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A FAS approach for linear time-varying multi-input systems: transformation and control

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Obtaining the fully actuated system (FAS) model from other descriptions of a system is the prerequisite and basis for applying existing FAS approaches. The systematic transformation method for linear time-invariant (LTI) systems has been studied in [1], showing that an LTI system can be equivalently converted into a FAS if and only if it is controllable. It has also been shown that the transformation is closely associated with system controllability, which similarly applies to linear time-varying (LTV) systems [2]. The background and notations are given in Appendix A.

Problem formulation. Consider the multi-input LTV system as follows:

$$\dot{x} = A(t)\,x + B(t)\,u,\ \ t \geqslant t_0, \tag{1}$$

where $u \in \mathbb{R}^r$, $x \in \mathbb{R}^n$ are the input and the state, respectively, t_0 is the initial time, $A : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^+ \to \mathbb{R}^{n\times r}$ are matrix functions.

The model transformation problem should be solved in the sense of Lyapunov to ensure kinematic equivalence, thereby preserving key dynamical properties of linear systems such as stability and controllability. Otherwise, the proposed controller may be invalid for system (1), since its stability is not necessarily implied by the stability of the obtained FAS. The Lyapunov transformation can be described as $\bar{x} = P(t)\,x$ with $P : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ being nonsingular and continuously differentiable, such that $P(t)$, $P^{-1}(t)$ and $\dot{P}(t)$ are bounded.

Define the following two matrix operators:

$$\begin{cases} \Pi\left[F(t)\right] \triangleq A(t)\,F(t) - \frac{\mathrm{d}}{\mathrm{d}t}F(t)\,, \\ \Xi\left[F(t)\right] \triangleq F(t)\,A(t) + \frac{\mathrm{d}}{\mathrm{d}t}F(t) \end{cases} \tag{2}$$

with $F(t)$ being any analytic matrix function, and define

$$\Pi^n\left[F(t)\right] \triangleq \underbrace{\Pi\ldots\Pi}_{\text{n times}}\left[F(t)\right],\ \ \Xi^n\left[F(t)\right] \triangleq \underbrace{\Xi\ldots\Xi}_{\text{n times}}\left[F(t)\right],$$

where $n \in \mathbb{N}$ and $\Xi^0\left[F(t)\right] = \Pi^0\left[F(t)\right] = F(t)$ by default. Then, the well-known criterion for assessing controllability of system (1) is given as follows.

Lemma 1 ([3]). Let $A(t)$ and $B(t)$ be analytic in $J \triangleq [t_0, \infty)$ and let the controllability matrix $Q(t)$ be defined as

$$Q(t) = \begin{bmatrix} Q_1(t) & Q_2(t) & \cdots & Q_n(t) \end{bmatrix} \tag{3}$$

with $Q_i(t) = \Pi^{i-1}\left[B(t)\right]$, $i = 1, 2, \ldots, n$. If $\operatorname{rank} Q(t_1) = n$ holds for any $t_1 \in J$, then system (1) is uniformly controllable in the interval J.

Analogous to the LTI case, $Q(t)$ can be reordered as

$$\bar{Q}(t) = \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 & \cdots & \bar{Q}_r \end{bmatrix}, \tag{4}$$

where $\bar{Q}_i = \begin{bmatrix} b_i & \Pi\left[b_i\right] & \cdots & \Pi^{n_i-1}\left[b_i\right] \end{bmatrix}$, $i = 1, 2, \ldots, r$ with b_i being the i-th column of $B(t)$ and n_i the controllability index corresponding to b_i. Different from LTI systems, the controllability indices n_i, $i = 1, 2, \ldots, r$, depend on t, meaning they may not remain fixed for all $t \geqslant t_0$. But there must exist a maximum finite number μ_i within $[t_0, \infty)$, namely, $\mu_i = \max_{t\geqslant t_0} n_i(t)$. Clearly, $\sum_{i=1}^r \mu_i \geqslant n$ holds. For controllability indices, we introduce a somewhat outdated concept called lexicographically-fixed controllability, which has also been recently discussed in [4].

