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A FAS approach for linear time-varying multi-input systems: transformation and control

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Appendix A Background and notations

The transformation of state equations plays a pivotal role in the control theory community, particularly when it comes to linear time-invariant (LTI) systems. Partly this is because the effective transformation can facilitate designing controllers, and partly it is because the canonical form is fascinating in its own right. Some canonical transformations or canonical forms, such as Frobenius form [1], Brunovsky form [2,3], and Jordan form [4], are profoundly renowned and well documented. These canonical transformations significantly simplify the relevant design processes. For similar purposes, parallel to the state-space model, Duan has recently originated the fully actuated system (FAS) approach, introducing mathematical FAS models heuristically [5]. As pointed out in [5], numerous underactuated systems can be converted into FASs as long as they satisfy some kind of controllability. Note that the mathematical FASs serve really as control-oriented models for dynamic systems, which might be looked upon as certain canonical forms of nonlinear systems, rather than represent physical ones only. To date, under this framework, quite a few control design issues have been effectively resolved [6, 7], and numerous applications have also been implemented [8, 9]. The prerequisite and basis for applying the existing FAS approaches is that an equivalent FAS model can be directly derived from certain physical laws or other descriptions of control systems. The systematic conversion criteria for different types of systems are still being studied.

For a specific system, the key point regarding the FAS model transformation problem lies in two aspects: i) how to check the existence of the corresponding FAS model as well as to provide necessary and sufficient conditions if possible; ii) how to derive its FAS model systematically if it exists. The pioneering work is conducted by Duan [5], concluding that the existence is closely related to certain controllability of a system [10]. Most directly, all controllable LTI systems possess FAS models, which is also necessary and sufficient. A systematic transformation method has also been supplemented via a frequency-domain approach in [11]. For linear time-delay systems, Refs. [12] and [13] have tried to give solutions based on differential flatness and right coprime factorization, respectively. Provided that one of state equations has a solution on a simply connected set containing the origin, the FAS models for four general types of nonlinear underactuated systems can also be obtained [14]. As a matter of fact, we can also approximate the nonlinear system in practice around the given trajectory, typically resulting in a linear time-varying (LTV) system [15]. However, to the best of our knowledge, the unique result regarding the FAS method of LTV systems was reported in [16], which addressed the single-input case only.

Unlike the single-input case, it is more difficult for multi-input LTV systems due to the lack of uniqueness. The researches regarding canonical forms of LTV continuous- and discrete-time systems widely report on this fact [17–22]. Specifically, Silverman has proved that a single-input LTV system can be converted into a phase-variable form, also known as Frobenius canonical form, if and only if it is uniformly controllable. The above result is analogous to that derived by Luenberger for LTI cases [1]. Unfortunately, it cannot be directly extended to multi-input cases because even though the considered LTV system is controllable, its controllability indices may not remain fixed. For all controllable LTI systems, most of canonical transformations depend on controllability indices, while these indices for the LTV case may be time-varying, resulting in the structure of canonical form changing with time. This forces the design engineer to determine the best form from the several possibilities. Most directly, provided that the LTV system is controllable with fixed controllability indices, namely, lexicography-fixedly controllable, Refs. [19,20,23,24] established phase-variable forms successfully, and then obtained time-invariant closed-loop systems whose eigenvalues were arbitrarily assignable via state feedback. Recently, Hernández et al. designed a higher-order sliding-mode controller under the assumption of lexicography-fixed controllability, achieving finite-time stability for perturbed LTV multi-input systems [25]. On the other hand, the treatment of non-lexicographically-fixed systems also appeared in [15, 26, 27]. The key idea is to construct an auxiliary system, ensuring that the augmented system possesses lexicographically-fixed controllability. Then, the treatment of the augmented system is trivial using the existing schemes. Very recently, Dong and Zhou proposed a prescribed-time fault-tolerant controller for lexicographically-fixed systems, and then extended the achieved result to non-lexicographically-fixed ones through introducing an auxiliary system [28].

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However, the biggest limitation in the previous research lies in a scotoma that whether lexicographically-fixed systems or non-lexicographically-fixed systems are only considered in the problem of pole placement [26, 27, 29] and lack attention to other issues. Due to the unpopularity of pole placement issues, it is unfortunate that some fundamental definitions and brilliant ideas have gradually become lifeless in the long history of control theory. Compared with canonical form or pole placement of LTV systems, recent advances and interest lie in data-driven methods [30,31] and output-feedback stabilization using normal forms [32]. By solving convex feasibility/optimization problems involving data-dependent LMIs, a controller based on a model-free, data-driven representation of the closed-loop LTV system is designed in [30]. To avoid the need for sufficient data to start the control process, a completely online data-driven adaptive control method is proposed in [31]. Compared with the mature model-based theory, a unified research paradigm and theoretical analysis framework have not been established for data-driven strategies.

As mentioned above, to pursue a more general research paradigm involving recent advancements, Duan has proposed a terminology, that is, the FAS model, from a new and higher perspective. It has attracted much attention to various hot topics. The benefits are evident with the development of FAS theories and applications. Both the previous research and the FAS approach complement each other, improving and developing in tandem. This paper confirms the originality of invoking lexicographically-fixed/non-lexicographically-fixed controllability into the framework of the FAS theory, establishing a basis for FAS approaches regarding LTV systems. The contributions are summarized below.

1. The relationship between the existence of a FAS model and controllability of an LTV system is indicated. It is identified that the single-input LTV system can be converted into a FAS if and only if it is uniformly controllable, while lexicographically-fixed controllability is a sufficient condition for the multi-input case;

2. A systematic transformation scheme from the state-space description of LTV systems to FAS models is introduced to fill the vacuum in the framework of the FAS theories;

3. For non-lexicographically-fixed systems, a stabilizing controller based on the FAS approach can still work in the sense that the derived FAS model is not of the original system but of the augmented one.

