control gain to be designed. The sequence $\{k_t : t = 1, 2, \cdots\}$ represents when the system state is sampled. Let $h_t := k_{t+1} - k_t$ and suppose that $1 \leqslant \underline{h} \leqslant h_t \leqslant \bar{h}$ with $\underline{h}$ and $\bar{h}$ being two known positive integers.

The system matrices $A_{\rm un}$ and $B_{\rm un}$ are unknown, but the pair $[A_{\rm un}\ B_{\rm un}]$ belongs to the set $\mathscr{S}_6$, defined in (44). From Lemma 5, one has

$$[A_{\rm un}\ B_{\rm un}] \in \mathscr{S}_6 = \mathscr{S}_7 = \left\{ [A\ B] \ \Big|\ \begin{bmatrix} I_n \\ [A\ B]^{\rm T} \end{bmatrix}^{\rm T} \mathcal{M}_3 \begin{bmatrix} I_n \\ [A\ B]^{\rm T} \end{bmatrix} \geqslant 0 \right\}, \tag{72}$$

where $\mathcal{M}_3$ is given in (46). The problem is then stated as follows.

Problem IV: Design control gain F such that the closed-loop system

$$x(k+1) = Ax(k) + BFx(k_t) \tag{73}$$

is asymptotically stable for $\forall [A\ B] \in \mathscr{S}_7$.

An IQC method is introduced in [38] to solve Problem IV.

5.2 An interconnection representation and stability analysis

Let $\rho(k) = k - k_t$. Then $\underline{h} \leqslant \rho(k) \leqslant \bar{h}$, and $x(k_t) = x(k - \rho(k))$, leading to

$$\begin{aligned} x(k+1) &= (A+BF)x(k) + BF[x(k-\rho(k)) - x(k)], \\ z(k) = x(k+1) - x(k) &= (A+BF-I)x(k) + BF[x(k-\rho(k)) - x(k)]. \end{aligned}$$

Set $\tau(k) = x(k) - x(k-\rho(k))$. Then

$$\tau(k) = \sum_{i=k-\rho(k)}^{k-1} [x(i+1) - x(i)] = \sum_{i=k-\rho(k)}^{k-1} z(i).$$

In this situation, an interconnection system is obtained as

$$\left[\begin{array}{c} x(k+1) \\ \hline z(k) \end{array}\right] = \left[\begin{array}{c|c} A+BF & BF \\ \hline A+BF-I & BF \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline \tau(k) \end{array}\right], \tag{74a}$$

$$\tau(k) = \Delta z(k) := \sum_{i=k-\rho(k)}^{k-1} z(i). \tag{74b}$$

In [38], it is proven that the operator Δ defined in (74b) is bounded and satisfies, for $\forall T_0 \in \mathbb{N}$

$$\sum_{k=0}^{T_0} [\Delta z(k)]^{\rm T} \Delta z(k) \leqslant (1/2)\bar{h}(\bar{h}-1) \sum_{k=0}^{T_0} [z(k)]^{\rm T} z(k). \tag{75}$$

The inequality (75) provides an upper bound $\sqrt{(1/2)\bar{h}(\bar{h}-1)}$ on the l_2 gain of the delay operator Δ, which can be regarded as a discrete-time counterpart to the continuous-time case reported in [39]. This bound implies that the delay operator Δ satisfies a hard static IQC. Specifically, for any $R \in \mathbb{S}_+^n$, the following inequality holds for all $T_0 \in \mathbb{N}$:

$$\sum_{k=0}^{T_0} \begin{bmatrix} z(k) \\ \Delta z(k) \end{bmatrix}^{\rm T} \Pi_{l_2} \begin{bmatrix} z(k) \\ \Delta z(k) \end{bmatrix} \geqslant 0, \quad \text{where} \quad \Pi_{l_2} := \begin{bmatrix} (1/2)\bar{h}(\bar{h}-1)R & 0 \\ 0 & -R \end{bmatrix}. \tag{76}$$

Since the inequality (76) is satisfied, as usual, we say that $\Delta \in {\rm IQC}(\Pi_{l_2})$. From the IQC theory [40], the system (74) is asymptotically stable if there exist a positive definite storage function (e.g., a Lyapunov function $V(x) = x^{\rm T}P^{-1}x$ with $P \in \mathbb{S}_+^n$) and a scalar $\lambda \geqslant 0$ such that the following dissipation inequality holds

$$V(x(k+1)) - V(x(k)) + \lambda \begin{bmatrix} z(k) \\ \tau(k) \end{bmatrix}^{\rm T} \Pi_{l_2} \begin{bmatrix} z(k) \\ \tau(k) \end{bmatrix} \leqslant -\epsilon \|x(k)\|^2 \tag{77}$$

for some $\epsilon > 0$, which leads to the following conclusion.

