

Global control for a class of chained nonholonomic systems with input saturations

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Appendix A Proof of Theorem 1

The closed-loop system can be formulated as

$$\dot{x}_0(t) = -\text{sat}_{u_{\max}}(\lambda x_0(t) - \beta e^{-\alpha t}). \quad (\text{A1})$$

Let $X = \lambda x_0(t) - \beta e^{-\alpha t}$. Then the \dot{X} along the trajectory (A1) can be given as

$$\begin{aligned} \dot{X} &= \lambda \dot{x}_0(t) + \alpha \beta e^{-\alpha t} \\ &= -\lambda \text{sat}_{u_{\max}}(X) + \alpha \beta e^{-\alpha t}. \end{aligned} \quad (\text{A2})$$

Consider the Lyapunov function

$$V(X) = \frac{1}{2} X^2.$$

Along the trajectory of (A2), the time derivative of $V(X(t))$ is

$$\dot{V}(X) = -\lambda X \text{sat}_{u_{\max}}(X) + \alpha \beta e^{-\alpha t} X.$$

Since $\lim_{t \rightarrow \infty} \beta e^{-\alpha t} = 0$, there exists a constant $t_1 > 0$ such that, for $t \geq t_1$, $|\alpha \beta| e^{-\alpha t} < \lambda u_{\max}$. Assume that $|X| > u_{\max}$, for $t \geq t_1$. Therefore, we have

$$\begin{aligned} \dot{V}(X) &= -\lambda X \text{sign}(X) |\text{sat}_{u_{\max}}(X)| + \alpha \beta e^{-\alpha t} X \\ &= -\lambda u_{\max} |X| + \alpha \beta e^{-\alpha t} X < 0. \end{aligned}$$

Then, according to Lyapunov stability theory, there exists a positive constant $T_0 > t_1$ such that, for $t \geq T_0$, $|X| \leq u_{\max}$, and thus the controller can be further written as

$$\text{sat}_{u_{\max}}(u_0(t)) = -\lambda x_0(t) + \beta e^{-\alpha t}. \quad (\text{A3})$$

For $t \geq T_0$, (A1) can be described as

$$\dot{x}_0(t) = -\lambda x_0(t) + \beta e^{-\alpha t},$$

whose solution can be calculated as

$$\begin{aligned} x_0(t) &= e^{-\lambda t} x_0(T_0) + \beta \int_{T_0}^t e^{-\lambda(t-s)} e^{-\alpha s} ds \\ &= e^{-\lambda t} x_0(T_0) + \beta e^{-\lambda t} \frac{1}{\lambda - \alpha} (e^{(\lambda - \alpha)t} - e^{(\lambda - \alpha)T_0}) \\ &= e^{-\alpha t} \left(e^{-(\lambda - \alpha)t} x_0(T_0) + \beta \frac{1}{\lambda - \alpha} (1 - e^{-(\lambda - \alpha)t} e^{(\lambda - \alpha)T_0}) \right), \end{aligned}$$

substituting which into (A3) yields

$$\begin{aligned} \text{sat}_{u_{\max}}(u_0(t)) &= -\lambda x_0(t) + \beta e^{-\alpha t} \\ &= e^{-\alpha t} \left(-\lambda e^{-(\lambda - \alpha)t} x_0(T_0) - \frac{\lambda \beta}{\lambda - \alpha} (1 - e^{-(\lambda - \alpha)t} e^{(\lambda - \alpha)T_0}) + \beta \right) \end{aligned}$$

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$$\begin{aligned}
&= e^{-\alpha t} \left(-\lambda e^{-(\lambda-\alpha)t} x_0(T_0) + \frac{\lambda\beta}{\lambda-\alpha} e^{-(\lambda-\alpha)t} e^{(\lambda-\alpha)T_0} - \frac{\lambda\beta}{\lambda-\alpha} + \beta \right) \\
&= e^{-\alpha t} (\theta(t) + \delta),
\end{aligned}$$

where

$$\theta(t) = e^{-(\lambda-\alpha)t} \left(-\lambda x_0(T_0) + \frac{\lambda\beta}{\lambda-\alpha} e^{(\lambda-\alpha)T_0} \right), \quad \delta = -\frac{\lambda\beta}{\lambda-\alpha},$$

which proves (3) and $\lim_{t \rightarrow \infty} \theta(t) = 0$. The proof is finished.