Notations: I_r represents the identity matrix, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}^r$ and $\mathbb{R}^{r\times m}$ are the spaces of r-dimensional vectors and $r \times m$ dimensional matrices, respectively. The inverse, transpose and determinant of a matrix A are denoted by A^{-1}, A^{T} and $\det(A)$, respectively. $\operatorname{blockdiag}(A_k, k = 1, 2, \ldots, \mu)$ stands for a block diagonal matrix with its diagonal blocks being A_k, $k = 1, 2, \ldots, \mu$. $\exp(\cdot)$ denotes the matrix exponential function. The superscript (i) denotes i-th time derivative for a vector function, and the superscript or subscript "$\sim$" stands for the traversal sequence, which is a habitual symbol in the FAS theory and somewhat different from the traditional mathematical notation. In particular, for a set of square matrices $A_i \in \mathbb{R}^{r\times r}$ and vectors x_i, $\mu_i \in \mathbb{N}$, $i = 0, 1, \ldots, m-1$, we introduce the following notations in the FAS theories [5, 10]:

$$x^{(0\sim\mu_0)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\mu_0)} \end{bmatrix},$$

$$x_{i\sim j}^{(\mu_0\sim\mu_1)} = \begin{bmatrix} x_i^{(\mu_0\sim\mu_1)} \\ x_{i+1}^{(\mu_0\sim\mu_1)} \\ \vdots \\ x_j^{(\mu_0\sim\mu_1)} \end{bmatrix}, \quad j \geqslant i, \quad \mu_1 \geqslant \mu_0,$$

$$x_k^{(\mu_0\sim\mu_k)}\Big|_{k=i\sim j} = \begin{bmatrix} x_i^{(\mu_0\sim\mu_i)} \\ x_{i+1}^{(\mu_0\sim\mu_{i+1})} \\ \vdots \\ x_j^{(\mu_0\sim\mu_j)} \end{bmatrix}, \quad j \geqslant i, \quad \mu_k \geqslant \mu_0,$$

$$[A_i]_{0\sim m-1} = \begin{bmatrix} A_{i,0} & A_{i,1} & \cdots & A_{i,m-1} \end{bmatrix},$$

and

$$\Phi(A_{0\sim m-1}) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_{m-1} \end{bmatrix}.$$

Appendix B Proofs of the main results

Appendix B.1 Proof of Theorem 1

To facilitate the process of deriving Theorem 1, the following proposition is introduced.

Proposition B1 ([17]). Given two matrix operators defined in (2) , the following relations hold

$$\Xi^k\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]b_j=\bar{q}_{i,n_i-1}^{\mathrm{T}}\Pi^k\left[b_j\right],\ i,j=1,2,\ldots,r,\ k=0,1,\ldots,n_i-1. \tag{B1}$$

Using Proposition B1, let us prove Theorem 1:

Under Assumption 1, we suppose $n_1\geqslant n_2\geqslant\cdots\geqslant n_r\geqslant 1$ without loss of generality. In light of the specific expression of $T(t)$, for each $i=1,2,\ldots,r$, we have

$$z_i=\bar{q}_{i,n_i-1}^{\mathrm{T}}x. \tag{B2}$$

Taking derivative of the above and using the operators defined in (2), yield

$$\begin{aligned}\dot{z}_i &= \left(\dot{\bar{q}}_{i,n_i-1}^{\mathrm{T}}+\bar{q}_{i,n_i-1}^{\mathrm{T}}A(t)\right)x+\bar{q}_{i,n_i-1}^{\mathrm{T}}B(t)\,u\\ &= \Xi\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]x+\bar{q}_{i,n_i-1}^{\mathrm{T}}\Pi^0\left[B(t)\right]u.\end{aligned} \tag{B3}$$

Noticing Proposition B1 and $\bar{Q}^{-1}(t)\,\bar{Q}(t)=I_n$, we have

$$\bar{q}_{i,n_i-1}^{\mathrm{T}}\Pi^k\left[b_i\right]=\Xi^k\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]b_i=0,\ k=0,1,\ldots,n_i-2, \tag{B4}$$

and

$$\bar{q}_{i,n_i-1}^{\mathrm{T}}\Pi^{n_i-1}\left[b_j\right]\ =\ \Xi^{n_i-1}\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]b_j=\begin{cases}1, & i=j\\ 0, & i>j\\ X_{i,j}, & i<j,\end{cases} \tag{B5}$$

where $X_{i,j}$ is a uniquely specified function of time. Substituting the above into (B3) gives

$$\dot{z}_i=\Xi\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]x. \tag{B6}$$

Then, calculating higher order derivatives until the n_i-th order gives

$$\begin{cases} z_i^{(2)}=\Xi^2\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]x\\ z_i^{(3)}=\Xi^3\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]x\\ \quad\vdots\\ z_i^{(n_i)}=\Xi^{n_i}\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]x+u_i+\sum_{j=i+1}^{r}X_{i,j}u_j.\end{cases} \tag{B7}$$

Additionally, leveraging the definition of the controllability indices, it is not hard to see that the row vector $\Xi^{n_i}\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]$ is a linear combination of the rows of $P(t)$ given by (7). In other words, there exists the linear coefficients $l_{j,k}^i(t)$, $j=1,2,\ldots,r$, $k=0,1,\ldots,n_j-1$, such that

$$\Xi^{n_i}\left[\bar{q}_{i,n_i-1}^{\mathrm{T}}\right]=\sum_{j=1}^{r}\sum_{k=0}^{n_j-1}l_{j,k}^i(t)\,\Xi^k\left[\bar{q}_{j,n_j-1}^{\mathrm{T}}\right].$$

