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Semiglobal output feedback control for uncertain minimum-phase nonlinear systems

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Abstract Semiglobal practical output feedback stabilization is achieved for a class of uncertain minimum-phase nonlinear systems involving an unknown control gain, matched uncertainty, and unknown internal dynamics. Notably, neither boundedness conditions are imposed on the nonlinear model functions, nor are the input-to-state stability and bounded-input-to-bounded-state conditions imposed on the system's internal dynamics. Therefore, the proposed output feedback approach serves as a somewhat universal (model-free) controller because it exclusively requires the system output and structural information (e.g., the relative degree r). In addition, our results demonstrate that a (Byrnes-Isidori) minimum-phase normal form with $r \leqslant 2$ is semiglobally practically stabilizable using output feedback control.

Keywords semiglobal stabilization, extended observer, unknown control gain, unknown internal dynamics, underactuated system

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1 Introduction

Recent studies have increasingly focused on underactuated systems, such as the underactuated flexible joint robot [1], the four-degree-of-freedom crane system [2], the translational oscillator with rotational actuator (TORA) system [3], the aerial refueling system [4], and the underactuated autonomous underwater vehicle [5]. Many of these underactuated systems can be formulated using the following (Byrnes-Isidori) normal form [6]

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi, \dot{\xi}, \ldots, \xi^{(r-1)}), \\ \xi^{(r)} = f(\eta, \xi, \dot{\xi}, \ldots, \xi^{(r-1)}) + g(\eta, \xi, \dot{\xi}, \ldots, \xi^{(r-1)})u, \\ y = \xi, \end{cases} \tag{1}$$

where $\eta \in \mathbb{R}^l$, $\xi \in \mathbb{R}$, $\xi^{(i)}$ is the i-th derivative of ξ, $u \in \mathbb{R}$ is the input, and $y \in \mathbb{R}$ is the output. As pointed out in [6] and Chapter 4.5 in [7], the global or semiglobal stabilization of the interconnected system described in (1) using (dynamic) partial state feedback $u(\xi, \dot{\xi}, \ldots, \xi^{(r-1)})$ or further (dynamic) output feedback $u(y)$ remains a challenge, even under the minimum-phase assumption.

Before reviewing the output feedback approaches for system (1), it is important to introduce a specific cascade system:

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi, \dot{\xi}, \ldots, \xi^{(r-1)}), \\ \xi^{(r)} = f(\xi, \dot{\xi}, \ldots, \xi^{(r-1)}) + g(\xi, \dot{\xi}, \ldots, \xi^{(r-1)})u, \\ y = \xi. \end{cases} \tag{2}$$

For the above system (2), high-gain observer approaches [8–15] are typically employed to achieve output feedback stabilization. As elaborated in [9, 10, 13], this cascade system (2) is semiglobally stabilizable using output feedback if it is globally or semiglobally stabilizable by a (dynamic) partial state feedback $u(\xi, \dot{\xi}, \ldots, \xi^{(r-1)})$. This raises a crucial question about the existence of such partial state feedback. Note that the ξ-subsystem is a fully linearizable

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and naturally equipped with a globally stabilizing state feedback $u_{\rm h}(\xi,\dot{\xi},\ldots,\xi^{(r-1)})$. However, even under the minimum-phase assumption, global or semiglobal stabilization for system (2) may not be achieved merely using $u_{\rm h}$. This is because of the peaking phenomenon [16] in the responses of states $\xi,\dot{\xi},\ldots,\xi^{(r-1)}$, as reported in [6] and Chapter 4.5 in [7]. Therefore, supplementary assumptions, including the input-to-state stability (ISS) condition (see [15, 17, 18]), the bounded-input-to-bounded-state (BIBS) condition (see [19]), the global growth conditions on f_0 (see [20]), and the special structural restrictions on the η-subsystem (see [21–24]) are usually imposed. When imposing exclusively structural restrictions on the η-subsystem, low-gain feedback [22, 23] and small signal feedback [21] can be effectively used. For example, in [23], semiglobal stabilization for the minimum-phase system (2) was achieved using linear high-and-low gain feedback (a class of partial state feedback), where the ξ-subsystem was in the linear form and the η-subsystem presented $\dot{\eta}=f_0(\eta,\varphi(\xi^{(r_0-1)})\xi^{(r_0)})$ with $r_0\in\{1,2,\ldots,r\}$.

When it concerns system (1), extended high-gain observer methods [11, 19, 25–27] were employed to achieve output feedback stabilization. In [11, 19, 27], semiglobally practical output feedback stabilization was achieved for system (1), where the η-subsystem was BIBS system with respect to states $\xi,\dot{\xi},\ldots,\xi^{(r-1)}$, was achieved. However, the ISS and BIBS assumption is more stringent than the minimum-phase assumption (as discussed in the example (36)), making it less elegant both theoretically and practically.

In the following, we focus on the minimum-phase system

$$\begin{cases}\dot{\eta}=f_0(\eta,\psi(\xi^{(r_0-1)},\xi^{(r_0)})\xi^{(r_0)}),\\ \xi^{(r)}=f(\eta,\xi,\dot{\xi},\ldots,\xi^{(r-1)})+g(\eta,\xi,\dot{\xi},\ldots,\xi^{(r-1)})u,\\ y=\xi,\end{cases} \tag{3}$$

where $u\in\mathbb{R}$ is the input, $y\in\mathbb{R}$ is the output, $\eta\in\mathbb{R}^l$, $\xi\in\mathbb{R}$, $\xi^{(i)}$ is the i-th derivative of ξ, $\xi^{(r_0-1)}$ and $\xi^{(r_0)}$ are two consecutive derivatives of ξ with $r_0\in\{1,2,\ldots,r\}$, and f_0 and ψ are unknown sufficiently smooth functions with $f_0(0)=0$, f and g are unknown $\mathcal{C}^1$ functions, with $f(0)=0$ and $g(\cdot)>0$ for all its arguments. Besides, system (3) satisfies the following minimum-phase assumption [18, 28, 29].

Assumption 1. The equilibrium $\eta=0$ of the zero dynamics $\dot{\eta}=f_0(\eta,0)$ is globally asymptotically stable (GAS).

Readers may be interested in the special structure of the η-subsystem in (3). On the one hand, for system (3) with the relative degree $r\leqslant 2$, this structural restriction is nearly nonexistent, since ψ is a general function. As reported in [30], this type of normal form can be applied to some underactuated systems with appropriate outputs, such as the inertia wheel pendulum system, TORA system, and planar vertical takeoff and landing aircraft system. On the other hand, the structural restriction seems not so stringent since any two consecutive derivatives of ξ are allowed in the cross term $\psi(\xi^{(r_0-1)},\xi^{(r_0)})\xi^{(r_0)}$.

Based on the concept of semiglobally practical stabilization reported in [31], we state our control objective.

Problem 1 (Semiglobally practical stabilization)**.** Let Assumption 1 be met. Find an observer-based output feedback $u=u(y,\zeta)$ with $\dot{\zeta}=\varPsi(y,\zeta)$, independent of f_0, ψ, f, and g, for system (3) such that the trajectory $(\eta(t),\xi(t),\ldots,\xi^{(r-1)}(t),\zeta(t))$ of the closed-loop system, starting from an arbitrarily large compact set, enters an arbitrarily small compact set in the finite time and then remains in it thereafter.

Motivated by the (extended) high-gain observer approaches [10, 11, 19], the semiglobal stabilization approaches [31–34], and the properties of the parametric Lyapunov equation [35, 36], we intend to offer a concise and universal output feedback approach for system (3) under less amount of model information. The contributions of our study are outlined below.

- Semiglobally practical stabilization for system (3) is achieved by output feedback in the absence of the exact form of model functions f_0, ψ, f, and g. In this way, the developed controller effectively operates under any practical initial conditions and is somewhat considered a universal (model-free) controller (as detailed in Remark 5). Our results show that a minimum-phase normal form with the relative degree $r\leqslant 2$ is semiglobally practically stabilizable by output feedback.
- System (3) involves the unknown control gain, the matched uncertainties, and the unknown internal dynamics due to the unknown f_0, ψ, f, and g. These features differentiate our method from those in [9, 10, 37]. In addition, any bounded conditions, such as the globally Lipschitz condition [38, 39], the linear growth condition (see [36]) and polynomial growth condition (see [40, 41]), are not imposed on the nonlinear functions f_0, ψ, f, and g.
- Except for the basic assumptions such as the equilibrium assumption, only the minimum-phase assumption (Assumption 1) is imposed on system (3). The minimum-phase assumption is less stringent than the BIBS assumption (used in [19] and Chapter 6 in [11]) or the ISS assumption (used in [15, 17, 18]). Moreover, our model assumption is different from that in [30], and the employed nonlinear ξ-subsystem of system (3) is more general than the linear one in [34].

• System (3) exhibits an interconnected form and can be nonuniformly (completely) observable (see the exact definition in [33] or [42]), indicating a difference from the related studies [9, 10, 32–34, 42, 43].

Notation. A function $f(x)\colon D \to \mathbb{R}^m$ is said to be $\mathcal{C}^r$ (denoted by $f(x) \in \mathcal{C}^r$) on a domain $D \subset \mathbb{R}^n$ if it is r orders continuously differentiable for some integer $r \geqslant 1$. $\Omega \backslash \Theta$ represents the complement of the set Θ in Ω. We use $\mathrm{diag}(c_1, c_2, \dots, c_n)$ to denote a diagonal matrix whose i-th diagonal element is the scalar c_i.

2 Preliminaries

2.1 Design tools

Let us recall the properties of the parametric Lyapunov equation (PLE). Denote $(A, b, c) \in (\mathbb{R}^{n\times n}, \mathbb{R}^{n\times 1}, \mathbb{R}^{1\times n})$ as

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix},\ b = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}. \tag{4}$$

Lemma 1 ([35, 36]). Let (A, c) be given by (4). Then the following PLE:

$$AQ + QA^{\mathrm{T}} - Qc^{\mathrm{T}}cQ = -\gamma Q \tag{5}$$

has a unique positive definite solution $Q = Q(\gamma)$ if and only if $\gamma > 0$. Assumed that this is satisfied.

(1) The solution $Q(\gamma) = W^{-1}(\gamma)$ is solved from the Lyapunov equation $W(\gamma)(A+\frac{\gamma}{2}I_n)+(A+\frac{\gamma}{2}I_n)^{\mathrm{T}}W(\gamma) = c^{\mathrm{T}}c$. Moreover, $Q(\gamma)$ can be given by

$$Q(\gamma) = \gamma^{2n-1}L_n^{-1}(\gamma)Q_nL_n^{-1}(\gamma), \tag{6}$$

where $Q_n = Q(1)$ and $L_n(\gamma) = \mathrm{diag}(\gamma^{n-1}, \gamma^{n-2}, \dots, 1)$.

(2) $A - Q(\gamma)c^{\mathrm{T}}c$ (respectively, $A - Q_nc^{\mathrm{T}}c$) is Hurwitz and its eigenvalues are $-\gamma$ (respectively, -1).