Theorem 10 ([38, Theorem 10])**.** For $\bar{h} \geqslant 2$, the system (74) (or (73)) is asymptotically stable if there exist $P, R \in \mathbb{S}_+^n$ such that

$$\left[\begin{array}{cc} A+BF & BF \\ I & 0 \\ \hline A+BF-I & BF \\ 0 & I \end{array}\right]^{\mathrm{T}} \left[\begin{array}{cc|cc} P^{-1} & 0 & 0 & 0 \\ 0 & -P^{-1} & 0 & 0 \\ \hline 0 & 0 & (1/2)\bar{h}(\bar{h}-1)R^{-1} & 0 \\ 0 & 0 & 0 & -R^{-1} \end{array}\right] \left[\begin{array}{cc} A+BF & BF \\ I & 0 \\ \hline A+BF-I & BF \\ 0 & I \end{array}\right] < 0. \tag{78}$$

5.3 Control design

It is clear that Theorem 10 depends on the unknown matrices A and B. In order to solve them from Theorem 10, the full-block S-procedure (Lemma 6) and the following dualization lemma are useful.

Lemma 7 (Dualization lemma [41, Lemma 4.8])**.** Let $\Phi \in \mathbb{S}^n$ be invertible, and $\mathscr{U}$ and $\mathscr{V}$ be two complementary subspaces whose sum equals $\mathbb{R}^n$. Then $x^{\mathrm{T}}\Phi x < 0$ for $\forall x \in \mathscr{U}\backslash\{0\}$ is equivalent to $x^{\mathrm{T}}\Phi^{-1}x > 0$ for $\forall x \in \mathscr{U}^{\perp}\backslash\{0\}$.

Applying Lemma 7 to (78), one has

$$\beth_1 := \left[\begin{array}{cc} I & 0 \\ -(A+BF)^{\mathrm{T}} & -(A+BF-I)^{\mathrm{T}} \\ \hline 0 & I \\ -(BF)^{\mathrm{T}} & -(BF)^{\mathrm{T}} \end{array}\right]^{\mathrm{T}} \left[\begin{array}{cc|cc} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ \hline 0 & 0 & \frac{-2}{\bar{h}(\bar{h}-1)}R & 0 \\ 0 & 0 & 0 & R \end{array}\right] \left[\begin{array}{cc} I & 0 \\ -(A+BF)^{\mathrm{T}} & -(A+BF-I)^{\mathrm{T}} \\ \hline 0 & I \\ -(BF)^{\mathrm{T}} & -(BF)^{\mathrm{T}} \end{array}\right] < 0. \tag{79}$$

Note that

$$\begin{bmatrix} I & 0 \\ -(A+BF)^{\mathrm{T}} & -(A+BF-I)^{\mathrm{T}} \\ 0 & I \\ -(BF)^{\mathrm{T}} & -(BF)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -\left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right)^{\mathrm{T}} \\ 0 & I & 0 \\ 0 & 0 & -\left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ [A\ B]^{\mathrm{T}} & [A\ B]^{\mathrm{T}} \end{bmatrix}. \tag{80}$$

Then $\beth_1$ in (79) can be written as $\beth_1 = \Gamma_0^{\mathrm{T}}\daleth_1\Gamma_0$, where

$$\Gamma_0 = \begin{bmatrix} I & 0 \\ 0 & I \\ [A\ B]^{\mathrm{T}} & [A\ B]^{\mathrm{T}} \end{bmatrix}, \quad \daleth_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -\left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right)^{\mathrm{T}} \\ 0 & I & 0 \\ 0 & 0 & -\left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \frac{-2}{\bar{h}(\bar{h}-1)}R & 0 \\ 0 & 0 & 0 & R \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & -\left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right)^{\mathrm{T}} \\ 0 & I & 0 \\ 0 & 0 & -\left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right)^{\mathrm{T}} \end{bmatrix}. \tag{81}$$