Definition 1 ([5]). Given the truncated controllability matrix $\bar{Q}(t)$, if there exists a set of fixed controllability indices n_i, $i = 1, 2, \ldots, r$, satisfying $\sum_{i=1}^r n_i = n$ within the time interval $[t_0, \infty)$, then system (1) is said to be lexicography-fixedly controllable in $[t_0, \infty)$.

To determine the solvability criteria, the following assumptions on system (1) are introduced.

Assumption 1. The LTV system (1) is lexicography-fixedly controllable in $[t_0, \infty)$.

Assumption 2. For all $t \geqslant t_0$, all elements of $A(t)$ and $B(t)$ are at least n-times continuously differentiable.

Problem 1. Find a transformation described by $z_k^{(0\sim n_k-1)}\Big|_{k=1\sim r} = P(t)\,x$ such that system (1), under Assumptions 1 and 2, is converted into a FAS in the form of

$$z_k^{(n_k)}\Big|_{k=1\sim r} = L(t)\,z_k^{(0\sim n_k-1)}\Big|_{k=1\sim r} + G(t)\,u, \tag{5}$$

where $L : \mathbb{R}^+ \to \mathbb{R}^{r\times n}$, $G : \mathbb{R}^+ \to \mathbb{R}^{r\times r}$ are matrix functions.

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Problem 2. Construct a controller based on the FAS model (5) such that the resulting closed-loop system of the original LTV system (1) is exponentially stable.

Transformation from LTV systems into FASs. In view of Assumption 1 and $\bar{Q}(t)$ defined in (4), we know $\bar{Q}(t)$ is nonsingular on $t \in [t_0, \infty)$. Its inverse can be partitioned as

$$\bar{Q}^{-1}(t) = \begin{bmatrix} \bar{q}_{1,0} & \cdots & \bar{q}_{1,n_1-1} & \cdots & \bar{q}_{r,0} & \cdots & \bar{q}_{r,n_r-1} \end{bmatrix}^{\mathrm{T}} \tag{6}$$

with $\bar{q}_{i,j} \in \mathbb{R}^n$, $i = 1, 2, \ldots, r$, $j = 0, 1, \ldots, n_i - 1$. Letting $\sigma_k = \sum_{i=1}^{k} n_i$, we take all the σ_k-th rows of (6) and construct the transformation matrix as follows:

$$P(t) = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_r \end{bmatrix}, \; P_i = \begin{bmatrix} \bar{q}_{i,n_i-1}^{\mathrm{T}} \\ \Xi\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right] \\ \vdots \\ \Xi^{n_i-1}\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right] \end{bmatrix}. \tag{7}$$

Based on this construction, we have the following result.

Theorem 1. Let the LTV system (1) satisfy Assumptions 1 and 2, and define $z_k|_{k=1\sim r} = T(t)\,x$ where

$$T(t) = \begin{bmatrix} \bar{q}_{1,n_1-1} & \bar{q}_{2,n_2-1} & \cdots & \bar{q}_{r,n_r-1} \end{bmatrix}^{\mathrm{T}}. \tag{8}$$

Then system (1) can be converted into a FAS defined in (5), where $G(t)$ is given by

$$G(t) = \begin{bmatrix} 1 & X_{1,2} & \cdots & X_{1,r} \\ & 1 & \cdots & X_{2,r} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \tag{9}$$

with $X_{i,j}, i = 1, 2, \ldots, r-1, j = 2, 3, \ldots, r$, being uniquely specified functions, and $L(t)$ is provided in Appendix B.