We proceed with two classes of saturation functions. Given a positive scalar M, a $\mathcal{C}^0$ saturation function $\sigma_M : \mathbb{R} \to [-M, M]$ is defined as

$$\sigma_M(x) = \mathrm{sign}(x)\min\{|x|, M\}.$$

Another $\mathcal{C}^1$ saturation function $\rho_{(M,\gamma)}\colon \mathbb{R} \to [-M - 1/(2\gamma), M + 1/(2\gamma)]$ is defined as

$$\rho_{(M,\gamma)}(x) = \begin{cases} M + \dfrac{1}{2\gamma}, & x \geqslant M + 1/\gamma, \\ x - \dfrac{1}{2}\gamma(x-M)^2, & M < x < M + 1/\gamma, \\ x, & -M \leqslant x \leqslant M, \\ x + \dfrac{1}{2}\gamma(x+M)^2, & -M - 1/\gamma < x < -M, \\ -M - \dfrac{1}{2\gamma}, & x \leqslant -M - 1/\gamma, \end{cases}$$

where M and γ are two positive scalars. Such a $\rho_{(M,\gamma)}(x)$ satisfies

$$\left|\rho_{(M,\gamma)}(x) - \sigma_M(x)\right| \leqslant \frac{1}{2\gamma},\ \forall x \in \mathbb{R}. \tag{7}$$

These two saturation functions are depicted in Figure 1.

2.2 Analysis tools

In what follows, we omit the uniformity with respect to time t for the time-varying system and let $t_0 = 0$ without loss of generality. Thus, the initial condition is denoted as $x_0 = x(0) = x(t_0)$.

Lemma 2. Consider the singularly perturbed system

$$\dot{x} = F_0(x) + \varDelta_1(x, e, t),\ x_0 = x(0), \tag{8}$$

$$\dot{e} = \gamma\varPhi(x, e, t) + \varDelta_2(x, e, t),\ e_0 = e(0) = e_0(\gamma), \tag{9}$$

where $x \in \mathbb{R}^m$, $e \in \mathbb{R}^n$, $\gamma \in [1, \infty)$ is an adjustable parameter, the initial condition $e_0 = e_0(\gamma)$ satisfies $\|e_0(\gamma)\| \leqslant e_{\mathrm{c}}\gamma^p$, $e_{\mathrm{c}} > 0$ is an arbitrarily large scalar independent of γ, $p \geqslant 0$ is an arbitrary (finite) constant independent of

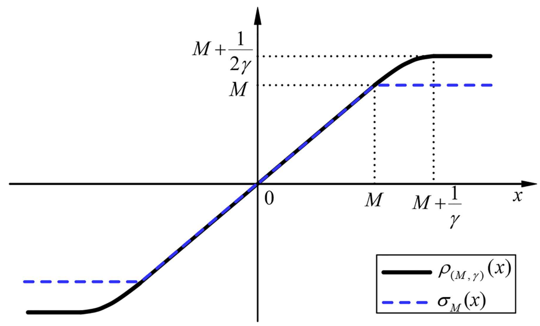

Figure 1 (Color online) Saturation functions.

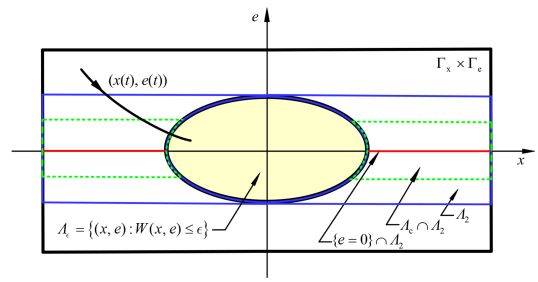

Figure 2 (Color online) Local convergence of the system trajectory.

γ, the initial condition x_0 is independent of γ, $F_0(x)$ and $\Phi(x,e,t)$ are piecewise continuous functions independent of γ, with $F_0(0)=0$ and $\Phi(0,0,t)=0$, $\forall t \geqslant 0$, $\Delta_1(x,e,t)$ and $\Delta_2(x,e,t)$ are piecewise continuous functions, with $\Delta_1(0,0,t)=0$, $\Delta_2(0,0,t)=0$, $\forall t \geqslant 0$. Suppose that

(1) The equilibrium $x=0$ of the nominal x-subsystem $\dot{x}=F_0(x)$ is locally asymptotically stable (LAS), with region of attraction (ROA) denoted by $\mathcal{R} \subset \mathbb{R}^m$;

(2) Given any subset $\Gamma_{\rm x} \subset \mathcal{R}$, the reduced e-subsystem $\dot{e}=\Phi(x,e,t)$ has a $\mathcal{C}^1$ Lyapunov function $U(e): \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ satisfying

$$c_1 \|e\|^2 \leqslant U(e) \leqslant c_2 \|e\|^2, \ \forall e \in \mathbb{R}^n, \tag{10}$$

$$\frac{\partial U(e)}{\partial e} \Phi(x,e,t) \leqslant -c_3 \|e\|^2, \ \forall (x,e,t) \in \Gamma_{\rm x} \times \mathbb{R}^n \times [0,\infty), \tag{11}$$

$$\left\| \frac{\partial U(e)}{\partial e} \right\| \leqslant c_4 \|e\|, \ \forall e \in \mathbb{R}^n, \tag{12}$$

where c_1, c_2, c_3, c_4 are positive constants independent of γ;

(3) Given any subset $\Gamma_{\rm x} \subset \mathcal{R}$, there exist positive constants δ_0, δ_1, δ_2, δ_3 independent of γ such that

$$\|\Delta_1(x,e,t)\| \leqslant \delta_0, \ \forall (x,e,t) \in \Gamma_{\rm x} \times \mathbb{R}^n \times [0,\infty), \tag{13}$$

$$\|\Delta_2(x,e,t)\| \leqslant \delta_1 \|e\| + \delta_2, \ \forall (x,e,t) \in \Gamma_{\rm x} \times \mathbb{R}^n \times [0,\infty), \tag{14}$$

$$\|\Delta_1(x,e,t)\| \leqslant \delta_3 \|e\|, \ \forall (x,e,t) \in \Gamma_{\rm x} \times \Gamma_{\rm e} \times [0,\infty), \tag{15}$$

where $\Gamma_{\rm e} \triangleq \{e \in \mathbb{R}^n \colon \|e\|^2 \leqslant \vartheta^2/\gamma^2\}$ and ϑ is some positive constant independent of γ.

Then, given any compact subset $\Omega_{\rm x} \subseteq \Gamma_{\rm x}$ and an arbitrarily small compact set $\Lambda_\epsilon \subset \mathbb{R}^{m+n}$, both centered at the origin, there exists a $\gamma_{\rm a} \in [1,\infty)$ such that, for any $\gamma \in [\gamma_{\rm a},\infty)$, the trajectory $(x(t),e(t))$ of system (8) and (9), with its initial condition $(x_0,e_0) \in \Omega_{\rm x} \times \mathbb{R}^n$, enters the set Λ_ϵ in the finite time and then remains in it thereafter.

Proof. See Appendix A.

Remark 1. The proof in Appendix A introduces a composite Lyapunov function $W(x,e)$ on the set $(\Gamma_\chi \times \Gamma_{\rm e}) \backslash \Lambda_\epsilon$, which is challenging to establish due to the interconnection of two subsystems. This Lyapunov function directly implies that the trajectory $(x(t),e(t))$ starting from $(\Gamma_\chi \times \Gamma_{\rm e}) \backslash \Lambda_\epsilon$ asymptotically enters the arbitrarily small set Λ_ϵ (as depicted in Figure 2). In addition, such a Lyapunov function $W(x,e)$ will facilitate the stability analysis of interconnected systems consisting of multiple subsystems in the form of (8) and (9).

Lemma 3. Consider the well-defined singularly perturbed system (8) and (9). Suppose that

(1) The conditions in Lemma 2 are satisfied;
(2) The equilibrium $x=0$ of the nominal x-subsystem $\dot{x}=F_0(x)$ is locally exponentially stable (LES);
(3) There exist positive constants δ_4 and δ_5 independent of γ such that

$$\|\varDelta_2(x,e,t)\| \leqslant \delta_4\|e\|+\delta_5\|x\|,\ \forall (x,e,t)\in\varLambda_\epsilon\times[0,\infty), \tag{16}$$

where $\varLambda_\epsilon\subset\mathbb{R}^{m+n}$ is an arbitrarily small compact set.

Then, given any compact subset $\varOmega_{\rm x}\subseteq\varGamma_{\rm x}$, centered at the origin, there exists a $\gamma_{\rm b}\in[\gamma_{\rm a},\infty)$ such that, for any $\gamma\in[\gamma_{\rm b},\infty)$, the equilibrium $(x,e)=(0,0)$ of system (8) and (9) is LAS with ROA containing $\varOmega_{\rm x}\times\mathbb{R}^n$, where $\gamma_{\rm a}$ is the one in Lemma 2.

Proof. This proof follows a similar procedure as the proof of Theorem 5 in [10].

We mention that the LES condition in Item 2 can be relaxed to other ones (see Assumption 4 in [10]).

3 Main results

Rewrite system (3) in the subsequent state-space form

$$\begin{cases}\dot{\eta}=f_0(\eta,\psi(x_{r_0},x_{r_0+1})x_{r_0+1}),\\ \dot{x}=A_{\rm c}x+b_{\rm c}\left(f(\eta,x)+g(\eta,x)u\right),\\ y=c_{\rm c}x,\end{cases} \tag{17}$$

where $x=[\,x_1\ x_2\ \cdots\ x_r\,]^{\rm T}$ is the state with $x_i=\xi^{(i-1)}$ and the triple $(A_{\rm c},b_{\rm c},c_{\rm c})\in(\mathbb{R}^{r\times r},\mathbb{R}^{r\times 1},\mathbb{R}^{1\times r})$ shares the similar form as (A,b,c) in (4).

3.1 Ideal target system

An ideal state feedback is constructed as

$$u=\frac{1}{g(\eta,x)}\left(v(x)-f(\eta,x)\right)\triangleq\phi(\eta,x), \tag{18}$$

$$v(x)=-\sum_{i=1}^{r_0}\frac{1}{k_{\rm s}}a_ix_{r-r_0+i}-\sum_{i=1}^{r-r_0}k_{\rm s}b_iw_i(x), \tag{19}$$

$$w_i(x)=x_{r_0-i+2}+\sum_{j=1}^{r_0}\frac{1}{k_{\rm s}}a_jx_{i+j-1},\ i=1,2,\ldots,r-r_0,$$

where $k_{\rm s}\in[1,\infty)$ is the scalar parameter determined by the following Lemma 4, a_i, $i=1,2,\ldots,r_0$ and b_i, $i=1,2,\ldots,r-r_0$ are parameters chosen such that companion polynomials $s^{r_0}+a_{r_0}s^{r_0-1}+\cdots+a_1$ and $s^{r-r_0}+b_{r-r_0}s^{r-r_0-1}+\cdots+b_1$ are Hurwitz. We point out that the partial state feedback $v(x)$ is in the linear form and designed based on the linear high-and-low-gain approach [22, 24]. However, as the nonlinear input transformation in (18) is employed, the entire state feedback $\phi(\eta,x)$ is no longer linear.

Thus, the closed-loop system consisting of (17) and (18) presents

$$\begin{bmatrix}\dot{\eta}\\ \dot{x}\end{bmatrix}=\begin{bmatrix}f_0(\eta,\psi(x_{r_0},x_{r_0+1})x_{r_0+1})\\ A_{\rm c}x+b_{\rm c}v(x)\end{bmatrix}\triangleq F_0(\eta,x). \tag{20}$$

We also view system (20) as the ideal target system to be recovered by our developed observer. Following a similar proof for Theorem 2 of [24], we have the following results.

Lemma 4. Let Assumption 1 be met. There exists a constant $k_{\rm s*}\in[1,\infty)$ such that, for any $k_{\rm s}\in[k_{\rm s*},\infty)$, the equilibrium $(\eta,x)=(0,0)$ of the ideal target system (20) is LAS, and its ROA, denoted by $\mathcal{R}\subset\mathbb{R}^{l+r}$, contains any compact subset $\varOmega_\chi\subset\mathbb{R}^{l+r}$.