From (72), the following holds:

$$\beth_2 := \begin{bmatrix} I \\ I \end{bmatrix} \begin{bmatrix} I \\ [A\ B]^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \mathcal{M}_3 \begin{bmatrix} I \\ [A\ B]^{\mathrm{T}} \end{bmatrix} [I\ I] = \Gamma_0^{\mathrm{T}}\daleth_2\Gamma_0 \geqslant 0, \tag{82}$$

where

$$\daleth_2 = \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \mathcal{M}_3 \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} B_w & X_1 \\ B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} B_w & X_1 \\ B_w & X_1 \\ 0 & -X_0 \\ 0 & -U_0 \end{bmatrix}^{\mathrm{T}}. \tag{83}$$

Thus, Problem IV is converted into the problem that $\beth_1 = \Gamma_0^{\mathrm{T}}\daleth_1\Gamma_0 < 0$ subject to $\beth_2 = \Gamma_0^{\mathrm{T}}\daleth_2\Gamma_0 \geqslant 0$. By the full-block S-procedure, the matrix inequality (79) is satisfied if $\daleth_1 + \daleth_2 < 0$. Direct computation yields $\daleth_1 = \daleth_{10} + \daleth_{11}$ with

$$\daleth_{10} = \begin{bmatrix} -P & 0 & 0 \\ \star & P - \frac{2}{\bar{h}(\bar{h}-1)}R & -\left(\begin{smallmatrix} P \\ Y \end{smallmatrix}\right)^{\mathrm{T}} \\ \star & \star & 0 \end{bmatrix}, \quad \daleth_{11} = \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & \left(\begin{smallmatrix} P \\ Y \end{smallmatrix}\right)P^{-1}(P+R)P^{-1}\left(\begin{smallmatrix} P \\ Y \end{smallmatrix}\right)^{\mathrm{T}} \end{bmatrix}, \tag{84}$$

where $Y = FP$. Applying the Schur complement to the matrix inequality $\daleth_1 + \daleth_2 < 0$, a solution to Problem IV is then obtained, which is given in the following.

Theorem 11. For a given scalar $\bar{h} \geqslant 2$, Problem IV is solvable with $F = YP^{-1}$ if there exist $P, R \in \mathbb{S}_+^n$ and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} \daleth_{10} + \daleth_2 & \begin{pmatrix} 0 \\ 0 \\ P \\ Y \end{pmatrix} \\ \star & -P(P+R)^{-1}P \end{bmatrix} < 0, \tag{85}$$

where $\daleth_{10}$ and $\daleth_2$ are given in (84) and (83), respectively.

Theorem 11 presents a data-based approach for solving Problem IV. However, the matrix inequality (85) is nonlinear due to the term $-P(P+R)^{-1}P$. Fortunately, similar to [42], three alternative strategies can be employed to obtain feasible solutions.

- *Scalar parameterization:* Let $R = \lambda_1 P$, where $\lambda_1 > 0$ is a given scalar. Then the nonlinear term simplifies as $-P(P+R)^{-1}P = -(1+\lambda_1)^{-1}P$. Under this substitution, the matrix inequality (85) is satisfied if the following LMI holds for some $\lambda_1 > 0$:

$$\begin{bmatrix} \daleth_{10} + \daleth_2 & (1+\lambda_1)\begin{pmatrix} 0 \\ 0 \\ P \\ Y \end{pmatrix} \\ \star & -(1+\lambda_1)P \end{bmatrix} < 0. \tag{86}$$

- *Inequality relaxation using a scalar bound:* Consider the inequality $[(P+R)^{-1}P - \lambda_2 I]^{\mathrm{T}}(P+R)[(P+R)^{-1}P - \lambda_2 I] \geqslant 0$ for any $\lambda_2 \in \mathbb{R}$, which implies $-P(P+R)^{-1}P \leqslant \lambda_2^2(P+R) - 2\lambda_2 P$. Then the matrix inequality (85) is satisfied if the following LMI holds for some $\lambda_2 \in \mathbb{R}$:

$$\begin{bmatrix} \daleth_{10} + \daleth_2 & \begin{pmatrix} 0 \\ 0 \\ P \\ Y \end{pmatrix} \\ \star & \lambda_2^2(P+R) - 2\lambda_2 P \end{bmatrix} < 0. \tag{87}$$