In the matrix form of (B6)-(B7), one can obtain

$$z_k^{(n_k)}\Big|_{k=1\sim r}=L(t)\,z_k^{(0\sim n_k-1)}\Big|_{k=1\sim r}+G(t)\,u, \tag{B8}$$

where $z_k^{(0\sim n_k-1)}\Big|_{k=1\sim r}=P(t)\,x$, $G(t)$ is given by (9) and

$$L(t)=\begin{bmatrix} l_{1,0}^1 & \cdots & l_{1,n_1-1}^1 & \cdots & l_{r,0}^1 & \cdots & l_{r,n_r-1}^1\\ l_{1,0}^2 & \cdots & l_{1,n_1-1}^2 & \cdots & l_{r,0}^2 & \cdots & l_{r,n_r-1}^2\\ \vdots & & \vdots & & \vdots & & \vdots\\ l_{1,0}^r & \cdots & l_{1,n_1-1}^r & \cdots & l_{r,0}^r & \cdots & l_{r,n_r-1}^r\end{bmatrix}. \tag{B9}$$

Note that $G(t)$ is commonly an upper-triangular matrix and nonsingular for $\forall t\geqslant t_0$. Finally, in terms of the control of system (1), the key step is to check whether $P(t)$ is a Lyapunov matrix. Assumption 2 cannot ensure that the above transformation is a Lyapunov one. If the elements of $A(t)$, $B(t)$ are further bounded with bounded derivatives, it might hold to a large extent. The proof is completed. □

Appendix B.2 Proof of Theorem 2

The decoupling controller (10) results in the following set of closed-loop subsystems

$$z_k^{(n_k)} + [A_k]_{0\sim n_k-1}\, z_k^{(0\sim n_k-1)} = 0,\ k = 1, 2, \ldots, r, \tag{B10}$$

Thus, we can select the matrices $[A_k]_{0\sim n_k-1}$, $k = 1, 2, \ldots, r$, to arbitrarily assign the eigenstructure of the closed-loop system

$$\left.\dot{z}_k^{(0\sim n_k-1)}\right|_{k=1\sim r} = \Phi_0\, \left.z_k^{(0\sim n_k-1)}\right|_{k=1\sim r}, \tag{B11}$$

whose response is given by

$$\left.z_k^{(0\sim n_k-1)}(t)\right|_{k=1\sim r} = \exp(\Phi_0 t)\, Z_0. \tag{B12}$$

Clearly, Assumption 1 directly ensures that $P(t)$ is nonsingular for $\forall t \geqslant t_0$. Combining (B12) with $\left.z_k^{(0\sim n_k-1)}\right|_{k=1\sim r} = P(t)\,x$ yields (11). Next, let us discuss the effectiveness of the designed controller (10) for system (1).

Rewrite the FAS (5) into its state-space form

$$\left.\dot{z}_k^{(0\sim n_k-1)}\right|_{k=1\sim r} = \bar{A}(t)\, \left.z_k^{(0\sim n_k-1)}\right|_{k=1\sim r} + \bar{B}(t)\, u, \tag{B13}$$

where

$$\bar{A}(t) = \left[\bar{A}_{i,j}\right],\ \bar{B}(t) = \begin{bmatrix} \bar{B}_1(t) \\ \bar{B}_2(t) \\ \vdots \\ \bar{B}_r(t) \end{bmatrix}, \tag{B14}$$

with

$$\bar{A}_{i,i} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \\ l_{i,0}^i & l_{i,1}^i & \cdots & l_{i,n_i-1}^i \end{bmatrix} \in \mathbb{R}^{n_i\times n_i},$$

$$\bar{A}_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ l_{j,0}^i & l_{j,1}^i & \cdots & l_{j,n_j-1}^i \end{bmatrix} \in \mathbb{R}^{n_i\times n_j},\ i \neq j,$$

$$\bar{B}_i(t) = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & X_{i,i+1} & \cdots & X_{i,r} \end{bmatrix} \in \mathbb{R}^{n_i\times r}. \tag{B15}$$

It can be verified that the above equations, together with $P(t)$ given by (7), satisfy

$$\begin{cases} \bar{A}(t) = \left(P(t)\,A(t) + \dot{P}(t)\right) P^{-1}(t) \\ \bar{B}(t) = P(t)\,B(t), \end{cases} \tag{B16}$$

implying that system (B13) is equivalent to system (1) as long as translation $\left.z_k^{(0\sim n_k-1)}\right|_{k=1\sim r} = P(t)\,x$ is in the sense of Lyapunov. Once the above holds and Φ_0 is taken as a Hurwitz matrix, controller (10) ensures exponential stability of the closed-loop system in the sense of eliminating the time-varying elements in the system matrix and arbitrarily assigning the eigenvalues of the closed-loop subsystems (B10) via state feedback. Additionally, the proof of Corollary 1 is trivial and identical except that the considered system is changed into the augmented system (11). Problem 2 is handled properly and the proof is done. □

Appendix C Additional remarks

Due to the limitation of the length of letter articles, we supplement some important demonstrations to facilitate readers' understanding.