Let $k_\mathrm{s} \in [k_{\mathrm{s}*}, \infty)$ be satisfied. It then follows from the converse Lyapunov theorem (Theorem 4.17 in [44]) that there exists a $\mathcal{C}^\infty$ positive definite function $V(\eta, x)\colon \mathcal{R} \to \mathbb{R}_{\geqslant 0}$ such that

$$\frac{\partial V(\eta,x)}{\partial(\eta,x)} F_0(\eta,x) \leqslant 0,\ \forall\,(\eta,x) \in \mathcal{R}, \tag{21}$$

$$\lim_{(\eta,x)\to\partial\mathcal{R}} V(\eta,x) = \infty,\ \forall\,(\eta,x) \in \mathcal{R}, \tag{22}$$

where $\partial\mathcal{R}$ denotes the boundary of $\mathcal{R}$. In view of the positive definiteness of $V(\eta, x)$ and (22), we can define a set

$$\mathit{\Gamma}_\chi \triangleq \{(\eta,x) \in \mathbb{R}^{l+r}\colon V(\eta,x) \leqslant \upsilon_0\},$$

such that $\mathit{\Omega}_\chi \subseteq \mathit{\Gamma}_\chi \subset \mathcal{R}$, where υ_0 is some positive constant. It follows from (21) and (22) that the set $\mathit{\Gamma}_\chi$ is a positively invariant compact set for the ideal target system (20), that is,

$$(\eta_0, x_0) \in \mathit{\Gamma}_\chi \Rightarrow (\eta(t), x(t)) \in \mathit{\Gamma}_\chi,\ \forall t \geqslant 0. \tag{23}$$

3.2 Output feedback

We then introduce some preliminaries for the extended observer design. Taking a control gain $g_0 \in (0,\infty]$ (selecting later) as the approximation to $g(\eta, x)$, the approximation error is denoted by

$$\tilde{g}(\eta,x) \triangleq g(\eta,x) - g_0.$$

Instead of employing η in the state feedback $\phi(\eta, x)$ (see (18)), we tend to find an extended state ϖ to offset the influence of the unusable state η. In other words, it is expected to construct a new state feedback relying on the states x and ϖ:

$$\varphi(x,\varpi) \triangleq \frac{v(x) - \varpi}{g_0} = \phi(\eta,x), \tag{24}$$

where $v(x)$ is given by (19). Thus the extended state ϖ can be expressed as

$$\begin{aligned}\varpi &= -g_0\phi(\eta,x) + v(x) = f(\eta,x) + \tilde{g}(\eta,x)\phi(\eta,x)\\ &= f(\eta,x) + \tilde{g}(\eta,x)\varphi(x,\varpi).\end{aligned} \tag{25}$$

We define $n \triangleq r+1$ and $b_\mathrm{d} \triangleq [\,0\ \cdots\ 0\ 1\ 0\,]^\mathrm{T} \in \mathbb{R}^n$. Then along with (A, b, c) is given by (4), we have

$$A = \begin{bmatrix} A_\mathrm{c} & b_\mathrm{c} \\ 0_{1\times r} & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ b_\mathrm{c} \end{bmatrix},\ c = \begin{bmatrix} c_\mathrm{c} & 0 \end{bmatrix},\ b_\mathrm{d} = \begin{bmatrix} b_\mathrm{c} \\ 0 \end{bmatrix}. \tag{26}$$

The PLE-based extended observer and the output feedback are designed as

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\varpi}} \end{bmatrix} = A \begin{bmatrix} \hat{x} \\ \hat{\varpi} \end{bmatrix} + Q(\gamma)c^\mathrm{T}\left(y - c\begin{bmatrix} \hat{x} \\ \hat{\varpi} \end{bmatrix}\right) + b_\mathrm{d} g_0 u, \tag{27}$$

$$u = \sigma_M(\varphi(\hat{x},\hat{\varpi})) \triangleq \varphi^s(\hat{x},\hat{\varpi}), \tag{28}$$

where $\hat{x}$ and $\hat{\varpi}$ are the estimations of x and ϖ, respectively, $g_0 \in (0,\infty]$ is selected later, $Q(\gamma) \in \mathbb{R}^{n\times n}$ is the solution to the PLE (5) with $\gamma \in [1,\infty)$ to be determined, $\varphi(\hat{x}, \hat{\varpi})$ shares the same form as $\varphi(x, \varpi)$ in (24) and its parameter k_s is well selected, and the scalar parameter M in the saturation function $\sigma_M(\varphi(\hat{x}, \hat{\varpi}))$ (defined in Subsection 2.1) is chosen such that

$$M > \max_{(\eta,x)\in\mathit{\Gamma}_\chi} |\varphi(x,\varpi)| \triangleq M_0. \tag{29}$$

Such M and M_0 are independent of γ and always exist due to (24) and (25).

Remark 2. The saturation operation in (28), motivated by the similar operation in [10, 11, 19], is used to prevent the incorporation of undesired information (particularly peaking information). As elaborated in (23), the trajectory $(\eta(t), x(t))$ of the ideal target system (20), starting from the arbitrarily large compact set $\mathit{\Gamma}_\chi$, always remains in the set $\mathit{\Gamma}_\chi$. This indicates that $\mathit{\Gamma}_\chi$ is the set of interest, containing all the necessary information for semiglobal stabilization. Thus we can choose a scalar M as described in (29) and saturate $\varphi(\hat{x}, \hat{\varpi})$ to $[-M, M]$.

Remark 3. As the observability or detectability of system (17) having the internal dynamics usually fails to hold, designing an observer for the η-subsystem is difficult (if not possible). Then without the estimation $\hat{\eta}$ for the state η, not only the control gain $g(\hat{\eta},\hat{x})$ and the matched uncertainty $f(\hat{\eta},\hat{x})$ are unknown, but also the estimation $\phi(\hat{\eta},\hat{x})$ for the preliminary state feedback $\phi(\eta,x)$ remains unknown. Fortunately, by recognizing the extended state ϖ and using the extended observer, we can obtain $\varphi(\hat{x},\hat{\varpi})$ to substitute the unattainable $\phi(\hat{\eta},\hat{x})$. From this perspective, our approach naturally incorporates the idea of the unknown input observer in a sense.

Motivated by [19], define an auxiliary state as

$$\omega = f(\eta,x) + \tilde{g}(\eta,x)\rho_{(M,\gamma)}\left(\varphi\left(x,\hat{\varpi}\right)\right) \triangleq f(\eta,x) + \tilde{g}(\eta,x)\varphi^{\rho}\left(x,\hat{\varpi}\right), \tag{30}$$

where $\rho_{(M,\gamma)}$ is the $\mathcal{C}^1$ saturation function defined in Subsection 2.1. Then, the scaled estimation-error (discussed latter) is defined as

$$e \triangleq L_n(\gamma)\begin{bmatrix} x-\hat{x} \\ \omega-\hat{\varpi}\end{bmatrix}, \tag{31}$$

where $L_n(\gamma)=\mathrm{diag}(\gamma^{n-1},\gamma^{n-2},\ldots,1)$. In view of the plant system (17) and observer (27), the e-subsystem can be evaluated as

$$\begin{aligned}\dot{e} &= L_n\left(A\begin{bmatrix} x \\ \omega\end{bmatrix} + b_{\mathrm{d}}\left(f(\eta,x)+g(\eta,x)u-\omega\right) + b\dot{\omega}\right) - L_n\left(A\begin{bmatrix}\hat{x} \\ \hat{\varpi}\end{bmatrix} + Q(\gamma)c^{\mathrm{T}}\left(y - c\begin{bmatrix}\hat{x} \\ \hat{\varpi}\end{bmatrix}\right) + b_{\mathrm{d}}g_0u\right) \\ &= L_n\left(A-Q(\gamma)c^{\mathrm{T}}c\right)\begin{bmatrix} x-\hat{x} \\ \omega-\hat{\varpi}\end{bmatrix} + b\dot{\omega} + \gamma b_{\mathrm{d}}\left(g(\eta,x)u - \tilde{g}(\eta,x)\varphi^{\rho}\left(x,\hat{\varpi}\right) - g_0u\right),\end{aligned}$$

where $L_n=L_n(\gamma)$, (26), (30), $L_nb_{\mathrm{d}}=\gamma b_{\mathrm{d}}$, and $L_nb=b$ are used. Along with $L_nA=\gamma AL_n$, (6), and $L_n^{-1}c^{\mathrm{T}}=c^{\mathrm{T}}/\gamma^{n-1}$, we have

$$L_n\left(A-Q(\gamma)c^{\mathrm{T}}c\right)\begin{bmatrix} x-\hat{x} \\ \omega-\hat{\varpi}\end{bmatrix} = \gamma AL_n\begin{bmatrix} x-\hat{x} \\ \omega-\hat{\varpi}\end{bmatrix} - L_n\left(\gamma^{2n-1}L_n^{-1}Q_nL_n^{-1}\right)c^{\mathrm{T}}c\begin{bmatrix} x-\hat{x} \\ \omega-\hat{\varpi}\end{bmatrix} = \gamma Ae - \gamma Q_nc^{\mathrm{T}}ce.$$

Similarily, we have $\dot{\hat{\varpi}}=\gamma b^{\mathrm{T}}Q_nc^{\mathrm{T}}ce=\gamma q_{n1}ce$ and subsequently,

$$\dot{\omega} = \dot{f}(\eta,x) + \dot{\tilde{g}}(\eta,x)\varphi^{\rho}\left(x,\hat{\varpi}\right) + \tilde{g}(\eta,x)\rho'\frac{\partial\varphi\left(x,\hat{\varpi}\right)}{\partial x}\dot{x} - \frac{\tilde{g}(\eta,x)}{g_0}\rho'\gamma q_{n1}ce,$$

where $Q_n=Q(1)$, q_{ij} is the i-th row and j-th column element of Q_n, and $\rho' \triangleq \mathrm{d}\varphi^{\rho}\left(x,\hat{\varpi}\right)/\mathrm{d}\varphi\left(x,\hat{\varpi}\right)$.

Subsequently, in view of $\varphi(x,\varpi)=\phi(\eta,x)$ given by (24), the closed-loop system consisting of the plant system (17) and the output feedback (27) and (28) can be formulated as

$$\dot{\chi} = F_0(\chi) + \mathit{\Delta}_1(\chi,\hat{x},\hat{\varpi}), \tag{32}$$

$$\dot{e} = \gamma\left(\mathcal{A}e - Bk(t)ce\right) + \mathit{\Delta}_2(\chi,\hat{x},\hat{\varpi}), \tag{33}$$

where $\chi \triangleq \mathrm{col}(\eta,x)$, $F_0(\chi)$ is the one in (20), and

$$\begin{aligned}&\mathcal{A} = A - Q_nc^{\mathrm{T}}c,\ B = q_{n1}b,\ k(t) = \frac{\tilde{g}(\chi)}{g_0}\rho', \\ &\mathit{\Delta}_1(\chi,\hat{x},\hat{\varpi}) = b_{\mathrm{a}}g(\chi)\left(\varphi^{s}(\hat{x},\hat{\varpi}) - \varphi(x,\varpi)\right), \\ &\mathit{\Delta}_2(\chi,\hat{x},\hat{\varpi}) = \gamma b_{\mathrm{d}}\left(g(\chi)\varphi^{s}(\hat{x},\hat{\varpi}) - \tilde{g}(\chi)\varphi^{\rho}\left(x,\hat{\varpi}\right) - g_0\varphi^{s}(\hat{x},\hat{\varpi})\right) \\ &\qquad\qquad + b\left(\dot{f}(\chi) + \dot{\tilde{g}}(\chi)\varphi^{\rho}\left(x,\hat{\varpi}\right) + \tilde{g}(\chi)\rho'\frac{\partial\varphi\left(x,\hat{\varpi}\right)}{\partial x}\dot{x}\right), \\ &b_{\mathrm{a}} = \left[\,0\ \cdots\ 0\ 1\,\right]^{\mathrm{T}} \in \mathbb{R}^{l+r}.\end{aligned}$$

Readers may be interested in the actual estimation error $\varpi-\hat{\varpi}$, rather than $\omega-\hat{\varpi}$ in (31). To clarify this issue, let us introduce Lemma 5.