Appendix B Proof of Theorem 2

For simplicity, we denote

$$u_1(t) = -\varepsilon_2 \text{sat} \left(\frac{\lambda_2 Y_2(t)}{\varepsilon_2} \right) + u_{1,1}(t), \quad u_{1,1}(t) = -\varepsilon_1 \text{sat} \left(\frac{\lambda_1 Y_1(t)}{\varepsilon_1} \right).$$

For $t \in [0, T_0]$, the state and controller are bounded. For $t \geq T_0$, consider Y_2 -subsystem in (8), namely, $\dot{Y}_2(t) = \text{sat}_{u_{\max}}(u_1(t)) = u_1(t)$. Choose the function

$$W_2(Y_2(t)) = \frac{1}{2} Y_2^2(t),$$

whose time-derivative along the trajectory of (8) can be calculated as

$$\begin{aligned}
\dot{W}_2(t) &= Y_2(t) u_1(t) \\
&= -\varepsilon_2 Y_2(t) \text{sat} \left(\frac{\lambda_2 Y_2(t)}{\varepsilon_2} \right) + Y_2(t) u_{1,1}(t) \\
&= -\varepsilon_2 Y_2(t) \text{sign}(Y_2(t)) \left| \text{sat} \left(\frac{\lambda_2 Y_2(t)}{\varepsilon_2} \right) \right| + Y_2(t) u_{1,1}(t).
\end{aligned}$$

Assume that $|\lambda_2 Y_2| > \varepsilon_2$. Since $\varepsilon_2 > \varepsilon_1$, we have

$$\dot{W}_2(t) = -\varepsilon_2 |Y_2| + |Y_2| u_{1,1}(t) \leq -\varepsilon_2 |Y_2| + \varepsilon_1 |Y_2| < 0.$$

Therefore, there exists a positive constant $T_1 > T_0$ such that, for $t \geq T_1$, $|\lambda_2 Y_2| \leq \varepsilon_2$, and

$$u_1(t) = -\varepsilon_2 \text{sat} \left(\frac{\lambda_2 Y_2(t)}{\varepsilon_2} \right) + u_{1,1}(t) = -\lambda_2 Y_2(t) + u_{1,1}(t).$$

Thus (8) can be written as

$$\begin{aligned}
\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} &= \begin{bmatrix} 0 & \lambda_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \theta(t) \begin{bmatrix} 0 & \frac{\lambda_2}{\delta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-\lambda_2 Y_2 + u_{1,1}(t)) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \theta(t) \begin{bmatrix} 0 & \frac{\lambda_2}{\delta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_{1,1}(t).
\end{aligned} \tag{B1}$$

Consider Y_1 -subsystem in (B1), namely,

$$\dot{Y}_1 = \frac{\lambda_2}{\delta} \theta(t) Y_2 + u_{1,1}(t). \tag{B2}$$

Choose the function

$$W_1(Y_1) = \frac{1}{2} Y_1^2,$$

whose time-derivative along the trajectory of (B2) can be calculated as

$$\begin{aligned}
\dot{W}_1(Y_1) &= Y_1 \left(\frac{\lambda_2}{\delta} \theta(t) Y_2 + u_{1,1}(t) \right) \\
&= \frac{\lambda_2}{\delta} \theta(t) Y_1 Y_2 + Y_1 u_{1,1}(t) \\
&= \frac{\lambda_2}{\delta} \theta(t) Y_1 Y_2 - \varepsilon_1 Y_1 \text{sat} \left(\frac{\lambda_1 Y_1}{\varepsilon_1} \right) \\
&\leq \frac{\varepsilon_2}{\delta} |\theta(t)| |Y_1| - \varepsilon_1 Y_1 \text{sat} \left(\frac{\lambda_1 Y_1}{\varepsilon_1} \right).
\end{aligned}$$

Since $\lim_{t \rightarrow \infty} \theta(t) = 0$, let $T_2 \geq T_1$ be a constant such that $\frac{\varepsilon_2}{\delta} |\theta(t)| < \varepsilon_1$, for $t \geq T_2$. Assume that $|\lambda_1 Y_1| > \varepsilon_1$, we have

$$\dot{W}_1(Y_1) \leq \frac{\varepsilon_2}{\delta} |\theta(t)| |Y_1| - \varepsilon_1 Y_1 \text{sat} \left(\frac{\lambda_1 Y_1}{\varepsilon_1} \right)$$

$$= \frac{\varepsilon_2}{\delta} |\theta(t)| |Y_1| - \varepsilon_1 |Y_1| \\ < 0.$$