Appendix C.1 Transformation from LTV systems into FASs

Remark 1. Lexicographically-fixed controllability is a somewhat outdated concept in the control community, resulting in unfamiliarity for readers. Let us elaborate on it based on the algebraic geometric description of linear systems [33, 34]. It is well known that an n-dimensional LTI system is completely controllable if and only if its controllable subspace is $\mathbb{R}^n$. In other words, for a controllable LTI system, we can always construct a set of bases in $\mathbb{R}^n$, which are chosen from column vectors of its controllability matrix Q. Although the corresponding controllability matrix $Q(t)$ is also full rank for LTV systems, there may be a situation that we can never select a fixed set of column vectors of $Q(t)$ to span $\mathbb{R}^n$ for all $t \geqslant t_0$, meaning that another different set of column vectors has to be reselected from $Q(t)$ at some moments. The above feature of controllable LTV systems just leads to the formation of two concepts, namely, lexicographically-fixed systems and non-lexicographically-fixed systems, where lexicographically-fixed ones are directly extended from LTI cases. Additionally, lexicographically-fixed controllability is not hard to verify for a specific LTV system. Specifically, one just needs to verify whether there exist $n-r$ column vectors of $Q(t)$ combining with all columns of $B(t)$ such that the matrix constructed by them is full rank for $\forall t \geqslant t_0$. Traversing all possible situations, we know there are $C_{(n-1)r}^{n-r}$ cases in total. Thus, the verification process is only tedious but not difficult for multi-input systems.

Remark 2. Assumptions 1 and 2 seem too strict at first sight, but they are quite necessary to solve Problem 1. By constructing the auxiliary system (13), Assumption 1 can be relaxed to Assumption 3, and the generalization of Theorem 1 then becomes Corollary 1. The combination of the two theorems actually covers all controllable LTV systems. As for Assumption 2, the condition of n-times differentiability is imposed in Lemma 1, namely, Controllability Criterion. If it is not satisfied, the controllability matrix cannot be well-defined. Once controllability of the system is ambiguous, the solution to Problem 1 may not exist, because Duan declared explicitly that systems having FAS models should obey a certain kind of controllability property [10]. To date, there exists no recognized controllability criterion for a general non-smooth LTV system, causing great obstacles to extending the result of this paper to non-smooth systems.

Appendix C.2 Implement of controller

Remark 3. The key goal of the paper is to provide a systematic model transformation method from LTV systems to FASs. The given controller (10) is the most typical one within the framework of the FAS theories and is just to demonstrate the control-oriented feature of the FAS model. The simulation is applied to validate the kinematic equivalence (preserving key dynamical properties of systems, such as stability and controllability) between the original system and its FAS description. Thus, the performance of controller (10) is not good compared to other advanced controllers based on the FAS approach. Besides, controller (10) is not the final form in applications, and its realization should be conducted via substituting $\left. z_k^{(0\sim n_k-1)}\right|_{k=1\sim r} = P(t)\,x$ into (10), which is in the form of

$$u = -G^{-1}(t)\left(L(t) + \mathrm{blockdiag}\left([A_k]_{0\sim n_k-1},\ k=1,2,\ldots,r\right)\right)P(t)\,x. \tag{C1}$$

Thereby, the measurement of high-order derivatives $\left. z_k^{(1\sim n_k-1)}\right|_{k=1\sim r}$, is not necessary. Thanks to the above FAS method, we directly preform the eigenvalue assignment for LTV system (1). This result can be looked upon as a generalized version of the multi-input LTI cases [5] or the single-input LTV systems [16].

Remark 4. Almost all the controller designs based on the FAS model possess a similar structure like Eqs. (10) or (C1). To make full use of the full-actuation feature, it is inevitable to compute the inverse of the input matrix $G(t)$ and to cancel the undesired term $L(t)$. Note that $G(t)$ given by (9) is commonly an upper-triangular matrix with all diagonal elements being one. Thereby, the nonsingularity of $G(t)$ is independent of the time variable t and the numerical stability of computing the inverse is definitely guaranteed. However, the specific expressions of $G(t)$ and $L(t)$ cannot be determined quickly before a series of matrix transformation operations, which is quite a common drawback among the numerous FAS methods. Additionally, the high computational cost can be looked upon as a trade-off for obtaining a constant linear closed-loop system to an extent. There is no good method, but it can be optimized as much as possible. The back-substitution method, whose time complexity is $O(r^3)$, can be applied to derive the inverse of an upper-triangular matrix, since the corresponding constant factor is less than the inverse of the general matrix. The Strassen algorithm might also be used to decrease the time complexity of matrix multiplication [35].

Let us end this sub-appendix with an algorithm:

Algorithm C1 A FAS method for the LTV system

Given the LTV system (1),
1. check its lexicographically-fixed controllability;
2. obtain the matrix $\bar{Q}(t)$ using the controllability indices n_i, $i=1,2,\ldots,r$;
3. calculate the inverse of $\bar{Q}(t)$ and construct $T(t)$, $P(t)$;
4. convert system (1) into the FAS (B8), according to the proof of Theorem 1, after confirming that $P(t)$ is a Lyapunov matrix;
5. design a controller in the form of (10).

Appendix C.3 The auxiliary system

The generalization for non-lexicographically-fixed systems needs to construct an auxiliary system such that the augmented system satisfies Assumption 1. For a given system, the augmentation dimension $n_e = \sum_{i=1}^{r}\mu_i - n$ is unique, but the introduced auxiliary system is not unique. Before elaborating on it, the following mathematical result is invoked.

Proposition C1. Any matrix Q of the dimension $n_1 \times n_2$ $(n_1 < n_2)$ of rank n_1 can be extended into a matrix Q_g of the dimension $n_2 \times n_2$ with full rank n_2.