Lemma 5. If the condition

$$\left|\frac{\tilde{g}(\eta,x)}{g_0}\right| < 1,\ \forall(\eta,x)\in\varGamma_\chi \tag{34}$$

is satisfied, then there holds

$$e=0\Rightarrow\hat{\varpi}=\omega=\varpi,\ \forall(\eta,x)\in\varGamma_\chi.$$

Proof. The fact $e=0\Rightarrow\hat{\varpi}=\omega$ follows by (31). It remains to show $e=0\Rightarrow\omega=\varpi$, $\forall(\eta,x)\in\varGamma_\chi$. Along with (25) and (30), we have

$$\varpi-\omega=\tilde{g}(\eta,x)\left(\varphi(x,\varpi)-\varphi^\rho\left(x,\omega\right)\right),$$

as long as $e=0$. Then by using the definition of the saturation function $\rho_{(M,\gamma)}$ and its parameter M chosen in (29), we have $\varphi(x,\varpi)=\varphi^\rho(x,\varpi)$, $\forall(\eta,x)\in\varGamma_\chi$ and subsequently,

$$\varpi-\omega=\tilde{g}(\eta,x)\left(\varphi^\rho(x,\varpi)-\varphi^\rho\left(x,\omega\right)\right),\ \forall(\eta,x)\in\varGamma_\chi,$$

as long as $e=0$. It then follows from the globally Lispchitz property of $\rho_{(M,\gamma)}$ and (24) that

$$|\varpi-\omega|\leqslant|\tilde{g}(\eta,x)|\left|\varphi(x,\varpi)-\varphi\left(x,\omega\right)\right|=\left|\frac{\tilde{g}(\eta,x)}{g_0}\right|\left|\varpi-\omega\right|,\ \forall(\eta,x)\in\varGamma_\chi,$$

as long as $e=0$. Thus the lemma is valid by noting condition (34).

Remark 4. When condition (34) is removed, $\hat{\varpi}=\varpi=\omega$ is not always satisfied even when $(\eta,x)\in\varGamma_\chi$ and $e=0$. Therefore, condition (34) is essential for guaranteeing that $e=0\Rightarrow\varphi^s(\hat{x},\hat{\varpi})=\varphi(x,\varpi)$, $\forall(\eta,x)\in\varGamma_\chi$. This further ensures the successful recovery of the ideal target system (20) when $e\to0$, namely, $e\to0\Rightarrow\varDelta_1(\eta,x,\hat{x},\hat{\varpi})\to0$. The above property is crucial because the χ-subsystem (32), despite having an asymptotically stable $\dot{\chi}=F_0(\chi)$, may exhibit finite-time-escaping if $\varDelta_1(\eta,x,\hat{x},\hat{\varpi})$ is merely bounded but failing to converge to zero.

We proceed to investigate the properties of the closed-loop system (32) and (33). Firstly, under the condition (34), the equilibrium of the closed-loop system (32) and (33) lies on the point $(\chi,e)=(0,0)$. Secondly, the properties of the reduced e-subsystem and the cross-terms $\varDelta_1(\chi,\hat{x},\hat{\varpi})$, $\varDelta_2(\chi,\hat{x},\hat{\varpi})$ are concluded in Lemmas 6, whose proofs are conducted in Appendixes B and C, respectively.

Lemma 6. Let the condition (34) be met and consider the system

$$\dot{e}=\mathcal{A}e-Bk(t)ce, \tag{35}$$

where $(\mathcal{A},B,c)$ and $k(t)$ are same as those in (33). Then when $(\eta,x)\in\varGamma_\chi$, there exists a Lyapunov function $U(e)=e^{\mathrm{T}}\mathcal{Q}e$ such that the time derivative of $U(e)$ along the trajectory of system (35) satisfies

$$\dot{U}(e)|_{(35)}\leqslant-\mu U(e),\ \forall e\in\mathbb{R}^n,$$

where $\mathcal{Q}\in\mathbb{R}^{n\times n}$ is a positive definite matrix and $\mu>0$ is a constant, both independent of γ.

Lemma 7. Let the condition (34) be met. There exist a sufficiently large constant $\gamma_1\in[1,\infty)$ and some positive constants δ_0, δ_1, δ_2, δ_3, δ_4, δ_5 independent of γ such that, for any $\gamma\in[\gamma_1,\infty)$,

$$\begin{aligned}
&\|\varDelta_1(\chi,\hat{x},\hat{\varpi})\|\leqslant\delta_0,\ \forall\,(\chi,e)\in\varGamma_\chi\times\mathbb{R}^n,\\
&\|\varDelta_2(\chi,\hat{x},\hat{\varpi})\|\leqslant\delta_1\|e\|+\delta_2,\ \forall\,(\chi,e)\in\varGamma_\chi\times\mathbb{R}^n,\\
&\|\varDelta_1(\chi,\hat{x},\hat{\varpi})\|\leqslant\delta_3\|e\|,\ \forall\,(\chi,e)\in\varGamma_\chi\times\varGamma_{\mathrm{e}},\\
&\|\varDelta_2(\chi,\hat{x},\hat{\varpi})\|\leqslant\delta_4\|e\|+\delta_5\|\chi\|,\ \forall\,(\chi,e)\in\varGamma_\chi\times\varGamma_{\mathrm{e}},
\end{aligned}$$

where $\varGamma_{\mathrm{e}}\triangleq\{e\in\mathbb{R}^n:\ \|e\|^2\leqslant\vartheta^2/\gamma^2\}$ and ϑ is some positive constant independent of γ.

Until now, our main result can be summarized as follows.

Theorem 1 (Semiglobally practical stabilization)**.** Let Assumption 1 and condition (34) be met. There exists a $\gamma_{\mathrm{a}}\in[\gamma_1,\infty)$ such that, for any $\gamma\in[\gamma_{\mathrm{a}},\infty)$, system (3) is semiglobally practically stabilized by the output feedback (27) and (28), that is, given an arbitrary compact set $\varOmega_\chi\subset\mathbb{R}^{l+r}$ and an arbitrarily small set $\varLambda_\epsilon\subset\mathbb{R}^{l+r+n}$, both centered at the origin, the trajectory $(\chi(t),e(t))$ of the closed-loop system (32) and (33), with the initial condition $(\chi_0,e_0)\in\varOmega_\chi\times\mathbb{R}^n$, enters the set $\varLambda_\epsilon$ in the finite time and then remains in it thereafter.

Proof. This proof follows directly by using Lemma 2. The initial condition χ_0 is independent of γ, and e_0 satisfies $\|e_0\| \leqslant \gamma^{n-1} \|(x_0 - \hat{x}_0, \omega_0 - \hat{\varpi}_0)\|$, with the initial conditions $x_0 - \hat{x}_0$ and $\omega_0 - \hat{\varpi}_0$ independent of γ. Then, the remaining conditions of Lemma 2 are verified individually. Firstly, it follows from Lemma 4 that the nominal x-subsystem $\dot{x} = F_0(x)$ is LAS with ROA $\mathcal{R}$ containing any compact sets. Without loss of generality, we employ Γ_χ to represent these compact sets. Secondly, it follows from Lemma 6 that the reduced e-subsystem has a Laypunov function $U(e) = e^{\mathrm{T}}\mathcal{Q}e$ satisfying (10)–(12) when $\chi \in \Gamma_\chi$. Thirdly, the first three conditions in Lemma 7 directly imply the conditions (13)–(15). Finally, applying Lemma 2 for the closed-loop system (32) and (33) obtains that for an arbitrary compact subset $\Omega_\chi \subseteq \Gamma_\chi \subset \mathcal{R}$, the trajectory $(\chi(t), e(t))$, with the initial condition $(\chi_0, e_0) \in \Omega_\chi \times \mathbb{R}^n$, enters the set Λ_ϵ in the finite time and then remains in it thereafter. Furthermore, in view of the arbitrary large set $\mathcal{R}$ and the definition of Γ_χ, the set Ω_χ can be set as any compact set of $\mathbb{R}^{l+r}$.

Readers may be interested in the applicability of the condition (34). It is worth mentioning that condition (34) can be satisfied by choosing an appropriate g_0. For instance, if we choose $g_0 > \max_{(\eta,x)\in\Gamma_\chi} g(\eta, x)$, it then follows from $g(\eta, x) > 0$ and $g_0 > 0$ that $0 < g(\eta, x)/g_0 < 1$ holds for any $(\eta, x) \in \Gamma_\chi$, which is exactly the condition (34). Then we obtain a corollary from Theorem 1 for practical applications.

Corollary 1. Let Assumption 1 and $g_0 > \max_{(\eta,x)\in\Gamma_\chi} g(\eta, x)$. There exists a $\gamma_{\mathrm{a}} \in [\gamma_1, \infty)$ such that, for any $\gamma \in [\gamma_{\mathrm{a}}, \infty)$, system (3) is semiglobally practically stabilized by the output feedback (27) and (28). Thus, Problem 1 is solved.

Remark 5. The controller (27) and (28) is somewhat a universal (model-free) controller. To clarify this point, note the following:

- f_0, ψ, f, and g are unknown general functions without any bounded conditions;
- The minimum-phase property (see Assumption 1) is an inherent characteristic of physical systems with a prescribed output y and is independent of the model process;
- The controller (27) and (28) only employs the output $y = x_1$ and structural information (e.g., the relative degree r);
- The controller (27) and (28) works for any practical initial conditions;
- The parameters γ and g_0 can be chosen by trial.

Therefore, if physical systems share the same structure as system (3), the developed controller (27) and (28) operates in a universal (model-free) manner similar to a PID controller or an active disturbance rejection controller (see [45, 46]). More exactly, since the PLE-based extended observer (27) also serves as a high-gain differentiator like those in [47, 48], the developed output feedback (27) and (28) can be recognized as a generalized PD controller subject to saturation, where the derivative of the output y is obtained using the differentiator (27). Evidently, our controller is distinctive from the integral controller proposed in the early work [18] for nonlinear systems with ISS internal dynamics.

Remark 6. The implementation of the developed observer (27) is straightforward, as its gain matrix $Q(\gamma)c^{\mathrm{T}}$ is directly calculated by the explicit solution (6) of the PLE (5). In addition, the property (outlined in Item 2) of the PLE (5) plays a crucial role in establishing the strictly positive realness of the system $(\mathcal{A}, B, k_0c, 1)$ in the proof of Lemma 6 (see Appendix B). These properties of the PLE facilitate both the observer design and stability analysis. For further details on PLE properties, readers can refer to [35, 36]. Readers may be interested in the quantitative selection of the parameters γ, g_0, and M. It is difficult (if not impossible) to provide a quantitative selection method when considering the general system (3) with f_0, ψ, f, and g unknown. These parameters are typically determined by trial. Alternatively, for a specific system with certain model information, a systematic approach, such as the traversing method used by [37], can be employed to explore a quantitative selection procedure for these parameters.