- *Cone complementary linearization (CCL):* Introduce a matrix variable $S > 0$ such that $-P(P+R)^{-1}P \leqslant -S$. Define $\bar{P} = P^{-1}$, $\bar{R} = (P+R)^{-1}$ and $\bar{S} = S^{-1}$. Then the matrix inequality (85) is satisfied if the following conditions hold:

$$\begin{bmatrix} \daleth_{10} + \daleth_2 & \begin{pmatrix} 0 \\ 0 \\ P \\ Y \end{pmatrix} \\ \star & -S \end{bmatrix} < 0, \quad \begin{bmatrix} \bar{R} & \bar{P} \\ \star & \bar{S} \end{bmatrix} \geqslant 0, \ \bar{P}P = I, \ \bar{R}(P+R) = I, \ \bar{S}S = I. \tag{88}$$

Using the cone complementary linearization method proposed in [43], this nonconvex feasibility problem can be recast as a nonlinear optimization problem subject to LMIs.

This section presents an IQC method for addressing Problem IV, utilizing the interconnection system representation given in (74). If defining $\varpi(k) = \left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right) x(k) + \left(\begin{smallmatrix} 0 \\ F \end{smallmatrix}\right) \tau(k)$, and $\omega(k) = [A\ B]\varpi(k)$, then the system (74) can be reformulated as the following LFT representation:

$$\left[\begin{array}{c} x(k+1) \\ \hline z(k) \\ \varpi(k) \end{array}\right] = \left[\begin{array}{c|cc} 0 & 0 & I \\ \hline -I & 0 & I \\ \left(\begin{smallmatrix} I \\ F \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 \\ F \end{smallmatrix}\right) & 0 \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline \tau(k) \\ \omega(k) \end{array}\right], \tag{89a}$$

$$\tau(k) = (\Delta z)(k), \tag{89b}$$

$$\omega(k) = ([A\ B]\varpi)(k). \tag{89c}$$

This structure represents an LFT with two uncertainty channels: $z \mapsto \tau$ and $\varpi \mapsto \omega$. Based on this formulation, Problem IV can potentially be solved using the approach introduced in Section 4 (or as developed in [37]). This connection opens an interesting direction for further investigation.

6 Data-driven control from noisy input-output data

Most of the aforementioned results are based on input-state data, assuming that the full system state can be measured during experiments. However, in many real-world applications, only noisy input-output measurements are available, and direct access to the full state is not feasible. In such cases, data-driven control methods that rely on input-state information become inapplicable. This limitation has motivated growing interest in developing data-driven control strategies that can operate directly from noisy input-output data [44].

For SISO systems, a state-space representation is taken [21] to design controllers from noisy data. Consider an SISO system described as

$$\begin{aligned} &y(k) + a_n y(k-1) + \cdots + a_2 y(k-n+1) + a_1 y(k-n) \\ &\quad = b_n u(k-1) + \cdots + b_2 u(k-n+1) + b_1 u(k-n). \end{aligned} \tag{90}$$

Let $x(k) = \mathrm{col}\{y(k-n), y(k-n+1), \cdots, y(k-1), u(k-n), u(k-n+1), \cdots, u(k-1)\}$. Then the system (90) can be represented as the following state-space form:

$$x(k+1) = \mathcal{A}x(k) + \mathcal{B}u(k), \tag{91a}$$

$$\mathcal{A} = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} & 0_{(n-1)\times 1} & 0_{n-1} \\ -a_1 & -(a_2\ a_3\ \cdots\ a_n) & b_1 & (b_2\ \cdots\ b_n) \\ 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0_{n\times 1} \\ 0_{(n-1)\times 1} \\ 1 \end{bmatrix}. \tag{91b}$$

With the state-space representation in (91), the problem of data-driven control from noisy input-output data is transformed into one of data-driven control from noisy "input-state" data. This allows the application of existing data-driven methods to the SISO system described by (90). However, the resulting analysis may be conservative due to the following reasons.

- The state-space representation (91) is not a minimal realization of system (90). Specifically, the state vector $x(k)$ has dimension $2n$, which implies that a large amount of data are required to ensure sufficient excitation and control design.
- When applying those methods based on Willems et al.'s fundamental lemma or quadratic matrix inequalities to system (91), both matrices $\mathcal{A}$ and $\mathcal{B}$ are typically treated as fully unknown. In reality, $\mathcal{B}$ is known, and only $2n$ out of the total $4n^2$ entries in $\mathcal{A}$ are truly unknown. This mismatch leads to unnecessary conservatism in the resulting control design.
- Although the method proposed in [37] offers a framework for integrating prior knowledge with experimental data, accurately defining the uncertain matrix Δ to capture the true structure of the system's uncertainties remains a challenging task.

