SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

January 2026, Vol. 69, Iss. 1, 112206:1–112206:15
https://doi.org/10.1007/s11432-024-4655-1

A hybrid output feedback control scheme of Markovian jump systems via partly transition rates/probabilities design

Yufeng TIAN[1], Xiaojie SU[1*], Sam KWONG[2] & Chao SHEN[3]

[1] *College of Automation, Chongqing University, Chongqing 400044, China*
[2] *Department of Computing Decision Sciences, Lingnan University, Hong Kong 999077, China*
[3] *School of Electronic and Information Engineering, Xi'an Jiaotong University, Xi'an 710049, China*

Received 9 December 2024/Revised 26 May 2025/Accepted 14 July 2025/Published online 4 January 2026

Abstract The hybrid control scheme, involving the design of all transition rates/probabilities, has been extensively studied in the literature. However, there may be cases where certain transition rates/probabilities are *fixed a priori*, rendering existing methods inapplicable. In this paper, hybrid control schemes which consider the co-design of partly transition rates/probabilities and output feedback controller are respectively investigated for continuous-time and discrete-time Markovian jump systems by proposing a synchronous mode-dependent parametric method. Firstly, novel necessary and sufficient conditions are established to reconstruct the unfixed switching rates/probabilities that ensure the mean square stability of both continuous-time and discrete-time Markovian jump systems. Next, stabilization conditions are established via hybrid control design. Importantly, the decision matrices related to the fixed and unfixed transition rates/probabilities are strictly separated, resulting in reduced complexity demands and avoiding the requirement to solve complex parameters. Finally, two numerical examples are provided to demonstrate the effectiveness of the proposed methods.

Keywords Markovian jump system, hybrid control scheme, partly fixed transition rates, partly fixed transition probabilities, static output feedback control

Citation Tian Y F, Su X J, Kwong S, et al. A hybrid output feedback control scheme of Markovian jump systems via partly transition rates/probabilities design. Sci China Inf Sci, 2026, 69(1): 112206, https://doi.org/10.1007/s11432-024-4655-1

1 Introduction

Markovian jump systems, representing a category of stochastically switching models, are extensively employed in accurately representing various real-world systems due to their robust modeling capabilities (see [1–3]). These systems adeptly model plants undergoing sudden structural changes, characterized by phenomena like unforeseen component failures, rapid environmental shifts, alterations in subsystem connections, and significant operational deviations in nonlinear plants (see [4–6]). A large number of studies have investigated the stability, stabilization, and optimal control of Markovian jump systems, which can be found in the relevant literature (see [7–9]).

In recent years, most existing stabilization approaches for Markovian jump systems have been widely studied based on the assumption that the Markovian transition matrices are *fixed a priori* (see [10–13]). However, in practical cases, engineers often have the flexibility to choose or design Markovian transition matrices or general switching rules, which may deviate from the aforementioned assumption (see [14]). In such scenarios, the design of an appropriate switching rule has the potential to stabilize Markovian jump systems, even if none of the individual subsystems are inherently stable. Previous research has shown that by designing transition rates/probabilities, it is possible to achieve overall stability and improve the dynamic performance of Markovian jump systems whose subsystems are unstable (such as in [15–19]). For instance, within the continuous-time domain, Markovian jump systems have been explored through hybrid design methods involving transition rates and output feedback control [15]. This research establishes criteria for formulating transition rate matrices. Further expanding on these findings, the authors in [16] extend the concept of transition rate synthesis to time-delayed Markovian jump systems, delving into stochastic stabilization challenges using transition rate matrix design and state feedback control gain strategies. Conversely, in the discrete-time domain, the focus shifts towards exponential stabilization via an asynchronous mode-dependent parametric approach [18]. This method hinges on synthesizing both transition probabilities and output feedback