When system (3) has a relative degree $r \leqslant 2$, the involving structural restriction is nearly nonexistent, since ψ is a general function. We then present the following appealing result.

Corollary 2. The well-defined minimum-phase normal form (3) with the relative degree $r \leqslant 2$ is semiglobally practically stabilizable by the output feedback (27) and (28) with the sufficiently large γ and g_0.

The results presented in Corollary 2 are particularly interesting as the "observability" is not required in the discussion. This implies that system (3) may allow for unobservable internal dynamics. This observation further motivates us to investigate dynamic output feedback directly, rather than focusing on the observer, which may be unnecessary in certain contexts. Finally, we investigate the conditions required to achieve semiglobal stabilization for system (3).

Theorem 2 (Semiglobal stabilization)**.** In addition to Assumption 1 and condition (34), assume that the equilibrium $\eta = 0$ of the zero dynamics $\dot{\eta} = f_0(\eta, 0)$ of system (3) is LES. Then there exists a $\gamma_{\mathrm{b}} \in [\gamma_{\mathrm{a}}, \infty)$ such that, for any $\gamma \in [\gamma_{\mathrm{b}}, \infty)$, system (3) is semiglobally stabilized by the output feedback (27) and (28), that is, given an arbitrary compact set $\Omega_\chi \subset \mathbb{R}^{l+r}$ centered at the origin, the equilibrium $(\chi, e) = (0, 0)$ of the closed-loop system (32) and (33) is LAS with ROA containing $\Omega_\chi \times \mathbb{R}^n$.

Proof. The conditions of Lemma 3 are verified individually. Firstly, as detailed in the proof of Theorem 1, Item 1 in Lemma 3 has been verified. Secondly, with a linear stabilizing feedback $v(x)$ in (18), the x-subsystem of the ideal target system (20) is in the linear form and thus LES. Notice that $\dot{\eta} = f_0(\eta, 0, 0)$ is also LES. It then follows from Lemma 13.1 in [44] that the nominal system $\dot{\chi} = F_0(\chi)$ is LES. Thirdly, we can employ the fourth condition of Lemma 7 to prove that the condition (16) of Lemma 3 holds. Finally, this proof is finished by employing Lemma 3.

We note that Corollaries 1 and 2 are enhanced to be semiglobal stabilization result if the zero dynamics $\dot{\eta} = f_0(\eta, 0, 0)$ is further assumed to be LES. In addition, the LES condition can be relaxed to other ones (e.g., see Assumption 4 in [10]).

4 An illustrative example

Consider the following system:

$$\begin{cases} \dot{\eta} = -0.5(1 + x_2x_3)\eta^3, \\ \dot{x} = A_\mathrm{c}x + b_\mathrm{c}\left(x_2x_3\eta^3 + \left(1 + \eta^2x_1^2\right)u\right), \\ y = c_\mathrm{c}x, \end{cases} \quad A_\mathrm{c} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; b_\mathrm{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; c_\mathrm{c} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \tag{36}$$

We note that the η-subsystem is not BIBS stable with respect to x_2x_3, even with $\dot{\eta} = -0.5\eta^3$ being GAS. If we choose $x_2(t)x_3(t) = -k^2te^{-kt}$ for some $k > 0$, the solution to (36) presents $\eta^2(t) = \eta_0^2/(1 + \eta_0^2(t + (1 + kt)\mathrm{e}^{-kt} - 1))$, and if the initial condition $|\eta_0| > 1$, there exists a sufficiently large k (not infinite) such that $\eta(t)$ escapes to ∞ in the finite time. Clearly, the BIBS stability with respect to x is not true for the η -subsystem of (36). Therefore, such a system (36) is quite different from the listed examples in [19]. In addition, the nonlinear function $x_2x_3\eta^3$ violates the global Lipschitz condition (see [38,39]), the linear growth condition (see [36]), the polynomial growth condition (see [40,41]), the homogenous growth condition (see [49]), and other nonlinear growth conditions (see [50]).

We now state our approach. According to (27) and (28), the output feedback is designed as

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\varpi}} \end{bmatrix} = A \begin{bmatrix} \hat{x} \\ \hat{\varpi} \end{bmatrix} + Q(\gamma)c^\mathrm{T}\left(y - c\begin{bmatrix} \hat{x} \\ \hat{\varpi} \end{bmatrix}\right) + b_\mathrm{d}g_0u,$$

$$u = \sigma_M\left(\frac{v(\hat{x}) - \hat{\varpi}}{g_0}\right), \; v(\hat{x}) = -\frac{a_0b_0}{k_\mathrm{s}}\hat{x}_1 - \left(a_1b_0 + \frac{a_0}{k_\mathrm{s}}\right)\hat{x}_2 - \left(\frac{a_1}{k_\mathrm{s}} + k_\mathrm{s}b_0\right)\hat{x}_3,$$

where (A, b_d, c) is given by (26) with $n = 4$. The parameters $k_\mathrm{s} = 2.5$, $a_0 = 1$, $a_1 = 2$, $b_0 = 1$ are chosen according to Lemma 4. $Q(\gamma)$ is obtained by Lemma 1. Set the initial conditions of the plant system and observer as $(\eta(0), x_1(0), x_2(0), x_3(0)) = (2, -2, 2, 2)$ and $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0), \hat{\varpi}(0)) = (-2, 0, 0, 0)$, respectively. Choose $g_0 = 100$, $M = 31$ (the parameter in the saturation function σ_M), and $\gamma = 170$. Subsequently, the simulation results are given in Figure 3, which validates the results of Corollary 1. We note that ϖ in figure is the extended state defined in (25).

5 Conclusion

Semiglobally practical output feedback stabilization was achieved for a class of uncertain minimum-phase nonlinear systems involving an unknown control gain, matched uncertainty, and unknown internal dynamics. Notably, neither boundedness conditions are imposed on the nonlinear model functions, nor ISS and BIBS conditions were imposed on the system's internal dynamics. Once the system output and structural information (e.g., the relative degree r) are known, the developed output feedback effectively operates under any practical initial conditions. Therefore, it operates as a somewhat universal (model-free) controller. Our results showed that a minimum-phase normal form with the relative degree $r \leqslant 2$ is semiglobally practically stabilizable by output feedback.

The developed approach incorporates the extended high-gain observer approach and the high-and-low gain feedback approach. A class of high-and-low gain state feedback was employed to establish an ideal target system that is locally asymptotically stable, with an ROA encompassing any compact set. Subsequently, a PLE-based extended observer was designed to recover the asymptotical performance and ROA of this ideal target system. The observer

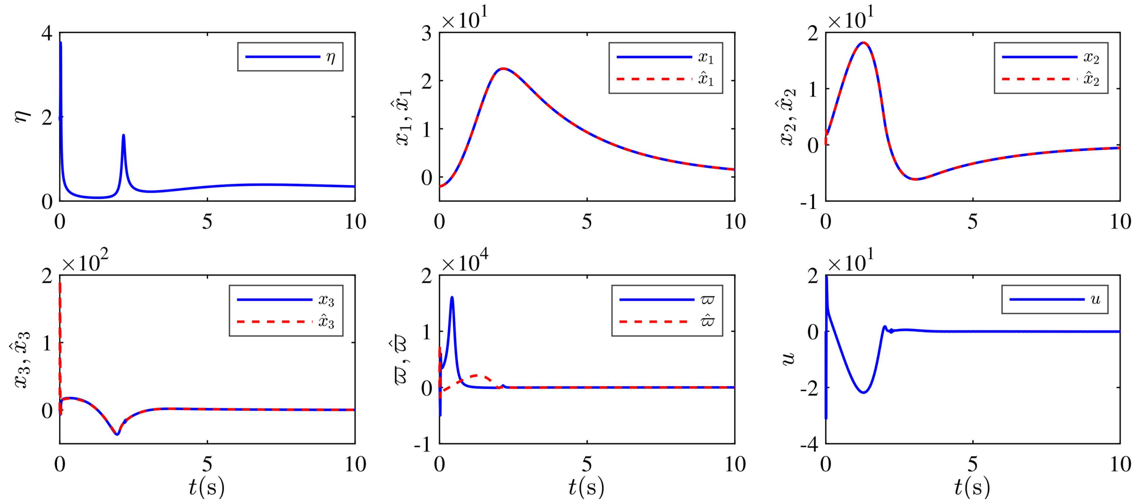

Figure 3 (Color online) Simulation results.

gain matrix was obtained from the solution to PLE, rather than using a pole assignment approach. Finally, an illustrative example featuring non-BIBS-stable internal dynamics was chosen to verify the superiority of the developed approach.

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Appendix A Proof of Lemma 2

This proof is motivated by the well-known work [10]. Except for some modifications in conditions, the main distinction lies in the asymptotical performance of the trajectory $(x(t), e(t))$ to the small set Λ_ϵ.

We omit the time domain $[0,\infty)$ without loss of generality and begin with some properties. It follows from Item 1 in Lemma 2 and the converse Lyapunov theorem (see Theorem 4.17 in [44]) that there exists a $\mathcal{C}^\infty$ positive definite function $V(x)\colon \mathcal{R}\to\mathbb{R}_{\geqslant 0}$ and a $\mathcal{C}^0$ positive definite function $V_0(x)\colon \mathcal{R}\to\mathbb{R}_{\geqslant 0}$ such that

$$\frac{\partial V(x)}{\partial x}F_0(x)\leqslant -V_0(x),\ \forall x\in\mathcal{R}, \tag{A1}$$

$$\lim_{x\to\partial\mathcal{R}} V(x)=\infty,\ \forall x\in\mathcal{R}, \tag{A2}$$

where $\partial\mathcal{R}$ is the boundary of $\mathcal{R}$. Since x_0 and $F_0(x)$ are independent of γ, the ROA $\mathcal{R}$ is also independent of γ. For any subset $\Gamma_{\rm x}\subset\mathcal{R}$, we define it without loss of generality as

$$\Gamma_{\rm x}\triangleq\{x\in\mathbb{R}^m\colon V(x)\leqslant\upsilon\},$$

where υ is some positive constant independent of γ, selected to guarantee $\Gamma_{\rm x}\subset\mathcal{R}$. Clearly, $\Gamma_{\rm x}$ is a compact set owing to (A2).

Step 1. We will show a fact that the set $\Gamma_{\rm x}\times\Gamma_{\rm e}$ can be a positively invariant set for system (8) and (9). Let $\partial\Gamma_{\rm x}\triangleq\{x\in\mathbb{R}^n\colon V(x)=\upsilon\}$ be the boundary of $\Gamma_{\rm x}$. It follows from (A1) and (15) that the time derivative of $V(x)$ along the trajectory of the x-subsystem (8) satisfies

$$\begin{aligned}\dot V(x)|_{(8)}&=\frac{\partial V(x)}{\partial x}F_0(x)+\frac{\partial V(x)}{\partial x}\Delta_1(x,e,t)\\&\leqslant -V_0(x)+\mu_1\delta_3\left\|e\right\|\\&\leqslant -\mu_0+\mu_1\delta_3\left\|e\right\|\\&\leqslant -\mu_0+\mu_1\delta_3\vartheta/\gamma,\ \forall\,(x,e)\in\partial\Gamma_{\rm x}\times\Gamma_{\rm e},\end{aligned}\tag{A3}$$

where $\mu_0\triangleq\min_{x\in\partial\Gamma_{\rm x}}V_0(x)$ and $\mu_1\triangleq\max_{x\in\Gamma_{\rm x}}\|\partial V(x)/\partial x\|$ are some positive constants independent of γ.