In [45], a behavioural approach to data-driven control is proposed from noisy input-output data for input-output systems described by higher order difference equations, also called autoregressive systems, as follows:

$$y(k+p) + L^y_{p-1} y(k+p-1) + \cdots + L^y_0 y(k) = L^u_{p-1} u(k+p-1) + \cdots + L^u_0 u(k) + w(k), \tag{92}$$

where p is an integer indicating the order of the system; $u \in \mathbb{R}^m, y \in \mathbb{R}^{n_y}$, and $w \in \mathbb{R}^{n_y}$ are the system input, output and unknown noise, respectively; $L^y_i \in \mathbb{R}^{n_y\times n_y}$ and $L^u_i \in \mathbb{R}^{n_y\times m}$ $(i = 0, 1, \cdots, p-1)$ are unknown system parameters. Assume that the noisy input-output data from the true system (92) are available, given by $u(0), u(1), \cdots, u(T)$; $y(0), y(1), \cdots, y(T)$. Let $v(k) = \mathrm{col}\{u(k), y(k)\}$ and $\tilde{L}_i = [-L^u_i\ L^y_i]$ $(i = 0, 1, \cdots, p-1)$. Then the system (92) can be rewritten as

$$y(k+p) + \tilde{L}_{p-1} v(k+p-1) + \cdots + \tilde{L}_1 v(k+1) + \tilde{L}_0 v(k) = w(k). \tag{93}$$

Set $k = 0, 1, 2, \cdots, T-p$. Then

$$\tilde{W} \triangleq [w(0)\ w(1)\ \cdots\ w(T-p)] = \left[\ \mathcal{L} \ \middle|\ I_{n_y}\ \right] \begin{bmatrix} \aleph_1(v) \\ \hline \aleph_2(v) \end{bmatrix}, \tag{94}$$

where $\mathcal{L} = [\tilde{L}_0 \ \tilde{L}_1 \ \cdots \ \tilde{L}_{p-1}]$ and

$$\aleph(v) = \begin{bmatrix} \aleph_1(v) \\ \hline \aleph_2(v) \end{bmatrix} = \left[\begin{array}{cccc} v(0) & v(1) & \cdots & v(T-p) \\ v(1) & v(2) & \cdots & v(T-p+1) \\ \vdots & \vdots & \ddots & \vdots \\ v(p-1) & v(p) & \cdots & v(T-1) \\ \hline y(p) & y(p+1) & \cdots & y(T) \end{array}\right]. \tag{95}$$

Assumption 7. The unknown noise matrix $\tilde{W}$ satisfies QMI as

$$\begin{bmatrix} I_{n_y} \\ \tilde{W}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tilde{\Phi} \begin{bmatrix} I_{n_y} \\ \tilde{W}^{\mathrm{T}} \end{bmatrix} \geqslant 0, \ \tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{12}^{\mathrm{T}} & \tilde{\Phi}_{22} \end{bmatrix}, \ \tilde{\Phi}_{11} \in \mathbb{S}^{n_y}, \ \tilde{\Phi}_{12} \in \mathbb{R}^{n_y \times (T-p+1)}, \ \tilde{\Phi}_{22} \in \mathbb{S}^{T-p+1} \tag{96}$$

with $\tilde{\Phi}_{22} < 0$ and $\tilde{\Phi}_{11} - \tilde{\Phi}_{12}\tilde{\Phi}_{22}^{-1}\tilde{\Phi}_{12}^{\mathrm{T}} \geqslant 0$.

Under Assumption 7, it is clear that the unknown matrix $\mathcal{L}$ satisfies the following QMI:

$$\begin{bmatrix} I_{n_y} \\ \mathcal{L}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix} \tilde{\Phi} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_{n_y} \\ \mathcal{L}^{\mathrm{T}} \end{bmatrix} \geqslant 0, \tag{97}$$

which allows us to define a set $\mathscr{S}_8$ as

$$\mathscr{S}_8 = \left\{ \mathcal{L}^{\mathrm{T}} : \ \mathcal{L}^{\mathrm{T}} \text{ satisfies (97)} \right\}. \tag{98}$$

Thus, the true system parameter matrix $\mathcal{L}^{\mathrm{T}}$ belongs to $\mathscr{S}_8$. In the following, it is shown that the control problem can also be described as a QMI.