Applying the above proposition, we can always extend $\tilde{Q}(t)$ defined in (12) into $\tilde{Q}_g(t)$ with full rank $(n+n_e)$, e.g., choosing all unit coordinate vectors of the space $\mathbb{R}^{n_e}$. Denote the augmentation vectors as $h_{i,j}$, $i = 1,2,\ldots,r$, $j = 1,2,\ldots,\mu_i$, satisfying $\begin{bmatrix} h_{1,1} & \cdots & h_{1,\mu_1} & \cdots & h_{r,1} & \cdots & h_{r,\mu_r} \end{bmatrix} \in \mathbb{R}^{n\times(n+n_e)}$. Using the theorem in [26], the auxiliary system can be formed as

$$\begin{aligned} \begin{bmatrix} A_{e,1} & A_{e,2} \end{bmatrix} &= \begin{bmatrix} h_{1,2}^{\mathrm{T}} + \frac{\mathrm{d}}{\mathrm{d}t} h_{1,1}^{\mathrm{T}} \\ \vdots \\ h_{1,\mu_1+1}^{\mathrm{T}} + \frac{\mathrm{d}}{\mathrm{d}t} h_{1,\mu_1}^{\mathrm{T}} \\ \vdots \\ h_{r,2}^{\mathrm{T}} + \frac{\mathrm{d}}{\mathrm{d}t} h_{r,1}^{\mathrm{T}} \\ \vdots \\ h_{r,\mu_r+1}^{\mathrm{T}} + \frac{\mathrm{d}}{\mathrm{d}t} h_{r,\mu_r}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tilde{Q}_g^{-1}(t), \\ B_e(t) &= \begin{bmatrix} h_{1,1} & h_{2,1} & \cdots & h_{r,1} \end{bmatrix}, \end{aligned} \tag{C2}$$

where h_{i,μ_i+1}, $i = 1,2,\ldots,r$, are freely chosen vectors. An illustrative example can be found in [26] to facilitate understanding (see also Example 2 in Appendix D.2).

Remark 5. We have to emphasize that although the augmented system (14) can be converted into a FAS such that the established controller is also effective and efficient, this does not directly imply that the FAS model of original system (1) cannot be found. The way to derive the FAS model is not unique. To the best knowledge of the authors, another effective way is to introduce the elementary module theory over principal ideal rings [36], because the essence of Problem 1 is to find flat outputs of controllable LTV systems and to form the corresponding differential homeomorphism between the original state variables and flat outputs [37,38]. It is unnecessary to build the auxiliary system within the framework of this purely abstract algebraic method (see [39] for more details).

Remark 6. The auxiliary system inevitably has an impact on the original system's performance. Admittedly, it is not easy to thoroughly analyze the influence because there are too many factors to consider. For instance, the construction of the auxiliary system is not unique; the eigenvalues of the obtained linear closed-loop system are arbitrarily chosen as long as the system is ensured to be stable; the initial values of the auxiliary system are selected freely to an extent. Additionally, the Lyapunov synthesis method is of little help in analyzing the transient response. In Ref. [26], the authors carried out a comparative simulation of different eigenvalues, concluding that arbitrarily assigning a stable pole of the auxiliary system did not noticeably influence the dynamics of the original system.

Appendix D Illustrative examples

Appendix D.1 Lexicographically-fixed case

Example 1. Let us consider the following example in [17] to validate Algorithm C1

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ \cos t & \sin t & 1+\mathrm{e}^{-t} \\ 1 & \sin t & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1+0.2\sin t \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u, \ t \geqslant 0, \tag{D1}$$

whose controllability matrix is given by

$$Q(t) = \begin{bmatrix} 0 & 0.2\sin t + 1 & 0 & 1-0.5\cos t & \mathrm{e}^{-t}+1 & \alpha_3 \\ 0 & 1 & \mathrm{e}^{-t}+1 & \alpha_1 & \alpha_2 & \alpha_4 \\ 1 & 0 & 1 & 1.2\sin t + 1 & \left(\mathrm{e}^{-t}+1\right)\sin t + 1 & \alpha_5 \end{bmatrix}, \tag{D2}$$

where

$$\begin{aligned} \alpha_1 &= \sin t + (0.2\sin t + 1)\cos t, \\ \alpha_2 &= 2\mathrm{e}^{-t} + \left(\mathrm{e}^{-t}+1\right)\sin t + 1, \\ \alpha_3 &= 0.8\sin t + (0.2\sin t + 1)\cos t, \\ \alpha_4 &= \mathrm{e}^{-t} + 0.2\cos t + 2.2\sin t + \sin t\cos t - 1.6\cos^2 t - 0.2\cos^3 t + 1.2\mathrm{e}^{-t}\sin t + 2.2, \\ \alpha_5 &= 1.2\sin t - 1.4\cos t + \sin^2 t + (0.2\sin t + 1)\cos t\sin t + 2. \end{aligned} \tag{D3}$$

From (D2), one can derive that system (D1) is uniformly controllable with the fixed controllability indices being $n_1 = 2$ and $n_2 = 1$. Then, it follows from (4) that

$$\bar{Q}(t) = \begin{bmatrix} b_1 & \Pi[b_1] & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.2\sin t + 1 \\ 0 & \mathrm{e}^{-t}+1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \tag{D4}$$

which is nonsingular for $\forall t \geqslant 0$. Its inverse is given by

$$\bar{Q}^{-1}(t) = \frac{1}{\varepsilon}\begin{bmatrix} 5\mathrm{e}^t & -\mathrm{e}^t(\sin t+5) & \varepsilon \\ -5\mathrm{e}^t & \mathrm{e}^t(\sin t+5) & 0 \\ 5(\mathrm{e}^t+1) & 0 & 0 \end{bmatrix}, \tag{D5}$$

with $\varepsilon = \left(\mathrm{e}^t+1\right)(\sin t+5)$.