Let $\partial\Gamma_{\rm e}\triangleq\{e\in\mathbb{R}^n\colon\|e\|^2=\vartheta^2/\gamma^2\}$ be the boundary of $\Gamma_{\rm e}$. It follows from (11), (12) and (14) that the time derivative of $U(e)$ along the trajectory of the e-subsystem (9) presents

$$\dot U(e)|_{(9)}=\frac{\partial U(e)}{\partial e}\gamma\Phi(e,t)+\frac{\partial U(e)}{\partial e}\Delta_2(x,e,t)$$

$$\begin{aligned} &\leqslant -c_3\gamma \|e\|^2 + c_4\delta_1 \|e\|^2 + c_4\delta_2 \|e\| \\ &= -c_3\vartheta^2/\gamma + c_4\delta_1\vartheta^2/\gamma^2 + c_4\delta_2\vartheta/\gamma \\ &= -c_3\vartheta^2/(2\gamma) + c_4\delta_1\vartheta^2/\gamma^2,\ \forall (x,e) \in \varGamma_{\rm x} \times \partial\varGamma_{\rm e}, \end{aligned} \tag{A4}$$

where ϑ is chosen as $\vartheta = 2c_4\delta_2/c_3$.

It can be concluded from (A3) and (A4) that there exists a $\gamma_1 \in [1,\infty)$ such that, for any $\gamma \in [\gamma_1,\infty)$,

$$\begin{aligned} &\dot V(x)|_{(8)} \leqslant 0,\ \forall (x,e) \in \partial\varGamma_{\rm x} \times \varGamma_{\rm e}, \\ &\dot U(e)|_{(9)} \leqslant 0,\ \forall (x,e) \in \varGamma_{\rm x} \times \partial\varGamma_{\rm e}, \end{aligned}$$

which implies that, under $\gamma \in [\gamma_1,\infty)$, the set $\varGamma_{\rm x} \times \varGamma_{\rm e}$ is a positively invariant set for system (8) and (9), that is,

$$(x_0,e_0) \in \varGamma_{\rm x} \times \varGamma_{\rm e} \Rightarrow (x(t),e(t)) \in \varGamma_{\rm x} \times \varGamma_{\rm e},\ \forall t \geqslant 0.$$

Step 2. We then prove that the trajectory $(x(t),e(t))$ ultimately enters the positively invariant set $\varGamma_{\rm x} \times \varGamma_{\rm e}$. We know from (8) and (13) that $\|\dot x\|$ is upper bounded by a constant independent of γ, as long as $x \in \varGamma_{\rm x}$. This, together with $\varOmega_{\rm x} \subseteq \varGamma_{\rm x}$, directly implies that there exists a finite time interval $[0,t_{\rm b}]$ with $t_{\rm b}$ being independent of γ such that

$$x_0 \in \varOmega_{\rm x} \Rightarrow x(t) \in \varGamma_{\rm x},\ \forall t \in [0,t_{\rm b}], \tag{A5}$$

that is, the trajectory $x(t)$ starting from the interior of $\varGamma_{\rm x}$ remains in the set $\varGamma_{\rm x}$ for all $t \in [0,t_{\rm b}]$, regardless of the trajectory $e(t)$. Following a similar procedure as (A4) and using (10), we can find a $\gamma_2 \in [1,\infty)$ such that, for any $\gamma \in [\gamma_2,\infty)$,

$$\begin{aligned} \dot U(e)|_{(9)} &\leqslant -c_3\gamma \|e\|^2 + c_4\delta_1 \|e\|^2 + c_4\delta_2 \|e\| \\ &\leqslant -\mu\gamma U(e),\ \forall (x,e) \in \varGamma_{\rm x} \times \mathbb{R}^n \backslash \varGamma_{\rm e}, \end{aligned}$$

where $\mu > 0$ is some constant independent of γ. This, together with (10), (A5), and the property of e_0, further implies that, for any $\gamma \in [\gamma_2,\infty)$,

$$\begin{aligned} \|e(t)\|^2 &\leqslant \frac{1}{c_1}U(e(t)) \leqslant \frac{1}{c_1}{\rm e}^{-\mu\gamma t}U(e_0) \leqslant {\rm e}^{-\mu\gamma t}\frac{c_2}{c_1}\|e_0\|^2 \\ &\leqslant \frac{c_2}{c_1}e_{\rm c}^2\gamma^{2p}{\rm e}^{-\mu\gamma t} \triangleq \alpha_0\gamma^{2p}{\rm e}^{-\mu\gamma t},\ \forall (x,e) \in \varGamma_{\rm x} \times \mathbb{R}^n \backslash \varGamma_{\rm e}. \end{aligned} \tag{A6}$$

In the following analysis, we suppose that the trajectory $x(t)$ starts from $\varOmega_{\rm x}$. Then combining (A5) and (A6) obtains that, for any $\gamma \in [\gamma_2,\infty)$ and $e \in \mathbb{R}^n \backslash \varGamma_{\rm e}$,

$$\|e(t)\|^2 \leqslant \alpha_0\gamma^{2p}{\rm e}^{-\mu\gamma t},\ \forall t \in [0,t_{\rm b}]. \tag{A7}$$

Let $t_{\rm c}(\gamma)$ be a time such that $e(t)$ first arrives at the boundary of $\varGamma_{\rm e}$ (namely, $\partial\varGamma_{\rm e}$). In view of (A6), we get $t_{\rm c}(\gamma) \leqslant t_{\rm c}^*(\gamma)$, where $t_{\rm c}^*(\gamma)$ satisfies $\alpha_0\gamma^{2p}{\rm e}^{-\mu\gamma t_{\rm c}^*(\gamma)} = \vartheta^2/\gamma^2$ and thus leads to

$$t_{\rm c}^*(\gamma) \triangleq \frac{1}{\mu\gamma}\ln\left(\frac{\alpha_0\gamma^{2(p+1)}}{\vartheta^2}\right).$$

Noting that $\lim_{\gamma\to\infty} t_{\rm c}^*(\gamma) = 0$ and α_0,μ,ϑ are independent of γ, there exists a $\gamma_3 \in [\gamma_2,\infty)$ such that $t_{\rm c} = t_{\rm c}(\gamma) \leqslant t_{\rm c}^*(\gamma) \leqslant t_{\rm b}$ for any $\gamma \in [\gamma_3,\infty)$. Thus, there is no contradiction to the necessary condition $t \in [0,t_{\rm b}]$ of (A7). Given (A5) and $t_{\rm c} \leqslant t_{\rm b}$, the trajectory $x(t)$ still remains in the set $\varGamma_{\rm x}$ until the time $t_{\rm c}$, while $e(t)$ enters the set $\varGamma_{\rm e}$ in the finite time $t_{\rm c}$, that is, $(x_0,e_0) \in \varOmega_{\rm x} \times \mathbb{R}^n \Rightarrow (x(t_{\rm c}),e(t_{\rm c})) \in \varGamma_{\rm x} \times \varGamma_{\rm e}$. Thereafter, the trajectory $(x(t),e(t))$ remains in the positively invariant set $\varGamma_{\rm x} \times \varGamma_{\rm e}$ (see the proof in Step 1) in this case. Overall, the above ultimately bounded result can be summarized as, for any $\gamma \in [\max\{\gamma_1,\gamma_3\},\infty)$,

$$(x_0,e_0) \in \varOmega_{\rm x} \times \mathbb{R}^n \Rightarrow (x(t),e(t)) \in \varGamma_{\rm x} \times \varGamma_{\rm e},\ \forall t \geqslant t_{\rm c}. \tag{A8}$$

Step 3. We then show the local performance of the trajectory $(x(t),e(t))$ after it remains in $\varGamma_{\rm x} \times \varGamma_{\rm e}$. Define a positive definite function $W(x,e)\colon \varGamma_{\rm x} \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ as

$$W(x,e) \triangleq V(x) + U(e).$$

Without loss of generality, we can define the arbitrarily small compact set $\varLambda_\epsilon$ as

$$\varLambda_\epsilon = \left\{(x,e) \in \varGamma_{\rm x} \times \varGamma_{\rm e} : W(x,e) \leqslant \epsilon\right\}.$$

Besides, define the following sets:

$$\begin{aligned} &\breve\varLambda_\epsilon \triangleq \left\{(x,e) \in \varGamma_{\rm x} \times \varGamma_{\rm e} : W(x,e) \geqslant \epsilon\right\}, \\ &\varLambda_1 \triangleq \left\{(x,e) \in \varGamma_{\rm x} \times \varGamma_{\rm e} : U(e) \geqslant \epsilon\right\}, \\ &\varLambda_2 \triangleq \left\{(x,e) \in \varGamma_{\rm x} \times \varGamma_{\rm e} : U(e) \leqslant \epsilon\right\} \cap \breve\varLambda_\epsilon. \end{aligned}$$

With the ultimately bounded result in (A8), the conditions (13)–(15) are applicable in the subsequent proof. Subsequently, the time derivative of $W = W(x,e)$ along the trajectory of system (8) and (9) presents

$$\begin{aligned}\dot{W}|_{(8)\text{ and }(9)} &= \frac{\partial V(x)}{\partial x}\left(F_0(x) + \Delta_1(x,e,t)\right) + \frac{\partial U(e)}{\partial e}\left(\gamma \Phi(x,e,t) + \Delta_2(x,e,t)\right) \\ &\leqslant -V_0(x) + \mu_1\delta_3 \|e\| - c_3\gamma \|e\|^2 + c_4\delta_1 \|e\|^2 + c_4\delta_2 \|e\| \\ &\leqslant -V_0(x) - c_3\gamma \|e\|^2 + c_4\delta_1 \|e\|^2 + (\mu_1\delta_3 + c_4\delta_2) \|e\|,\ \forall (x,e) \in \Lambda_1 \cup \Lambda_2, \end{aligned} \tag{A9}$$

where (A1), $\mu_1 = \max_{x\in\Gamma_{\rm x}} \|\partial V(x)/\partial x\|$, (15), (11), (12), and (14) are used.