Therefore, it can be concluded that there exists a positive constant $T_3 > T_2$ such that, for $t \geq T_3$, $|\lambda_1 Y_1| \leq \varepsilon_1$, and

$$u_{1,1}(t) = -\varepsilon_1 \text{sat} \left(\frac{\lambda_1 Y_1(t)}{\varepsilon_1} \right) = -\lambda_1 Y_1(t).$$

Then, for any $t \geq T_3$, we can get

$$\dot{Y} = A_{y,c} Y + \theta(t) A_2 Y, \quad (\text{B3})$$

where

$$A_{y,c} = \begin{bmatrix} -\lambda_1 & 0 \\ -\lambda_1 & -\lambda_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \frac{\lambda_2}{\delta} \\ 0 & 0 \end{bmatrix}.$$

Let $P > 0$ be an unique solution to

$$A_{y,c} P + P A_{y,c} = -P.$$

Choose the Lyapunov function

$$U(Y) = Y^T P Y,$$

whose time-derivative along the trajectory of (B3) can be calculated as

$$\begin{aligned} \dot{U}(Y) &= -Y^T P Y + 2\theta(t) Y^T P A_2 Y \\ &\leq -Y^T P Y + |\theta(t)| Y^T P Y + \frac{\lambda_{\max}(A_2^T P A_2)}{\lambda_{\min}(P)} |\theta(t)| Y^T P Y. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \theta(t) = 0$, there exists a constant $T_4 \geq T_3$ such that, for $t \geq T_4$,

$$\begin{aligned} \dot{U}(Y) &\leq -Y^T P Y + |\theta(t)| Y^T P Y + \frac{\lambda_{\max}(A_2^T P A_2)}{\lambda_{\min}(P)} |\theta(t)| Y^T P Y \\ &\leq -\varepsilon Y^T P Y, \end{aligned}$$

where $\varepsilon > 0$ is a constant. Therefore, according to Lyapunov stability theory, we have $\lim_{t \rightarrow \infty} Y(t) = 0$, which, together with (4) and (7), indicates that

$$\begin{aligned} \lim_{t \rightarrow \infty} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} &= \lim_{t \rightarrow \infty} \begin{bmatrix} \frac{\lambda_2}{\delta} & 1 \\ 0 & 1 \end{bmatrix}^{-1} Y(t) = \lim_{t \rightarrow \infty} \begin{bmatrix} \frac{\delta}{\lambda_2} & -\frac{\delta}{\lambda_2} \\ 0 & 1 \end{bmatrix} Y(t) = 0, \\ \lim_{t \rightarrow \infty} x_1(t) &= \lim_{t \rightarrow \infty} e^{-\alpha t} z_1(t) = 0, \\ \lim_{t \rightarrow \infty} x_2(t) &= \lim_{t \rightarrow \infty} z_2(t) = 0. \end{aligned}$$

The proof is finished by noting $\lim_{t \rightarrow \infty} u_1(t) = 0$.

Appendix C Extension to Other Nonholonomic Systems

In this section, we extend the methodologies outlined in Theorems 1 and 2 to a chained nonholonomic system comprising an integer subsystem and a bilinear subsystem.

Consider the following chained nonholonomic system

$$\begin{cases} \dot{x}_{01}(t) = x_{02}(t), \\ \dot{x}_{02}(t) = \text{sat}_{u_{\max}}(u_0(t)), \\ \dot{x}_1(t) = -\alpha x_1(t) + x_{02}(t)x_2(t), \\ \dot{x}_2(t) = \text{sat}_{u_{\max}}(u_1(t)), \end{cases} \quad (\text{C1})$$

where $\alpha > 0$ is a constant, $x_0(t) = [x_{01}(t), x_{02}(t)]^T$, and $x_1(t), x_2(t)$ are system states. The control inputs $u_0(t), u_1(t) \in \mathbf{R}$ are bounded through the saturation function

$$\text{sat}_{u_{\max}}(u) = \text{sign}(u) \cdot \min\{|u|, u_{\max}\},$$

where $u_{\max} > 0$ is a given constant.

Consider the state transformation

$$y(t) = T_{0,2} x_0(t) \triangleq \begin{bmatrix} \lambda_2 & 1 \\ 0 & 1 \end{bmatrix} x_0(t),$$

where $y(t) = [y_1(t), y_2(t)]^T$ and $\lambda_2 > 0$. According to [1], the x_0 -subsystem in (C1) can be reorganized as

$$\dot{y} = A_{T,2}y + b_{T,2}\text{sat}_{u_{\max}}(u_0), \quad (C2)$$

where

$$A_{T,2} = \begin{bmatrix} 0 & \lambda_2 \\ 0 & 0 \end{bmatrix}, \quad b_{T,2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (C3)$$

With the above preparations, we can now obtain the following result.