According to Algorithm C1, define $z_k|_{k=1\sim 2} \triangleq T(t)x$, with

$$T(t) = \frac{1}{\varepsilon}\begin{bmatrix} -5\mathrm{e}^t & \mathrm{e}^t(\sin t+5) & 0 \\ 5(\mathrm{e}^t+1) & 0 & 0 \end{bmatrix}, \tag{D6}$$

and construct a transformation matrix

$$P(t) = \frac{1}{\varepsilon}\begin{bmatrix} -5\mathrm{e}^t & \mathrm{e}^t(\sin t+5) & 0 \\ \frac{\beta_1}{\varepsilon} & \beta_2 & \varepsilon \\ 5(\mathrm{e}^t+1) & 0 & 0 \end{bmatrix}, \tag{D7}$$

where

$$\begin{aligned} \beta_1 &= \mathrm{e}^t\left(5\sin 2t + 31\cos t - 5\sin t + 31\mathrm{e}^t\cos t - \left(1+\mathrm{e}^t\right)\cos^3 t + 5\mathrm{e}^t\sin 2t - 25\right), \\ \beta_2 &= \frac{\mathrm{e}^t}{\mathrm{e}^t+1}\left(6\sin t - 5\mathrm{e}^t + 5\mathrm{e}^t\sin t + \sin^2 t + \mathrm{e}^t\sin^2 t\right). \end{aligned} \tag{D8}$$

After confirming $P(t)$ given by (D7) is a Lyapunov matrix, we have

$$z_k^{(n_k)}\Big|_{k=1\sim 2} = L(t)\, z_k^{(0\sim n_k-1)}\Big|_{k=1\sim 2} + G(t)u, \tag{D9}$$

where

$$L(t) = \begin{bmatrix} l_{1,0}^1 & l_{1,1}^1 & l_{2,0}^1 \\ l_{1,0}^2 & 0 & l_{2,0}^2 \end{bmatrix}, \quad G(t) = \begin{bmatrix} 1 & \frac{X_{12}}{5\varepsilon} \\ 0 & 1 \end{bmatrix}, \tag{D10}$$

with

$$\begin{aligned} l_{1,0}^2 &= \frac{5\left(\mathrm{e}^{-t}+1\right)}{\sin t+5},\ l_{2,0}^2 = \frac{5-\cos t}{\sin t+5}, \\ l_{1,1}^1 &= \frac{1}{\varepsilon}\left(7\sin t + 6\mathrm{e}^t\sin t + \left(1+\mathrm{e}^t\right)\sin^2 t + 5\right), \\ l_{1,0}^1 &= \frac{\mathrm{e}^{-t}}{\varepsilon^2}\left(\left(25.75 + 71.5\mathrm{e}^t + 65.75\mathrm{e}^{2t} + 30\mathrm{e}^{3t}\right)\sin t + \left(60.5\mathrm{e}^t + 121\mathrm{e}^{2t} + 60.5\mathrm{e}^{3t}\right)\cos t\right. \\ &\quad + \left(10\mathrm{e}^t + 20\mathrm{e}^{2t} + 10\mathrm{e}^{3t}\right)\sin 2t - \left(5 + 12\mathrm{e}^t + 9\mathrm{e}^{2t} + 2.5\mathrm{e}^{3t}\right)\cos 2t - 41\mathrm{e}^{2t} + 2.5\mathrm{e}^{3t} \\ &\quad \left. - \left(0.25 + 0.5\mathrm{e}^t + 0.25\mathrm{e}^{2t}\right)\sin 3t - \left(0.5\mathrm{e}^t + \mathrm{e}^{2t} + 0.5\mathrm{e}^{3t}\right)\cos 3t - 13\mathrm{e}^t + 5\right), \\ l_{2,0}^1 &= \frac{1}{40\varepsilon^2}\left(\left(1636 + 2982\mathrm{e}^t + 516\mathrm{e}^{2t}\right)\sin t + \left(2610\mathrm{e}^t + 1380\mathrm{e}^{2t}\right)\cos t\right. \\ &\quad + \left(300\mathrm{e}^t - 22\mathrm{e}^{2t}\right)\sin 2t + \left(124\mathrm{e}^{2t} - 260 - 336\mathrm{e}^t\right)\cos 2t + 1257\mathrm{e}^t \\ &\quad + \left(6\mathrm{e}^t + 28\mathrm{e}^{2t} - 12\right)\sin 3t + \left(20\mathrm{e}^{2t} - 10\mathrm{e}^t\right)\cos 3t + 797\mathrm{e}^{2t} \\ &\quad \left. + \mathrm{e}^{2t}\sin 4t - \left(\mathrm{e}^t + \mathrm{e}^{2t}\right)\cos 4t + 1260\right), \\ X_{1,2} &= \mathrm{e}^t\left(31\cos t + 25\sin t + 10\sin t\cos t - 5\cos^2 t - \cos^3 t - 20\right). \end{aligned} \tag{D11}$$

Then, applying Theorem 2, we design a controller in the form of (10) with $[A_1]_{0\sim n_1-1} = \begin{bmatrix} 6 & 5 \end{bmatrix}$, $[A_2]_{0\sim n_2-1} = 1$, whose specific realization is given by (C1).