The first case is on the set Λ_1. Since there exists a $\gamma_{4,1}(\epsilon) \in [1,\infty)$ such that, for any $\gamma \in [\gamma_{4,1}(\epsilon),\infty)$,

$$-\left(\frac{c_3}{2}\gamma - c_4\delta_1\right) \|e\|^2 + (\mu_1\delta_3 + c_4\delta_2) \|e\| \leqslant 0,\ \forall (x,e) \in \Lambda_1,$$

$\dot{W}|_{(8)\text{ and }(9)}$ in (A9) can be continued as, for any $\gamma \in [\gamma_{4,1}(\epsilon),\infty)$,

$$\dot{W}|_{(8)\text{ and }(9)} \leqslant -V_0(x) - \frac{c_3}{2}\gamma \|e\|^2,\ \forall (x,e) \in \Lambda_1. \tag{A10}$$

Another case is on the set Λ_2. Define a new function $E(x,e)\colon \Gamma_{\rm x} \times \mathbb{R}^n \to \mathbb{R}$ as

$$E(x,e) \triangleq -\frac{1}{2}V_0(x) + c_4\delta_1 \|e\|^2 + (\mu_1\delta_3 + c_4\delta_2) \|e\|.$$

We use a notation $\{e=0\} = \{(x,e) \in \Gamma_{\rm x} \times \Gamma_{\rm e}\colon e = 0\}$ for simplicity. By noting $W(x,0) = V(x) \geqslant \epsilon$ on the set Λ_2, it yields a constant $\nu(\epsilon) > 0$ independent of γ such that

$$(x,e) \in \{e=0\} \cap \Lambda_2 \Rightarrow E(x,e) = -V_0(x)/2 \leqslant -\nu(\epsilon),$$

which, together with the continuity of $E(x,e)$, implies that there exists an open set $\Lambda_{\rm c}$ containing $\{e=0\}$ such that

$$(x,e) \in \Lambda_{\rm c} \cap \Lambda_2 \Rightarrow E(x,e) \leqslant 0. \tag{A11}$$

Such a set $\Lambda_{\rm c}$ can be defined as $\Lambda_{\rm c} \triangleq \{(x,e) \in \Gamma_{\rm x} \times \Gamma_{\rm e}\colon U(e) < \rho(\epsilon)\}$, where $\rho(\epsilon) \leqslant \epsilon$ is some positive constant independent of γ. Illustrations of these sets are depicted in Figure 2. Focused on the set $\Lambda_{\rm c} \cap \Lambda_2$, substituting (A11) into (A9) yields

$$\dot{W}|_{(8)\text{ and }(9)} \leqslant -\frac{1}{2}V_0(x) - c_3\gamma \|e\|^2,\ \forall (x,e) \in \Lambda_{\rm c} \cap \Lambda_2. \tag{A12}$$

We then focus on the set $\Lambda_2 \backslash \Lambda_{\rm c}$. Following a similar procedure as for (A10), there exists a $\gamma_{4,2}(\epsilon) \in [1,\infty)$ such that $\dot{W}|_{(8)\text{ and }(9)}$ in (A9) can be continued as, for any $\gamma \in [\gamma_{4,2}(\epsilon),\infty)$,

$$\dot{W}|_{(8)\text{ and }(9)} \leqslant -V_0(x) - \frac{c_3}{2}\gamma \|e\|^2,\ \forall (x,e) \in \Lambda_2 \backslash \Lambda_{\rm c}. \tag{A13}$$

Notice that $\breve{\Lambda}_\epsilon \subset \Lambda_1 \cup \Lambda_2 = \Lambda_1 \cup (\Lambda_2 \backslash \Lambda_{\rm c}) \cup (\Lambda_{\rm c} \cap \Lambda_2)$ and choose $\gamma_4(\epsilon) = \max\{\gamma_{4,1}(\epsilon), \gamma_{4,2}(\epsilon)\}$. Then combining (A10), (A12), and (A13) yields, for any $\gamma \in [\gamma_4(\epsilon),\infty)$,

$$\dot{W}|_{(8)\text{ and }(9)} \leqslant -\frac{1}{2}\left(V_0(x) + c_3\gamma \|e\|^2\right) \leqslant -\mu_{\rm c} W,\ \forall (x,e) \in \breve{\Lambda}_\epsilon,$$

where $\mu_{\rm c}$ is some positive constant independent of γ. This means that $W(x,e)$ is a Lyapunov function on the set $\breve{\Lambda}_\epsilon$. It then follows that, for any $\gamma \in [\gamma_4(\epsilon),\infty)$,

$$\begin{aligned} W(x(t),e(t)) &\leqslant {\rm e}^{-\mu_{\rm c} t} W(x_0,e_0) = {\rm e}^{-\mu_{\rm c} t}\left(V(x_0) + U(e_0)\right) \\ &\leqslant \left(\upsilon + c_2\vartheta^2\right) {\rm e}^{-\mu_{\rm c} t} \triangleq \alpha_1 {\rm e}^{-\mu_{\rm c} t},\ \forall (x,e) \in \breve{\Lambda}_\epsilon, \end{aligned} \tag{A14}$$

where we have used $\breve{\Lambda}_\epsilon \subset \Gamma_{\rm x} \times \Gamma_{\rm e}$ and (10). Let $t_{\rm d}$ be the time such that the trajectory $(x(t),e(t))$ starting from $\Gamma_{\rm x} \times \Gamma_{\rm e}$ first arrives at the boundary of Λ_ϵ, namely, $\partial\Lambda_\epsilon \triangleq \{(x,e) \in \Gamma_{\rm x} \times \Gamma_{\rm e}\colon W(x,e) = \epsilon\}$. In view of (A14), we obtain $t_{\rm d} \leqslant t_{\rm d}^*$, where $t_{\rm d}^*$ satisfies $\alpha_1 {\rm e}^{-\mu_{\rm c} t_{\rm d}^*} = \epsilon$. This further implies

$$t_{\rm d} \leqslant t_{\rm d}^* = \frac{1}{\mu_{\rm c}} \ln\left(\frac{\alpha_1}{\epsilon}\right).$$

Clearly, such a $t_{\rm d}$ is independent of γ.

Overall, we define $\gamma_{\rm a} \triangleq \max\{\gamma_1, \gamma_3, \gamma_4(\epsilon)\}$ and choose $\gamma \in [\gamma_{\rm a},\infty)$. Then the trajectory $(x(t),e(t))$ with its initial condition $(x_0,e_0) \in \Omega_{\rm x} \times \mathbb{R}^n$ enters the positively invariant set $\Gamma_{\rm x} \times \Gamma_{\rm e}$ in the finite time $t_{\rm c}(\gamma)$ (see (A8)). Later, the trajectory $(x(t),e(t))$ starting from $\Gamma_{\rm x} \times \Gamma_{\rm e}$ enters the set Λ_ϵ in the finite time $t_{\rm d}$ and then remains in it thereafter. This proof is finished with the entire convergence time being less than $t_{\rm s}(\gamma) \triangleq t_{\rm c}(\gamma) + t_{\rm d}$.

Appendix B Proof of Lemma 6

Let the condition $(\eta, x) \in \mathit{\Gamma}_\chi$ be satisfied in this proof. Rewrite system (35) as

$$\dot{e} = \mathcal{A}e + Bu,\ u = -k(t)y \triangleq -\Xi(y),\ y = ce, \tag{B1}$$

where c is given in (4), u and y are viewed as the (time-varying) input and the output of the linear system $(\mathcal{A}, B)$, respectively. Along with condition (34) and $|\rho'| \leqslant 1$ (derived from the definition of $\rho_{(M,\gamma)}$), we deduce $|k(t)| = |\rho' \tilde{g}(\eta, x)/g_0| < 1$ for its all arguments. We focus on the case of $0 < k(t) < 1$ without loss of generality. Let k_0 be a constant satisfying $k(t) \leqslant k_0 < 1$. Such a k_0 exists due to $0 < k(t) < 1$. In this case, system (B1) is the well-known Lur'e system satisfying

$$\Xi(y)(\Xi(y) - k_0 y) \leqslant 0. \tag{B2}$$

In view of the definitions of $\mathcal{A} = A - Q_n c^{\mathrm{T}} c$, $B = q_{n1} b$, and c given in (4), the transfer function of system (B1) is $G_0(s) = q_{n1}/(s^n + q_{11}s^{n-1} + \cdots + q_{n1})$, where q_{ij} denotes the i-th row and j-th column element of Q_n. It then follows from Item 2 of Lemma 1 that the poles of $G_0(s)$ lie on $(-1, 0)$ of the s-plane, which further implies $q_{n1} = 1$ and $G_0(s) = 1/(s+1)^n$. We then obtain $\|G_0(s-\alpha)\|_\infty = \sup_{\omega\in\mathbb{R}} |G_0(\mathrm{j}\omega - \alpha)| = 1/(1-\alpha)^n$ for some constant α satisfying $0 < \alpha < 1$.

Let $\alpha = 1 - k_0^{1/n}$. Then we obtain from $|\mathrm{Re}\,(G_0(\mathrm{j}\omega - \alpha))| \leqslant \|G_0(s-\alpha)\|_\infty = 1/k_0$, $\forall \omega \in \mathbb{R}$ that the transfer function $1 + k_0 G_0(s-\alpha)$ is positive real with such an α, which further implies a strictly positive real transfer function $1 + k_0 G_0(s)$. By using the KYP lemma (Lemma 6.3 in [44]) to $1 + k_0 G_0(s)$ (whose minimal realization is $(\mathcal{A}, B, k_0 c, 1)$, it yields a positive definite matrix $\mathcal{Q} \in \mathbb{R}^{n\times n}$, a matrix $L \in \mathbb{R}^{n\times n}$, a vector $N \in \mathbb{R}^n$, and a positive constant $\mu = 2\alpha$ such that

$$\mathcal{A}^{\mathrm{T}}\mathcal{Q} + \mathcal{Q}\mathcal{A} = -L^{\mathrm{T}}L - \mu\mathcal{Q},\ \mathcal{Q}B = k_0 c^{\mathrm{T}} - L^{\mathrm{T}}N,\ N^{\mathrm{T}}N = 2.$$

With these equations, differentiating $U(e)$ along the trajectory of system (B1) yields

$$\begin{aligned}
\dot{U}(e)|_{(\mathrm{B1})} &= e^{\mathrm{T}}(\mathcal{A}^{\mathrm{T}}\mathcal{Q} + \mathcal{Q}\mathcal{A})e + 2e^{\mathrm{T}}\mathcal{Q}Bu \\
&= -\mu e^{\mathrm{T}}\mathcal{Q}e - e^{\mathrm{T}}L^{\mathrm{T}}Le + 2e^{\mathrm{T}}\left(k_0 c^{\mathrm{T}} - L^{\mathrm{T}}N\right)u \\
&= -\mu e^{\mathrm{T}}\mathcal{Q}e - (Le)^{\mathrm{T}}Le + 2(u + k_0 ce)u - (Nu)^{\mathrm{T}}Nu - 2(Le)^{\mathrm{T}}Nu \\
&= -\mu e^{\mathrm{T}}\mathcal{Q}e - (Le + Nu)^{\mathrm{T}}(Le + Nu) + 2(\Xi(y) - k_0 y)\Xi(y),\ \forall e \in \mathbb{R}^n,
\end{aligned}$$

which, together with (B2), leads to $\dot{U}(e)|_{(\mathrm{B1})} \leqslant -\mu e^{\mathrm{T}}\mathcal{Q}e,\ \forall e \in \mathbb{R}^n$. This proof is completed.

Appendix C Proof of Lemma 7

Let $\gamma \in [1, \infty)$ be satisfied by default in this proof.

The first growth condition. In view of the continuity and the saturation operation, it is clear that $\mathit{\Delta}_1(\chi, \hat{x}, \hat{\varpi})$ is bounded independent of γ, as long as $(\chi, e) \in \mathit{\Gamma}_\chi \times \mathbb{R}^n$.