* Corresponding author (email: suxiaojie@cqu.edu.cn)

control gains, incorporating various asynchronous parameters for enhanced system stability. These parameters are settled using a gridding technique. Based on the core of asynchronous mode-dependent parametric method, a hybrid sliding mode control scheme for Markovian jump systems is designed in [19], where iterative algorithms are employed to settle the asynchronous parameters. Furthermore, the problem of hybrid design optimization, incorporating adaptive event-triggered schemes and an asynchronous fault detection filter for stochastic Markovian jump systems, has been considered using genetic algorithm [20]. In [21], the co-design problem of scheduling protocol and sliding mode controller for interval type-2 T-S fuzzy systems has been investigated, in which a stochastic scheduling protocol established through a co-designed Markov chain is proposed for system state transmission. These techniques have provided inspiration for exploring the co-design of transition rates/probabilities and controllers. However, a challenge arises when fixed transition rates/probabilities are present, as the current techniques become inapplicable. Therefore, this paper addresses the following aspects to tackle this open issue. (i) Addressing the scenario where fixed transition rates/probabilities exist, and at least one individual subsystem is unstable, but none of them are output controllable by a single static output feedback. The goal is to co-design unfixed transition rates/probabilities and a static output feedback controller, enabling the transformation of "slow" and "unstable" subsystems into a "fast and stable" Markovian jump system. (ii) The asynchronous mode-dependent parametric method in the literature suggests complex optimization algorithms. The paper aims to solve complex parameter optimization problems in both continuous-time and discrete-time Markovian jump systems under partly fixed transition rates/probabilities.

With the above analysis, the paper investigates the hybrid control design problems for both continuous-time and discrete-time Markovian jump systems. Considering certain transition rates/probabilities *fixed a priori*, new sufficient and necessary conditions for the switching rate/probability matrix are established to ensure the mean square stability of both Markovian jump systems by the synchronous mode-dependent parametric method. Additionally, hybrid control conditions via co-designing transition rates/probabilities and a static output feedback controller are respectively proposed. The contributions are summarized as follows.

- This paper considers the co-design of partly transition rates/probabilities for both continuous-time and discrete-time Markovian jump systems, filling the gap where there are some transition rates/probabilities *fixed a priori*.
- The synchronous mode-dependent parametric method is proposed to strictly separate the fixed and unfixed transition rates/probabilities, which simultaneously avoids the generation of large decision variables and the requirement to solve asynchronous parameters.

2 System description and preliminaries

2.1 The overall framework

The overall framework of the hybrid control scheme is portrayed in Figure 1, which can be concluded as the following three parts.

- The black transition rates/probabilities elements in the "MJS (i.e., the abbreviation of Markov jump system) with fixed TR/Ps" mean that the elements are fixed and cannot be computed. These are the differences and difficulties (how to separate the fixed and unfixed parts) compared with the previous studies (such as in [13, 18]) (see the expression in (4)).
- The red transition rates/probabilities elements in the "MJS with fixed TR/Ps" mean that the elements are free and can be optimized (i.e., the condition in Theorems 1 and 2) (see the blue transition rates/probabilities elements in "MJS with reconstructed TR/Ps").
- If the inner optimized structure can stabilize the system, then stop the external feedback controller. If not, an external static output controller and the inner optimized structure are co-designed (i.e., the conditions in Theorems 3 and 4).

2.2 System description

Consider the continuous-time and discrete-time Markovian jump system represented by the following equations:

$$\dot{x}(t) = A(r_t)x(t) + B(r_t)u(t), \tag{1}$$
$$x(k+1) = A(r_k)x(k) + B(r_k)u(k), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ (or $x(k) \in \mathbb{R}^n$) represents the state vector of the system, and $u(t)$ (or $u(k)$) represents the control input. The switching between different system modes is governed by a jumping process, defined as r_t (or r_k), with values in the finite set $\ell = \{1, 2, ..., N\}$.