The stabilization problem of the system (92) is to design a controller of the following form:

$$u(k+p) + F_{p-1}^{u} u(k+p-1) + \cdots + F_0^{u} u(k) = F_{p-1}^{y} y(k+p-1) + \cdots + F_0^{y} y(k), \tag{99}$$

where $F_i^u \in \mathbb{R}^{m \times m}$ and $F_i^y \in \mathbb{R}^{m \times n_y}$ $(i = 0, 1, \cdots, p-1)$ are real matrices to be designed. Let $\tilde{F}_i = [F_i^u \ \ -F_i^y]$ $(i = 0, 1, \cdots, p-1)$. Then Eq. (99) can be rewritten as

$$u(k+p) + \tilde{F}_{p-1} v(k+p-1) + \cdots + \tilde{F}_1 v(k+1) + \tilde{F}_0 v(k) = 0. \tag{100}$$

The closed-loop system of (92) connecting with (99) can be given by

$$v(k+p) + \begin{bmatrix} \tilde{F}_{p-1} \\ \tilde{L}_{p-1} \end{bmatrix} v(k+p-1) + \cdots + \begin{bmatrix} \tilde{F}_1 \\ \tilde{L}_1 \end{bmatrix} v(k+1) + \begin{bmatrix} \tilde{F}_0 \\ \tilde{L}_0 \end{bmatrix} v(k) = \begin{bmatrix} 0 \\ w(k) \end{bmatrix}. \tag{101}$$

The system (101) with $w(k) \equiv 0$ is stable if $v(k) \to 0$ as $k \to \infty$ for all solutions v. Let $\mathcal{F} = [\tilde{F}_0 \ \tilde{F}_1 \ \cdots \ \tilde{F}_{p-1}]$. A necessary and sufficient condition on the stability of system (101) is that there exists $0 \leqslant \tilde{\mathcal{P}} \in \mathbb{S}^{p(m+n_y)}$ such that [45, Theorem 9]

$$\begin{bmatrix} I_{p(m+n_y)} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 0_{m+n_y} & 0 \\ 0 & \tilde{\mathcal{P}} \end{bmatrix} - \begin{bmatrix} \tilde{\mathcal{P}} & 0 \\ 0 & 0_{m+n_y} \end{bmatrix} \right) \begin{bmatrix} I_{p(m+n_y)} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix} < 0. \tag{102}$$

Then the stabilization problem based on noisy input-output data can be stated as follows.

Problem V: Design control gain $\mathcal{F}$ such that the QMI (102) holds for $\forall \mathcal{L}^{\mathrm{T}} \in \mathscr{S}_8$.

Although both Eqs. (97) and (102) are QMIs, they involve different matrix multipliers in their 'margins': $\mathrm{col}\{I_{n_y}, \mathcal{L}^{\mathrm{T}}\}$ in (97), and $\mathrm{col}\{I_{p(m+n_y)}, -\mathcal{F}, -\mathcal{L}\}$ in (102). As a result, the QMI method presented in Section 3 is not directly applicable to Problem V. To overcome this issue, the QMI (102) requires further reformulation. Note that with $\mathcal{J} = [0_{(p-1)(m+n_y)\times(m+n_y)}\ I_{(p-1)(m+n_y)}]$,

$$\begin{bmatrix} I_{p(m+n_y)} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0_{m+n_y} & 0 \\ 0 & \tilde{\mathcal{P}} \end{bmatrix} \begin{bmatrix} I_{p(m+n_y)} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix} = \begin{bmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}^{\mathrm{T}} \tilde{\mathcal{P}} \begin{bmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}, \quad \begin{bmatrix} I_{p(m+n_y)} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{\mathcal{P}} & 0 \\ 0 & 0_{m+n_y} \end{bmatrix} \begin{bmatrix} I_{p(m+n_y)} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix} = \tilde{\mathcal{P}}.$$