The second growth condition. We divide $\mathit{\Delta}_2(\chi, \hat{x}, \hat{\varpi})$ into two terms

$$\begin{aligned}
\mathit{\Delta}_{2\mathrm{d}}(\chi, \hat{x}, \hat{\varpi}) &\triangleq \gamma b_{\mathrm{d}}\left(g(\chi)\varphi^s(\hat{x}, \hat{\varpi}) - \tilde{g}(\chi)\varphi^\rho(x, \hat{\varpi}) - g_0\varphi^s(\hat{x}, \hat{\varpi})\right), \\
\mathit{\Delta}_{2\mathrm{c}}(\chi, \hat{x}, \hat{\varpi}) &\triangleq b\left(\dot{f}(\chi) + \dot{\tilde{g}}(\chi)\varphi^\rho(x, \hat{\varpi}) + \tilde{g}(\chi)\rho'\frac{\partial\varphi(x, \hat{\varpi})}{\partial x}\dot{x}\right).
\end{aligned}$$

It follows from (7) and (28) that

$$\begin{aligned}
\|\mathit{\Delta}_{2\mathrm{d}}(\chi, \hat{x}, \hat{\varpi})\| &= \gamma\left|g(\chi)\varphi^s(\hat{x}, \hat{\varpi}) - \tilde{g}(\chi)\varphi^\rho(x, \hat{\varpi}) - g_0\varphi^s(\hat{x}, \hat{\varpi})\right| \\
&= \gamma\left|g(\chi)\left(\varphi^s(\hat{x}, \hat{\varpi}) - \varphi^\rho(x, \hat{\varpi})\right) + g_0\varphi^\rho(x, \hat{\varpi}) - g_0\varphi^s(\hat{x}, \hat{\varpi})\right| \\
&= \gamma\left|(g_0 - g(\chi))\left(\varphi^\rho(x, \hat{\varpi}) - \varphi^s(x, \hat{\varpi})\right) + (g_0 - g(\chi))\left(\varphi^s(x, \hat{\varpi}) - \varphi^s(\hat{x}, \hat{\varpi})\right)\right| \\
&\leqslant \gamma(\kappa_1 + \kappa_2)\left|\varphi^\rho(x, \hat{\varpi}) - \varphi^s(x, \hat{\varpi})\right| + \gamma(\kappa_1 + \kappa_2)\left|\varphi^s(x, \hat{\varpi}) - \varphi^s(\hat{x}, \hat{\varpi})\right| \\
&\leqslant 2(\kappa_1 + \kappa_2) + \gamma(\kappa_1 + \kappa_2)\left|\varphi^s(x, \hat{\varpi}) - \varphi^s(\hat{x}, \hat{\varpi})\right|,\ \forall(\chi, e) \in \mathit{\Gamma}_\chi \times \mathbb{R}^n,
\end{aligned}$$

where $\kappa_1 = \max_{\chi\in\mathit{\Gamma}_\chi} |g(\chi)|$ and $\kappa_2 = g_0$. Subsequently, it follows from the special forms of $\varphi(x, \varpi) = (v(x) - \varpi)/g_0$ (see (24)) and $v(x)$ (see (19)) that $\varphi(x, \varpi)$ is globally Lipschitz with respect to (x, ϖ), which further implies

$$|\varphi(x, \hat{\varpi}) - \varphi(\hat{x}, \hat{\varpi})| \leqslant L_\varphi \|x - \hat{x}\|,\ \forall x \in \mathbb{R}^r, \hat{x} \in \mathbb{R}^r, \hat{\varpi} \in \mathbb{R}, \tag{C1}$$

where L_φ is the Lipschitz constant independent of γ. Moreover, in view of (31), we have

$$\|x - \hat{x}\| = \left\|\left[\begin{array}{cc} I_r & 0_{r\times 1} \end{array}\right] L_{\bar{r}}^{-1}(\gamma)e\right\| \leqslant \|e\|/\gamma. \tag{C2}$$

We then obtain from the globally Lipschitz property of σ_M, (C1) and (C2) that

$$|\varphi^s(x, \hat{\varpi}) - \varphi^s(\hat{x}, \hat{\varpi})| \leqslant |\varphi(x, \hat{\varpi}) - \varphi(\hat{x}, \hat{\varpi})| \leqslant L_\varphi \|x - \hat{x}\|$$

$$\leqslant L_{\varphi}\left\|e\right\|/\gamma,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\mathbb{R}^{n}.$$

Thus $\|\varDelta_{2\mathrm{d}}(\chi,\hat{x},\hat{\varpi})\|$ can be continued as

$$\|\varDelta_{2\mathrm{d}}(\chi,\hat{x},\hat{\varpi})\|\leqslant 2\left(\kappa_1+\kappa_2\right)+L_{\varphi}\left(\kappa_1+\kappa_2\right)\left\|e\right\|,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\mathbb{R}^{n}. \tag{C3}$$

On the other hand, it follows from (24) and (32) that

$$\varDelta_{2\mathrm{c}}(\chi,\hat{x},\hat{\varpi})=b\left(\dot{f}(\chi)+\dot{\tilde{g}}(\chi)\varphi^{\rho}\left(x,\hat{\varpi}\right)+\frac{\tilde{g}(\chi)}{g_0}\rho'\frac{\mathrm{d}v(x)}{\mathrm{d}x}\dot{x}\right), \tag{C4}$$

$$\dot{x}=\left[\begin{array}{cc}0_{r\times(n-r)} & I_r\end{array}\right]\dot{\chi}=\left[\begin{array}{cc}0_{r\times(n-r)} & I_r\end{array}\right]\left(F_0(\chi)+\varDelta_1(\chi,\hat{x},\hat{\varpi})\right). \tag{C5}$$

Then, along with $|\rho'|\leqslant 1$, the boundedness of $\varphi^{\rho}(\cdot)$ and $\varDelta_1(\chi,\hat{x},\hat{\varpi})$, and the continuity with respect to χ, there exists a positive constant κ_3 independent of γ such that

$$\|\varDelta_{2\mathrm{c}}(\chi,\hat{x},\hat{\varpi})\|\leqslant\kappa_3,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\mathbb{R}^{n}. \tag{C6}$$

Combining (C3) and (C6), we can deduce two positive constants $\delta_1=L_{\varphi}\left(\kappa_1+\kappa_2\right)$ and $\delta_2=2\left(\kappa_1+\kappa_2\right)+\kappa_3$ independent of γ such that

$$\|\varDelta_2(\chi,\hat{x},\hat{\varpi})\|\leqslant\delta_1\left\|e\right\|+\delta_2,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\mathbb{R}^{n}.$$

The third growth condition. When condition (34) is satisfied, we know from Lemma 5 and (31) that

$$e=0\Rightarrow\hat{x}=x;\ \hat{\varpi}=\omega=\varpi,\ \forall\chi\in\Gamma_{\chi},$$

which, in view of (29), implies

$$e=0\Rightarrow\max_{\chi\in\Gamma_{\chi}}\left|\varphi\left(x,\hat{\varpi}\right)\right|=\max_{\chi\in\Gamma_{\chi}}\left|\varphi\left(\hat{x},\hat{\varpi}\right)\right|=\max_{\chi\in\Gamma_{\chi}}\left|\varphi\left(x,\varpi\right)\right|=M_0<M. \tag{C7}$$

Subsequently, we know from definition of Γ_{e} that $\gamma\rightarrow\infty\Rightarrow e\rightarrow 0$. It then follows from the continuity and (C7) that there exists a constant $\gamma_1\in[1,\infty)$ such that, for any $\gamma\in[\gamma_1,\infty)$,

$$\max_{(\chi,e)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}}\left|\varphi\left(x,\hat{\varpi}\right)\right|<M,\quad\max_{(\chi,e)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}}\left|\varphi\left(\hat{x},\hat{\varpi}\right)\right|<M,$$

which further implies that, for any $\gamma\in[\gamma_1,\infty)$,

$$\varphi^{\rho}(x,\hat{\varpi})=\varphi\left(x,\hat{\varpi}\right),\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}, \tag{C8}$$

$$\varphi^{s}(\hat{x},\hat{\varpi})=\varphi(\hat{x},\hat{\varpi}),\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}, \tag{C9}$$

$$\varphi^{s}\left(x,\hat{\varpi}\right)=\varphi\left(x,\hat{\varpi}\right),\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}. \tag{C10}$$

This means that these terms are not saturated when e is sufficiently small and $\chi\in\Gamma_{\chi}$. Then the auxiliary state ω in (30) simplifies as, for any $\gamma\in[\gamma_1,\infty)$,

$$\omega=f(\eta,x)+\tilde{g}(\chi)\varphi\left(x,\hat{\varpi}\right),\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}. \tag{C11}$$

In this case, it follows from (24), (18), (C9), (C11), (C1), and (31) that, for any $\gamma\in[\gamma_1,\infty)$,

$$\begin{aligned}\|\varDelta_1(\chi,\hat{x},\hat{\varpi})\|&=|g(\chi)\varphi^{s}(\hat{x},\hat{\varpi})-g(\chi)\phi(\chi)|=|g(\chi)\varphi(\hat{x},\hat{\varpi})+f(\chi)-v(x)|\\&=|f(\chi)+\tilde{g}(\chi)\varphi\left(x,\hat{\varpi}\right)+g_0\varphi\left(x,\hat{\varpi}\right)-v(x)+g(\chi)\left(\varphi(\hat{x},\hat{\varpi})-\varphi\left(x,\hat{\varpi}\right)\right)|\\&\leqslant|\omega-\hat{\varpi}|+\kappa_1\left|\varphi(\hat{x},\hat{\varpi})-\varphi\left(x,\hat{\varpi}\right)\right|\leqslant|\omega-\hat{\varpi}|+\kappa_1L_{\varphi}\left\|x-\hat{x}\right\|\\&\leqslant\delta_3\left\|e\right\|,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}},\end{aligned} \tag{C12}$$

where $\delta_3\triangleq\max\{1,\kappa_1L_{\varphi}\}$.

The fourth growth condition. It follows from (C8)–(C10), (C1), and (C2) that, for any $\gamma\in[\gamma_1,\infty)$,

$$\begin{aligned}\|\varDelta_{2\mathrm{d}}(\chi,\hat{x},\hat{\varpi})\|&=\gamma\left|g(\chi)\varphi(\hat{x},\hat{\varpi})-\tilde{g}(\chi)\varphi\left(x,\hat{\varpi}\right)-g_0\varphi\left(\hat{x},\hat{\varpi}\right)\right|\\&=\gamma\left|\tilde{g}(\chi)\left(\varphi(\hat{x},\hat{\varpi})-\varphi\left(x,\hat{\varpi}\right)\right)\right|\\&\leqslant\left(\kappa_1+\kappa_2\right)L_{\varphi}\left\|e\right\|,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}.\end{aligned} \tag{C13}$$

On the other hand, it follows from (C4) and (C5), $|\rho'|\leqslant 1$, the boundedness of $\varphi^{\rho}(\cdot)$, $\dot{f}(\chi)=\left(\partial f(\chi)/\partial\chi\right)\dot{\chi}$, $\dot{\tilde{g}}(\chi)=\left(\partial\tilde{g}(\chi)/\partial\chi\right)\dot{\chi}$, and the continuity with respect to χ that there exists a positive constant κ_4 independent of γ such that

$$\|\varDelta_{2\mathrm{c}}(\chi,\hat{x},\hat{\varpi})\|\leqslant\kappa_4\left\|\dot{\chi}\right\|,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\mathbb{R}^{n},$$

which, together with (32), $F_0(0)=0$, and (C12), can be continued as, for any $\gamma\in[\gamma_1,\infty)$,

$$\begin{aligned}\|\varDelta_{2\mathrm{c}}(\chi,\hat{x},\hat{\varpi})\|&\leqslant\kappa_4\left\|F_0(\chi)-F_0(0)+\varDelta_1(\chi,\hat{x},\hat{\varpi})\right\|\\&\leqslant\kappa_4L_{F_0}\left\|\chi\right\|+\kappa_4\delta_3\left\|e\right\|,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}},\end{aligned} \tag{C14}$$

where L_{F_0} is the Lipschitz constant of $F_0(\chi)$ on the compact set Γ_{χ}. Combining (C13) and (C14), we can deduce two positive constants $\delta_4=\left(\kappa_1+\kappa_2\right)L_{\varphi}+\kappa_4\delta_3$ and $\delta_5=\kappa_4L_{F_0}$ independent of γ such that, for any $\gamma\in[\gamma_1,\infty)$,

$$\|\varDelta_2(\chi,\hat{x},\hat{\varpi})\|\leqslant\delta_4\left\|e\right\|+\delta_5\left\|\chi\right\|,\ \forall\left(\chi,e\right)\in\Gamma_{\chi}\times\Gamma_{\mathrm{e}}.$$

This proof is finished.