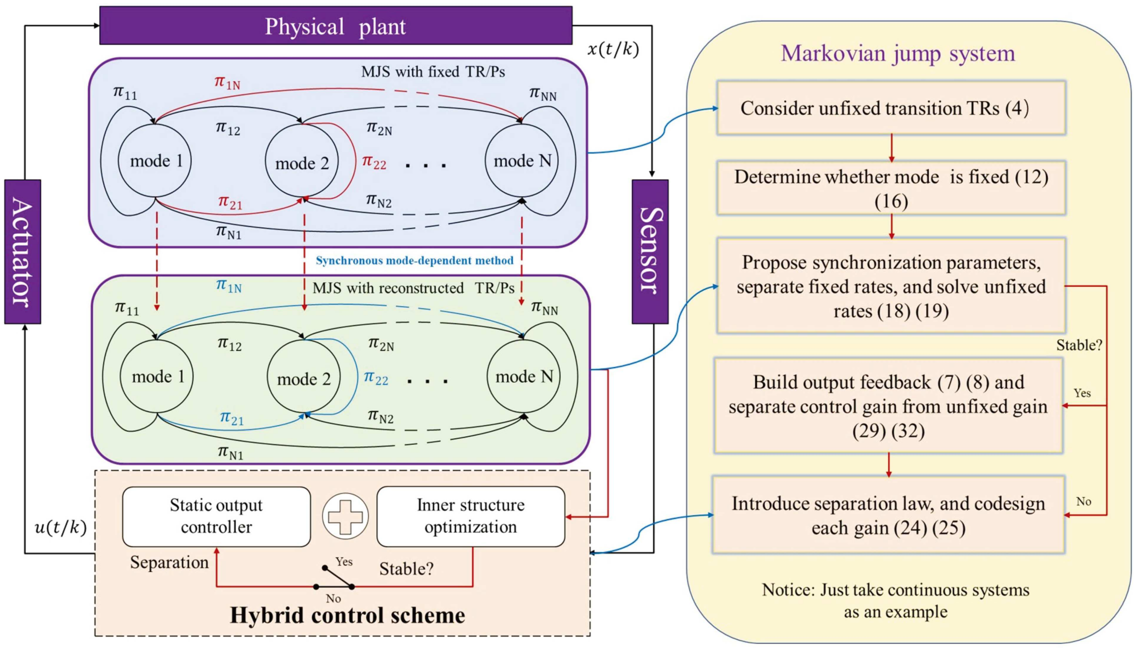

Figure 1 (Color online) The overall framework of hybrid control scheme.

In continuous-time scenarios, the jumping process, defined as r_t, manifests as a homogeneous Markov process characterized by continuous time and discrete states. The associated mode transition rates are specified as follows:

$$Pr(r_{t+h}=j|r_t=i) \triangleq \begin{cases} \pi_{ij}h+o(h), & \text{if } j\neq i, \\ 1+\pi_{ii}h+o(h), & \text{if } j=i. \end{cases}$$

Here, $h>0$ represents the time interval, and we have $\lim_{h\to 0}(\frac{o(h)}{h})=0$. The parameter $\pi_{ij}\geqslant 0$ $(i,j\in\ell, j\neq i)$ represents the switching rate from mode i at time t to mode j at time $t+h$. Additionally, for all $i\in\ell$, the condition $\pi_{ii}=-\sum_{j=1,j\neq i}^{N}\pi_{ij}$ holds.

In discrete-time scenarios, the process represented by $r_k, k\geqslant 0$ operates as a homogeneous Markov chain in discrete time, drawing values from a defined finite set ℓ. The probabilities governing mode transitions within this chain are delineated as follows:

$$Pr(r_{k+1}=j|r_k=i)\triangleq\lambda_{ij}.$$

Here, $\lambda_{ij}\geqslant 0$ for all $i,j\in\ell$, and the probabilities satisfy the condition $\sum_{j=1}^{N}\lambda_{ij}=1$.

Previous studies have extensively investigated hybrid control schemes, with a primary focus on the full reconstruction of transition probability matrices (e.g., [15, 18, 19, 21]). However, in practical scenarios, certain transition probabilities or rates are often fixed and cannot be freely adjusted, which is an open issue. For instance, when incorporating a Gossip Markov Chain into a network-on-chip system to allocate data packet flow rates across communication links (see [14]), the transition probability matrix adheres to a predefined structural form, as illustrated in Figure 2. Motivated by such practical considerations, this paper proposes a hybrid control framework that explicitly accommodates both fixed and unfixed (i.e., designable) transition probabilities/rates. The distinctive features of this framework are detailed in Remark 1, and the main difficulties and contributions are elaborated in Remarks 2–5.