Then the QMI (102) can be rewritten as (the first inequality below ensures $\tilde{\mathcal{P}} > 0$)

$$\begin{bmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}^{\mathrm{T}} \tilde{\mathcal{P}} \begin{bmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix} - \tilde{\mathcal{P}} < 0 \iff \begin{bmatrix} \tilde{\mathcal{P}} & \begin{pmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{pmatrix}^{\mathrm{T}} \\ \star & \tilde{\mathcal{P}}^{-1} \end{bmatrix} > 0 \iff \tilde{\mathcal{P}}^{-1} - \begin{bmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix} \tilde{\mathcal{P}}^{-1} \begin{bmatrix} \mathcal{J} \\ -\mathcal{F} \\ -\mathcal{L} \end{bmatrix}^{\mathrm{T}} > 0. \tag{103}$$

Let $\mathcal{C}_0 = \mathrm{col}\{0_{(p-1)(n_y+m)\times n_y}, 0_{m\times n_y}, I_{n_y}\}$. Then $\mathrm{col}\{\mathcal{J}, -\mathcal{F}, -\mathcal{L}\} = \mathrm{col}\{\mathcal{J}, -\mathcal{F}, 0\} - \mathcal{C}_0\mathcal{L}$. Substituting it into the last inequality in (103) gives

$$\begin{bmatrix} I_{p(n_y+m)} \\ \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \hat{\Xi} \begin{bmatrix} I_{p(n_y+m)} \\ \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \end{bmatrix} > 0, \quad \hat{\Xi} \triangleq \begin{bmatrix} \tilde{\mathcal{P}}^{-1} - \begin{pmatrix} \mathcal{J} \\ -\mathcal{F} \\ 0 \end{pmatrix} \tilde{\mathcal{P}}^{-1} \begin{pmatrix} \mathcal{J} \\ -\mathcal{F} \\ 0 \end{pmatrix}^{\mathrm{T}} & \begin{pmatrix} \mathcal{J} \\ -\mathcal{F} \\ 0 \end{pmatrix} \tilde{\mathcal{P}}^{-1} \\ \star & -\tilde{\mathcal{P}}^{-1} \end{bmatrix}. \tag{104}$$

On the other hand, from (97), the following holds

$$\mathcal{C}_0 \begin{bmatrix} I_{n_y} \\ \mathcal{L}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix} \tilde{\Phi} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_{n_y} \\ \mathcal{L}^{\mathrm{T}} \end{bmatrix} \mathcal{C}_0^{\mathrm{T}} \geqslant 0 \tag{105}$$

$$\iff \begin{bmatrix} I_{p(n_y+m)} \\ \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \hat{\Psi} \begin{bmatrix} I_{p(n_y+m)} \\ \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \end{bmatrix} \geqslant 0, \ \hat{\Psi} \triangleq \begin{bmatrix} \mathcal{C}_0 & 0 \\ 0 & I_{p(n_y+m)} \end{bmatrix} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix} \tilde{\Phi} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{C}_0^{\mathrm{T}} & 0 \\ 0 & I_{p(n_y+m)} \end{bmatrix}. \tag{106}$$

Define a set $\mathscr{S}_9$ as

$$\mathscr{S}_9 = \left\{\mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} :\ \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \text{ satisfies (106)}\right\}. \tag{107}$$

Note that $\mathcal{C}_0$ has full column rank. Then we have the following result.

Lemma 8. $\mathcal{L}^{\mathrm{T}} \in \mathscr{S}_8 \iff \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \in \mathscr{S}_9$.

Proof. '⇒' is straightforward. Suppose $\mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \in \mathscr{S}_9$. Then the inequality (105) holds. Choose $\tilde{\mathcal{C}}_0$ such that $[\tilde{\mathcal{C}}_0\ \mathcal{C}_0] = I_{p(n_y+m)}$. Then

$$\mathcal{C}_0\tilde{\Upsilon}\mathcal{C}_0^{\mathrm{T}} = [\tilde{\mathcal{C}}_0\ \mathcal{C}_0] \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Upsilon} \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{C}}_0^{\mathrm{T}} \\ \mathcal{C}_0^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Upsilon} \end{bmatrix} \geqslant 0, \ \tilde{\Upsilon} \triangleq \begin{bmatrix} I_{n_y} \\ \mathcal{L}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix} \tilde{\Phi} \begin{bmatrix} I_{n_y} & \aleph_2(v) \\ 0 & \aleph_1(v) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_{n_y} \\ \mathcal{L}^{\mathrm{T}} \end{bmatrix}, \tag{108}$$

which leads to $\mathcal{L}^{\mathrm{T}} \in \mathscr{S}_8$.

With Lemma 8, Problem V is equivalent to Problem V′.