Controller design. The controller can be designed as

$$u = -G^{-1}(t) \cdot \left(L(t)\, z_k^{(0\sim n_k-1)}\Big|_{k=1\sim r} + \begin{bmatrix} [A_1]_{0\sim n_1-1}\, z_1^{(0\sim n_1-1)} \\ [A_2]_{0\sim n_2-1}\, z_2^{(0\sim n_2-1)} \\ \vdots \\ [A_r]_{0\sim n_r-1}\, z_r^{(0\sim n_r-1)} \end{bmatrix} \right), \tag{10}$$

where $[A_k]_{0\sim n_k-1} \in \mathbb{R}^{1\times n_k}$, $k = 1, 2, \ldots, r$ are arbitrarily given vectors. The following result holds.

Theorem 2. If the transformation is in the sense of Lyapunov, the controller (10) based on the FAS model (5) solves Problem 2, such that the state response of the original LTV system (1) is determined by

$$x(t) = P^{-1}(t)\exp(\Phi_0 t)\, Z_0 \tag{11}$$

with $\Phi_0 = \mathrm{blockdiag}\left(\Phi\left([A_i]_{0\sim n_i-1}\right),\ i = 1, 2, \ldots, r\right)$ Hurwitz and $Z_0 = z_k^{(0\sim n_k-1)}(0)\Big|_{k=1\sim r}$.

Further generalizations. Assumption 1 may be too strict. For a non-lexicographically-fixed system, the key difference is that the matrix $\bar{Q}(t)$ given by (4) is not square and must be redefined as

$$\tilde{Q}(t) = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 & \cdots & \tilde{Q}_r \end{bmatrix}, \tag{12}$$

where $\tilde{Q}_i = \begin{bmatrix} b_i & \Pi[b_i] & \cdots & \Pi^{\mu_i-1}[b_i] \end{bmatrix}$, $i = 1, 2, \ldots, r$ with μ_i defined as before. Following the idea in [5], we can construct an auxiliary system, making the augmented system lexicography-fixedly controllable. Accordingly, Assumption 1 is replaced by the following.

Assumption 3. The LTV system (1) is non-lexicography-fixedly controllable on $t \in [t_0, \infty)$, but the maximum values of all controllability indices μ_i, $i = 1, 2, \ldots, r$ are known.

Define $n_e = \sum_{i=1}^{r} \mu_i - n$. The augmented system instead of (1) is then considered

$$\dot{\tilde{x}} = \tilde{A}(t)\,\tilde{x} + \tilde{B}(t)\,u, \tag{13}$$

where $\tilde{x} = \begin{bmatrix} x^{\mathrm{T}} & x_e^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+n_e}$ is the extended state and

$$\tilde{A}(t) = \begin{bmatrix} A(t) & 0_{n\times n_e} \\ A_{e,1}(t) & A_{e,2}(t) \end{bmatrix}, \; \tilde{B}(t) = \begin{bmatrix} B(t) \\ B_e(t) \end{bmatrix} \tag{14}$$

with $A_{e,1} : \mathbb{R}^+ \to \mathbb{R}^{n_e\times n}$, $A_{e,2} : \mathbb{R}^+ \to \mathbb{R}^{n_e\times n_e}$ and $B_e : \mathbb{R}^+ \to \mathbb{R}^{n_e\times r}$ being the auxiliary system matrices to be determined. We then provide the following extended results.

Lemma 2 ([5])**.** For system (1) satisfying Assumptions 2 and 3, there exists an auxiliary system of dimension n_e:

$$\dot{x}_e = A_{e,1}(t)\,x + A_{e,2}(t)\,x_e + B_e(t)\,u,$$

such that system (13) is lexicography-fixedly controllable over $t \in [t_0, \infty)$.

Corollary 1. Consider the LTV system (1) under Assumptions 2 and 3. A controller based on the FAS approach can still be established for the augmented system (13), provided the state transformation is in the sense of Lyapunov.

The proofs and remarks of Theorems 1 and 2 are provided in Appendixes B and C, respectively, and simulations are presented in Appendix D.

Conclusion. This work introduced a method for transforming multi-input LTV systems into FASs based on existing algebraic approaches. For LTV systems, the obtained FAS models streamline the control design process by enabling controllers to be formulated efficiently and directly. As control-oriented models, they are expected to play a crucial role in addressing a wide range of control problems.

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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