$$\Pi=\begin{bmatrix} p & q & 1-p_q & 0 \\ 1-p_q & p & 0 & q \\ q & 0 & p & 1-p_q \\ 0 & 1-p_q & q & p \end{bmatrix}, \tag{3}$$

where p and q are adjustable parameters within the range of $[0,1]$, and $p_q=p+q$. However, certain elements, such as 0, are *fixed a priori*. Consequently, existing approaches that aim to reconstruct the entire transition probability

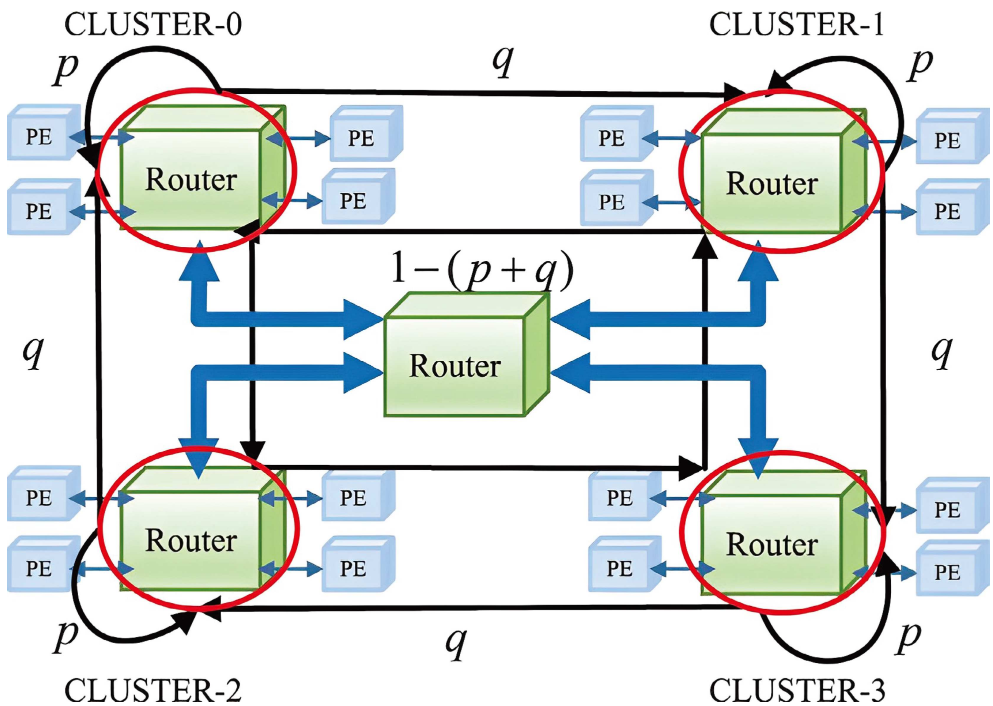

Figure 2 (Color online) The gossip Markov chain [14] ("PE" is the abbreviation of "processing element").

matrix are not applicable in this scenario. To address this gap, this paper presents a hybrid control scheme that incorporates the design of partly fixed transition rates/probabilities for both continuous-time and discrete-time Markovian jump systems.

To simplify the expression of transition rates/probabilities, we adopt two forms based on the methodology presented in [12]. These forms, defined as Π and Λ, are utilized to represent the transition rate and probability matrices that require reconstruction. For example, considering system (1) and system (2), matrices Π and Λ are represented as follows:

$$\Pi \triangleq \begin{bmatrix} \pi_{11} & \circ & \cdots & \pi_{1N} \\ \pi_{21} & \circ & \cdots & \pi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \circ & \pi_{N2} & \cdots & \circ \end{bmatrix}, \Lambda \triangleq \begin{bmatrix} \circ & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \circ & \cdots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N1} & \circ & \cdots & \circ \end{bmatrix}. \tag{4}$$

It is important to note that, unlike the representation in [12], in this paper, π_{ij} and λ_{ij} represent fixed elements, while the symbol "$\circ$" represents the unfixed elements that can be reconstructed.