Problem V′: Design control gain $\mathcal{F}$ such that the QMI (104) holds for $\forall \mathcal{L}^{\mathrm{T}}\mathcal{C}_0^{\mathrm{T}} \in \mathscr{S}_9$.

Problem V′ can be solved using the QMI approaches presented in Section 3. Specifically, by employing Lemma 3, Problem V′ is solvable if there exists a scalar $\alpha \geqslant 0$ such that $\hat{\Xi} - \alpha\hat{\Psi} > 0$. After further algebraic manipulations, we have the following result [45, Theorem 20].

Theorem 12. Suppose that $\aleph_1(v)$ has full row rank and Assumption 7 is satisfied. Problem V′ is solvable with $\mathcal{F} = YP^{-1}$ if and only if there exist $P \in \mathbb{S}_+^{p(n_y+m)}$ and $Y \in \mathbb{R}^{m\times p(n_y+m)}$ such that

$$\begin{bmatrix} \begin{pmatrix} P & \mathcal{Y} \\ \mathcal{Y}^{\mathrm{T}} & -P \end{pmatrix} - \hat{\Psi} & \begin{pmatrix} \mathcal{Y} \\ 0 \end{pmatrix} \\ \star & P \end{bmatrix} > 0, \quad \mathcal{Y} = \begin{bmatrix} \mathcal{J}P \\ -Y \\ 0 \end{bmatrix}. \tag{109}$$

Using a QMI-based approach, Theorem 12 provides a solution to Problem V for input-output systems described by (92). This method offers several key advantages. First, it does not require a state-space representation of the original system, making it applicable to both SISO and MIMO systems. Second, it establishes a necessary and sufficient condition in the form of an LMI for designing suitable feedback controllers directly from noisy input-output data. Notably, the decision variables P and Y are independent of the time horizon of the experimental data, thereby enhancing computational efficiency and scalability.

7 Conclusion and challenges

Recent advances in data-driven control from noisy data have been reviewed in this paper for linear discrete-time systems with unknown system matrices. Several recently developed methods, such as data-driven representations of system dynamics based on Willems et al.'s fundamental lemma, QMI-based techniques, IQC frameworks, and LFT representations combining prior knowledge with data, have been analyzed in depth with meaningful insights. Data-driven control approaches for both SISO and MIMO systems have also been briefly reviewed in the context of available input-output data. Despite the notable progress achieved in this area, several key challenges remain, some of which are outlined below.

- Developing a universal noise-robust framework to design controllers directly from noisy data remains a critical and challenging issue. Most existing methods rely on specific noise assumptions, such as energy-bounded disturbances or constraints described by QMIs, rather than accommodating arbitrary noise distributions. Moreover, the influence of the 'direction' of the noise on system performance is not yet fully understood and warrants further investigation.
- Determining the minimal dataset required from experiments to fully capture the essential properties of a physical system is an important and intriguing problem in data-driven control. In [21], a data-dependent representation of the open-loop or closed-loop system can be derived if a certain rank condition on the data matrix is satisfied. However, in [31], it is revealed that this rank condition is not necessary for establishing some system-theoretic properties. Therefore, investigating the minimal data requirements for data-driven control holds significant value, both theoretically and practically.
- Studying data-driven control using partial input-state or input-output noisy data is an important and challenging topic. Most existing approaches assume access to either full input-state data or sufficiently informative input-output data. However, in many practical scenarios, state information is not directly measurable, and output data may be sparse or incomplete. Moreover, designing controllers based on delayed partial (or even full) input-state or input-output data introduces further complexity and remains a critical area for further research.
- Security is another critical concern in data-driven control. Adversaries may compromise experimental data used for controller design, potentially leading to unsafe or unstable systems. Moreover, control signals transmitted from controllers to physical plants over communication networks are also vulnerable to cyber-attacks [46]. Addressing security issues in data-driven control is not only meaningful but essential for ensuring the reliability and resilience of modern control systems [47–49].
- Data-driven control for nonlinear and multi-agent systems remains an open and challenging area. Most existing methods are primarily designed for linear systems or specific classes of nonlinear dynamics. Developing DDC approaches that can be applied directly to general nonlinear or multi-agent systems is a highly desirable direction for future research [50–54]. Further, integrating DDC with real-time learning algorithms, like reinforcement learning, offers great potential for enabling adaptive control in time-varying and uncertain environments.

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