Theorem C1. Let $\lambda_i > \alpha > 0$, $\beta \neq 0$, and $\varepsilon_i > 0$, $i = 1, 2$, be constants satisfying

$$\varepsilon_1 + \varepsilon_2 \leq u_{\max}, \quad \varepsilon_2 > \varepsilon_1. \quad (C4)$$

Consider the time-varying nonhomogeneous feedback

$$u_0(t) = -\varepsilon_2 \text{sat}\left(\frac{\lambda_2 y_2}{\varepsilon_2}\right) - \varepsilon_1 \text{sat}\left(\frac{\lambda_1 y_1}{\varepsilon_1} - \frac{\beta e^{-\alpha t}}{\varepsilon_1}\right), \quad (C5)$$

with $\text{sat}(x) = \text{sat}_1(x)$. Then, the origin point is globally attractive, and the state converges locally to zero exponentially. In particular, there exists a positive constant T_2 such that $x_{02}(t)$ can be expressed as

$$x_{02}(t) = e^{-\alpha t}(\theta_1(t) + \delta_1), \quad t \geq T_2, \quad (C6)$$

where $\delta_1 = \beta\alpha / ((\alpha - \lambda_1)(\alpha - \lambda_2))$ is a constant and $\theta_1(t)$ is a time-varying function satisfying $\lim_{t \rightarrow \infty} \theta_1(t) = 0$.

Proof. For simplicity, we denote

$$\begin{aligned} u_0(t) &= -\varepsilon_2 \text{sat}\left(\frac{\lambda_2 y_2}{\varepsilon_2}\right) + u_{0,1}(t), \\ u_{0,1}(t) &= -\varepsilon_1 \text{sat}\left(\frac{\lambda_1 y_1}{\varepsilon_1} - \frac{\beta e^{-\alpha t}}{\varepsilon_1}\right). \end{aligned}$$

Consider y_2 -subsystem in (C2), namely, $\dot{y}_2 = \text{sat}_{u_{\max}}(u_0(t)) = u_0(t)$. Choose the function

$$V_2 = \frac{1}{2}y_2^2,$$

whose time-derivative along the trajectory of (C2) and (C5) can be calculated as

$$\begin{aligned} \dot{V}_2 &= y_2 u_0(t) \\ &= -\varepsilon_2 y_2 \text{sat}\left(\frac{\lambda_2 y_2}{\varepsilon_2}\right) + y_2 u_{0,1}(t) \\ &= -\varepsilon_2 y_2 \text{sign}(y_2) \left| \text{sat}\left(\frac{\lambda_2 y_2}{\varepsilon_2}\right) \right| + y_2 u_{0,1}(t). \end{aligned}$$

Assume that $|\lambda_2 y_2| > \varepsilon_2$. Since $\varepsilon_2 > \varepsilon_1$, we have

$$\dot{V}_2 = -\varepsilon_2 |y_2| + |y_2| u_{0,1}(t) \leq -\varepsilon_2 |y_2| + \varepsilon_1 |y_2| < 0.$$

Therefore, there exists a positive constant T_1 such that, for $t \geq T_1$, $|\lambda_2 y_2| \leq \varepsilon_2$, and

$$u_0(t) = -\varepsilon_2 \text{sat}\left(\frac{\lambda_2 y_2}{\varepsilon_2}\right) + u_{0,1}(t) = -\lambda_2 y_2 + u_{0,1}(t).$$

Thus, (C2) can be written as

$$\begin{aligned} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} 0 & \lambda_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-\lambda_2 y_2 + u_{0,1}(t)) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_{0,1}(t). \end{aligned}$$

Consider the y_1 -subsystem in the above equation, namely, $\dot{y}_1 = u_{0,1}(t)$. According to the proof of Theorem 1, it can be concluded that there exists a positive constant $T_2 \geq T_1$ such that, for $t \geq T_2$, $|\lambda_1 y_1 + \beta e^{-\alpha t}| \leq \varepsilon_1$, and

$$u_{0,1}(t) = -\varepsilon_1 \text{sat}\left(\frac{\lambda_1 y_1}{\varepsilon_1} - \frac{\beta e^{-\alpha t}}{\varepsilon_1}\right) = -\lambda_1 y_1 + \beta e^{-\alpha t}.$$