Remark 1. It should be noted that this paper introduces a novel perspective by considering two types of elements in transition rates/probabilities: (i) fixed transition elements and (ii) unfixed elements. The paper's unique contribution lies in the reconstruction of these unfixed elements, which diverges from the main idea in the known and unknown elements of the most existing studies [12, 13]. Concretely, the research on Markov jump systems with unknown rates/probabilities primarily attributes the uncertainty to the challenges or high costs associated with measuring transition rates/probabilities during the modeling process.

In this paper, our objective is to determine the transition rates/probabilities that are unfixed. To ensure clear notations, we define ℓ for each $i \in \ell$ as $\ell \triangleq \ell_k^i \cup \ell_{uk}^i$, where

$$\ell_k^i \triangleq \{j : \pi_{ij}/\lambda_{ij} \text{ is known}\}, \quad \ell_{uk}^i \triangleq \{j : \pi_{ij}/\lambda_{ij} \text{ is unknown}\}.$$

Furthermore, if $\ell_k^i \neq \emptyset$, it can be further described as

$$\ell_k^i \triangleq (k_1^i, k_2^i, \ldots, k_{m_1}^i), \text{ for } 1 \leqslant m_1 \leqslant N, \quad \ell_{uk}^i \triangleq (u_1^i, u_2^i, \ldots, u_{m_2}^i), \text{ for } 1 \leqslant m_2 \leqslant N,$$

where $k_{m_1}^i$ denotes the fixed element at position m_1 in the ith row of matrices Π or Λ. Correspondingly, $u_{m_2}^i$ signifies the element at position m_2 in the ith row, whose value is yet to be ascertained. Notably, $m_1 + m_2 = N$.

2.3 Preliminaries

The following preconditions are necessary for the proposed hybrid control scheme.

Lemma 1 ([22])**.** Positive definite matrices P_1, P_2, ..., P_N exist if and only if condition (5) is satisfied. Under this circumstance, the free system (1) achieves mean square stability with the specified transition rate matrix Π. Similarly, condition (6) being met ensures mean square stability for the free system (2) when associated with the transition rate matrix Λ, where

$$A_i^\top P_i + P_i A_i + \sum\nolimits_{j=1}^{N} \pi_{ij} P_j < 0, \tag{5}$$

$$A_i^\top \sum\nolimits_{j=1}^{N} \lambda_{ij} P_j A_i - P_i < 0. \tag{6}$$

Lemma 2 ([23])**.** The following two conditions are equivalent for matrices T, P, U, and A with appropriate dimensions and a scalar β:

$$\begin{bmatrix} T & \beta P + A^\top U^\top \\ \star & -\beta U - \beta U \end{bmatrix} < 0 \Leftrightarrow T < 0,\ T + A^\top P + PA < 0.$$

Proof. The detailed proof has been shown in [23], which is omitted here.

In this paper, we consider the possibility of designing or modifying the switching matrices Π and Λ. The primary goal is to delineate a condition that is both necessary and sufficient for the effective synthesis of these matrices, ensuring mean square stability for the unforced systems (1) and (2). The second objective is to co-design the transition rates/probabilities and the following static output feedback controllers:

$$u(t) \triangleq K_i y(t) = K_i C_i x(t), \tag{7}$$

$$u(k) \triangleq K_i y(k) = K_i C_i x(k), \tag{8}$$

to ensure mean square stability of the closed-loop systems, where $y(t)$ (or $y(k)$) represents the measured output, K_i is the controller gain to be determined.

3 Main results

This paper concentrates on the intricate co-design of transition rates/probabilities and static feedback control mechanisms within the frameworks of systems (1) and (2). For simplicity, setting $\mathcal{E}_{kj} \triangleq \sum\limits_{j\in\ell_k}$ and $\mathcal{E}_{uj} \triangleq \sum\limits_{j\in\ell_{u_k}}$ in the whole paper.

3.1 Stabilizing transition rates/probabilities

In the subsection, we will present the necessary and sufficient conditions for the synthesis of a stabilizing transition rate in Π and transition probabilities in Λ, respectively. To begin, we will focus on Π.