The simulation results, with initial condition $x_0 = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^{\mathrm{T}}$, are shown in Figs. D1–D3. It follows from Figs. D1–D2 that the strict equivalence in the Lyapunov sense is achieved. Notice that the system dimension of FAS (D9) is still 3, meaning that $\dot{z}_1$ should be looked upon as a new state variable rather than just a derivative of z_1. That is, if we denote $y_1 = z_1$, $y_2 = \dot{z}_1$, $y_3 = z_2$, then the state-space representation of FAS (D9) is equivalent to system (D1) in the sense of y_i, $i = 1, 2, 3$, being new state variables. We are just accustomed to directly using $\dot{z}_1$ in the legend of Fig. D1. The established controller ensures an exponential stability, which is in agreement with Theorem 2. Thanks to the framework of the FAS theories, the closed-loop system becomes an LTI system whose eigenvalues are -2, -3 and -1.

Appendix D.2 Non-lexicographically-fixed case

Example 2. To demonstrate the effectiveness of Corollary 1, a Brockett integrator form discussed in [40] is taken into account

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1. \end{cases} \tag{D12}$$

Assuming that the desired trajectory is

$$\begin{cases} x_1^* = -\frac{\cos 2t}{2} \\ x_2^* = \frac{\sin 2t}{4} \\ x_3^* = \frac{\sin 2t}{16} - \frac{\sin 6t}{48}, \end{cases} \tag{D13}$$

we have the following proposition.

Proposition D1. The linearization model of system (D12) around the given trajectory (D13) is

$$\Delta \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sin 2t & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\sin 2t}{4} & 0 \end{bmatrix} \Delta u, \tag{D14}$$

where $\Delta x = x - x^*$ and $\Delta u = u - u^*$.

Proof. In view of (D12), one can obtain its linearization model

$$\Delta \dot{x} = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x^*,u^*} \Delta x + \left.\frac{\partial f(x,u)}{\partial u}\right|_{x^*,u^*} \Delta u, \tag{D15}$$

where

$$\left.\frac{\partial f(x,u)}{\partial x}\right|_{x^*,u^*} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u_1^* & 0 \end{bmatrix}, \quad \left.\frac{\partial f(x,u)}{\partial u}\right|_{x^*,u^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_2^* & 0 \end{bmatrix}. \tag{D16}$$

Substituting the given trajectory (D13) into the above, yields (D14). The proof is done. □

The controllability matrix of system (D14) can be expressed as

$$Q(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\sin 2t}{4} & 0 & -\frac{\cos 2t}{2} & \sin 2t & -\sin 2t & -2\cos 2t \end{bmatrix}. \tag{D17}$$

Notice that system (D14) is typically non-lexicographically-fixed, because the controllability indices are $n_1 = 2$, $n_2 = 1$ over $t \in [0, \pi/4)$ but $n_1 = 1$, $n_2 = 2$ at $t = \pi/4$. Clearly, the maximum values are $\mu_1 = \mu_2 = 2$. Leveraging the technique proposed in [26], the generalized truncated controllability matrix $\tilde{Q}_g$ can be constructed as

$$\tilde{Q}_g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sin 2t}{4} & -\frac{\cos 2t}{2} & 0 & \sin 2t \\ h_{1,1} & h_{1,2} & h_{2,1} & h_{2,2} \end{bmatrix}, \tag{D18}$$

whose determinant is

$$\det\left(\tilde{Q}_g\right) = h_{1,2}\sin 2t + \frac{1}{2} h_{2,2}\cos 2t. \tag{D19}$$

Remarking that all elements of the extension row are freely chosen, we take

$$h_{1,2} = \sin 2t, \ h_{2,2} = 2\cos 2t, \tag{D20}$$

to ensure $\det\left(\tilde{Q}_g\right) \neq 0$ for all $t \geqslant 0$. Then, according to (C2), we set for simplicity

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{1,1} = -h_{1,2}, \ \frac{\mathrm{d}}{\mathrm{d}t} h_{1,2} = -h_{1,3}, \tag{D21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{2,1} = -h_{2,2}, \ \frac{\mathrm{d}}{\mathrm{d}t} h_{2,2} = -h_{2,3}, \tag{D22}$$

such that $\begin{bmatrix} A_{e,1} & A_{e,2} \end{bmatrix} = 0_{1\times 4}$. Thereby, the final augmented LTV system is in the form of

$$\Delta\dot{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sin 2t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Delta\tilde{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\sin 2t}{4} & 0 \\ \frac{\cos 2t}{2} & -\sin 2t \end{bmatrix} \Delta u, \tag{D23}$$

where $\Delta\tilde{x} = \begin{bmatrix} \Delta x^{\mathrm{T}} & x_e \end{bmatrix}^{\mathrm{T}}$. It can be readily verified that the augmented system (D23) is strictly lexicographically-fixed with $\mu_1 = \mu_2 = 2$. Applying Theorem 1, we obtain the single-order FAS model of system (D23) as

$$\ddot{\tilde{z}} = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tilde{z}^{(0\sim 1)} + \Delta u, \tag{D24}$$

where

$$\tilde{z}^{(0\sim 1)} = \begin{bmatrix} 0 & \sin^2 2t & -2\cos 2t & \sin 2t \\ 0 & \sin 4t & 4\sin 2t & 2\cos 2t \\ \frac{-1}{4} & \frac{\sin 4t}{4} & \sin 2t & \frac{\cos 2t}{2} \\ 0 & 1-\sin^2 2t & 2\cos 2t & -\sin 2t \end{bmatrix} \Delta\tilde{x}. \tag{D25}$$

Then, the controller in the form of (10), where $[A_1]_{0\sim\mu_1-1} = \begin{bmatrix} 6 & 5 \end{bmatrix}$, $[A_2]_{0\sim\mu_2-1} = \begin{bmatrix} 3 & 2 \end{bmatrix}$, is easily established. The simulation, with initial condition $\Delta\tilde{x}_0 = \begin{bmatrix} 0.2 & -0.1 & -0.3 & 0 \end{bmatrix}^{\mathrm{T}}$, has been conducted. Figs. D4 and D5 show the state evolutions of the original system and the auxiliary system, respectively. The simulation results are identical to the statement of Corollary 1.