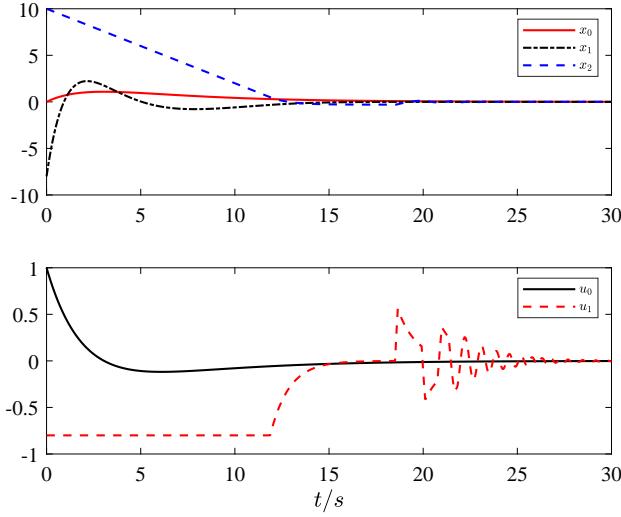


Figure D1 Time evolution of state variables and control inputs for system (1)

Then, for any $t \geq T_2$, we can get

$$\dot{y}(t) = A_{y,c}y(t) + b_{T,2}\beta e^{-\alpha t}, \quad (C7)$$

where

$$A_{y,c} = \begin{bmatrix} -\lambda_1 & 0 \\ -\lambda_1 & -\lambda_2 \end{bmatrix}.$$

Choose the function

$$Y(t) = e^{\alpha t}y(t) + \varepsilon_y,$$

where $Y(t) = [Y_1(t), Y_2(t)]^T$ and $\varepsilon_y = (A_{y,c} + \alpha I_2)^{-1}b_{T,2}\beta$. Then the time-derivative of $Y(t)$ along the trajectory of (C7) can be calculated as

$$\dot{Y}(t) = e^{\alpha t}\dot{y}(t) + \alpha e^{\alpha t}y(t) = (A_{y,c} + \alpha I_2)Y(t) - (A_{y,c} + \alpha I_2)\varepsilon_y + b_{T,2}\beta = (A_{y,c} + \alpha I_2)Y(t),$$

which means that $\lim_{t \rightarrow \infty} Y(t) = 0$ since $\lambda_1 > \alpha$ and $\lambda_2 > \alpha$. Hence, $x_0(t)$ can be expressed as

$$\begin{aligned} x_0(t) &= T_{0,2}^{-1}y(t) \\ &= e^{-\alpha t} \left(T_{0,2}^{-1}Y(t) - T_{0,2}^{-1}(A_{y,c} + \alpha I_2)^{-1}b_{T,2}\beta \right) \\ &= e^{-\alpha t} \left(\begin{bmatrix} \frac{1}{\lambda_2} & -\frac{1}{\lambda_2} \\ 0 & 1 \end{bmatrix} Y(t) - \frac{\beta}{(\alpha - \lambda_1)(\alpha - \lambda_2)} \begin{bmatrix} 1 \\ -\alpha \end{bmatrix} \right), \end{aligned}$$

which implies that

$$x_{02}(t) = e^{-\alpha t}(\theta_1(t) + \delta_1),$$

where

$$\theta_1(t) = Y_2(t), \quad \delta_1 = \frac{\beta\alpha}{(\alpha - \lambda_1)(\alpha - \lambda_2)}.$$

Therefore, (C6) has been proven, and it follows that $\lim_{t \rightarrow \infty} \theta_1(t) = 0$. The proof is finished.

For the saturation design of the controller $u_1(t)$ in $[x_1(t), x_2(t)]$ -subsystem of (C1), we can follow a similar process as outlined in Section of *Design of control law $u_1(t)$* , which is omitted here for brevity.

Appendix D Numerical Simulation

In this section, we conduct a numerical simulation for system (1) with the controllers $u_0(t)$ and $u_1(t)$ as described in Theorems 1 and 2.

For the simulation, to verify the effectiveness of the proposed method when $x_0(0) = 0$, we set the initial condition as $[x_0(0), x_1(0), x_2(0)] = [0, -8, 10]$. Additionally, the parameters are chosen as $\lambda = 0.5, \beta = 1, \alpha = 0.2, \varepsilon_1 = 0.3, \varepsilon_2 = 0.5, \lambda_1 = 0.2$, and $\lambda_2 = 1$. Figure D1 clearly illustrates that the proposed method successfully drives both the system states and the control inputs to zero in the presence of input saturation.

References

1 Zhou B, Duan G R. Global stabilization of linear systems via bounded controls. *Syst Control Lett*, 2009, 58(1), 54-61.