Theorem 1. The continuous-time Markovian jump system (1) is mean square stable if and only if there exist positive-definite symmetric matrices $P_{k_1^i}$, $P_{k_2^i}$, ..., $P_{k_{m_1}^i}$, $X_{u_1^i}$, $X_{u_2^i}$, ..., $X_{u_{m_2}^i}$, $\bar{X}_{u_1^i}$, $\bar{X}_{u_2^i}$, ..., $\bar{X}_{u_{m_2}^i}$, a set of scalars $\varepsilon_{u_1^i}$, $\varepsilon_{u_2^i}$, ..., $\varepsilon_{u_{m_2}^i}$, $\bar{\pi}_{iu_1^i}$, $\bar{\pi}_{iu_2^i}$, ..., $\bar{\pi}_{iu_{m_2}^i}$ such that the following conditions hold for $j \in \ell_{uk}^i$:

$$rank\left(\begin{bmatrix} X_j & I \\ I & \bar{X}_j \end{bmatrix}\right) < n, \tag{9}$$

$$\begin{bmatrix} \hat{\Upsilon}_{11i} & \Upsilon_{12i} & \Upsilon_{13i} \\ \star & -\Upsilon_{22i} & 0 \\ \star & \star & -\Upsilon_{22i} \end{bmatrix} < 0, \quad i \in \ell_k^i, \tag{10}$$

$$\begin{bmatrix} \Upsilon_{11i} & \Upsilon_{12i} & \Upsilon_{13i} & A_{\varepsilon i} \\ \star & -\Upsilon_{22i} & 0 & 0 \\ \star & \star & -\Upsilon_{22i} & 0 \\ \star & \star & \star & -\bar{X}_i \end{bmatrix} < 0, \quad i \in \ell_{u_k}^i, \tag{11}$$

where

$$\hat{\Upsilon}_{11i} \triangleq A_i^\top P_i + P_i A_i + \mathcal{E}_{kj}\pi_{ij}P_j - \mathcal{E}_{uj}X_j, \quad \Upsilon_{11i} \triangleq -\mathcal{E}_{uj}X_j - 2X_i + \mathcal{E}_{kj}\pi_{ij}P_j,$$
$$\Upsilon_{12i} \triangleq [(1+\bar{\pi}_{iu_1^i})I, (1+\bar{\pi}_{iu_2^i})I, ..., (1+\bar{\pi}_{iu_{m_2}^i})I], \quad \Upsilon_{13i} \triangleq [\varepsilon_{u_1^i}A_i^\top, \varepsilon_{u_2^i}A_i^\top, ..., \varepsilon_{u_{m_2}^i}A_i^\top],$$
$$\Upsilon_{22i} \triangleq diag\{\bar{X}_{u_1^i}, \bar{X}_{u_2^i}, ..., \bar{X}_{u_{m_2}^i}\}, \quad A_{\varepsilon i} \triangleq I + \varepsilon_i A_i^\top.$$

Then, the unfixed transition rates can be given by $\pi_{ij} = \frac{2}{\varepsilon_j}\bar{\pi}_{ij}$.

Proof. To simplify, decision matrices $P_{k_1^i}$, $P_{k_2^i}$, ..., $P_{k_{m_1}^i}$ are shown by $i \in \ell_k^i$ and $i \in \ell_{u_k}^i$. Then, two cases on $i \in \ell_k^i$ and $i \in \ell_{u_k}^i$ will be respectively discussed as follows.

Case 1: If $i \in \ell_k^i$, it yields from (5) and $\ell \triangleq \ell_k^i \cup \ell_{u_k}^i$

$$A_i^\top P_i + P_i A_i + \mathcal{E}_{kj}\pi_{ij}P_j + \mathcal{E}_{uj}\pi_{ij}P_j < 0. \tag{12}$$

Inequality (12) holds if and only if, for sufficiently small and mode-synchronous parameters $\varepsilon_j > 0$ with P_j (i.e., the parameter ε_j shares the same subscript j with the decision matrix P), the following inequality holds:

$$A_i^\top P_i + P_i A_i + \mathcal{E}_{kj}\pi_{ij}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\varepsilon_j\left(\frac{\pi_{ij}}{2}P_j\frac{\pi_{ij}}{2} + A_i^\top P_j A_i\right) < 0. \tag{13}$$