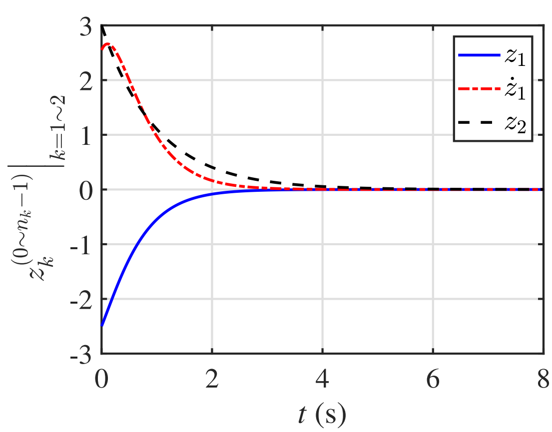

Figure D1 Evolution of $z_k^{(0\sim n_k-1)}\big|_{k=1\sim 2}$ for Example 1

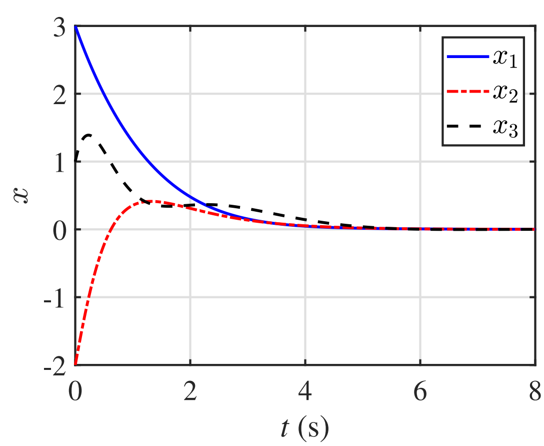

Figure D2 State responses for Example 1

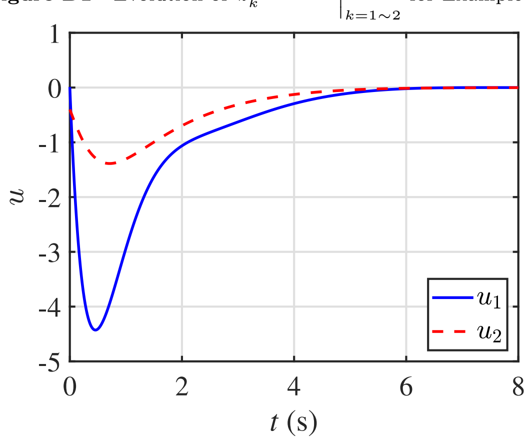

Figure D3 Control inputs for Example 1

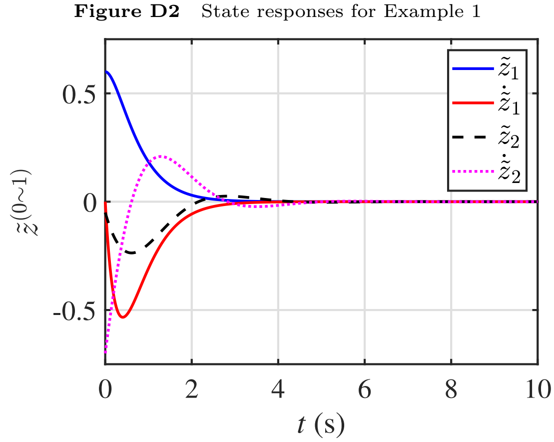

Figure D4 Evolution of $\tilde{z}^{(0\sim 1)}$ for Example 2

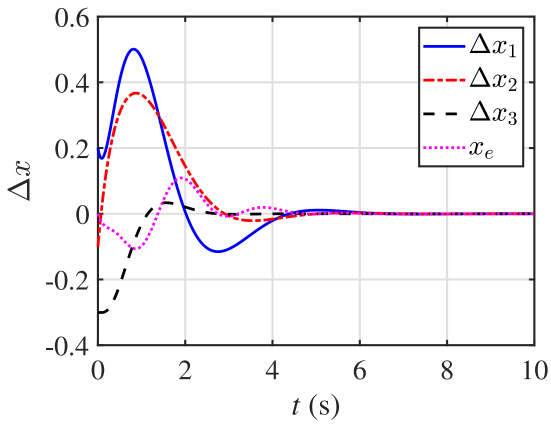

Figure D5 State responses for Example 2

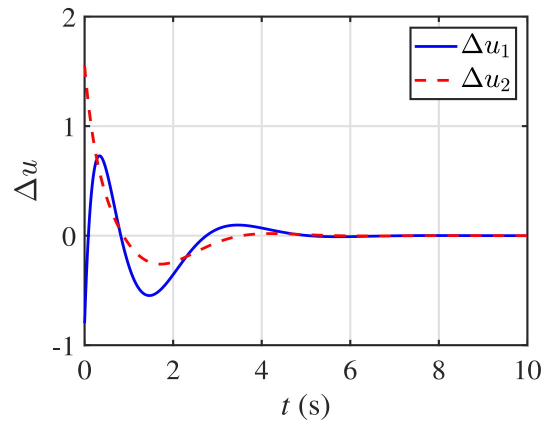

Figure D6 Control inputs for Example 2

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