Condition (13) can be further rewritten as

$$\begin{bmatrix} \bar{\Upsilon}_{11i} & \bar{\Upsilon}_{12i} & \Upsilon_{13i} \\ \star & -\bar{\Upsilon}_{22i} & 0 \\ \star & \star & -\bar{\Upsilon}_{22i} \end{bmatrix} < 0, \tag{14}$$

where

$$\bar{\Upsilon}_{11i} \triangleq A_i^\top P_i + P_i A_i + \mathcal{E}_{kj}\pi_{ij}P_j - \mathcal{E}_{uj}\varepsilon_j^{-1}P_j, \quad \bar{\Upsilon}_{12i} \triangleq \left[\left(1 + \frac{\varepsilon_{u_1^i}\pi_{iu_1^i}}{2}\right)I, ..., \left(1 + \frac{\varepsilon_{u_{m_2}^i}\pi_{iu_{m_2}^i}}{2}\right)I\right],$$
$$\bar{\Upsilon}_{22i} \triangleq diag\{\varepsilon_{u_1^i}P_{u_1^i}^{-1}, \varepsilon_{u_2^i}P_{u_2^i}^{-1}, ..., \varepsilon_{u_{m_2}^i}P_{u_{m_2}^i}^{-1}\}.$$

Defining

$$X_j \triangleq \varepsilon_j^{-1}P_j, \quad \bar{X}_j \triangleq \varepsilon_j P_j^{-1}, \quad \bar{\lambda}_{ij} \triangleq \frac{\varepsilon_j\lambda_{ij}}{2}, \tag{15}$$

we have (10).

Case 2: If $i \in \ell_{u_k}^i$, it yields from (5)

$$A_i^\top P_i + P_i A_i + \mathcal{E}_{kj}\pi_{ij}P_j + \mathcal{E}_{uj}\pi_{ij}P_j < 0. \tag{16}$$

Inequality (16) holds if and only if, for sufficiently small and mode-synchronous parameters $\varepsilon_j > 0$ with P_j and $\varepsilon_i > 0$ with P_i, the following inequality holds:

$$A_i^\top P_i + P_i A_i + \mathcal{E}_{kj}\pi_{ij}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\varepsilon_j\left(\frac{\pi_{ij}}{2}A_i^\top P_j\frac{\pi_{ij}}{2}A_i + A_i^\top P_j A_i\right) + \varepsilon_i A_i^\top P_i A_i < 0,$$

which can be rewritten as

$$\begin{bmatrix} \acute{\Upsilon}_{11i} & \bar{\Upsilon}_{12i} & \Upsilon_{13i} & I + \varepsilon_i A_i^\top \\ \star & -\bar{\Upsilon}_{22i} & 0 & 0 \\ \star & \star & -\bar{\Upsilon}_{22i} & 0 \\ \star & \star & \star & -\varepsilon_i P_i^{-1} \end{bmatrix} < 0, \tag{17}$$

where $\acute{\Upsilon}_{11i} \triangleq -\mathcal{E}_{uj}\varepsilon_j^{-1}P_j - 2\varepsilon_i^{-1}P_i + \mathcal{E}_{kj}\pi_{ij}P_j$. From (15), (17) with the similar procedure to (14), it yields (11). This completes the proof.

We will proceed to delineate the essential condition that is both necessary and sufficient for the synthesis of stabilizing transition probabilities within Λ.

Theorem 2. The discrete-time Markovian jump system (2) is mean-square stable if and only if there exist symmetric and positive-definite matrices $P_{k_1^i}$, $P_{k_2^i}$, ..., $P_{k_{m_1}^i}$, $X_{u_1^i}$, $X_{u_2^i}$, ..., $X_{u_{m_2}^i}$, $\bar{X}_{u_1^i}$, $\bar{X}_{u_2^i}$, ..., $\bar{X}_{u_{m_2}^i}$, a set of scalars $\varepsilon_{u_1^i}$, $\varepsilon_{u_2^i}$, ..., $\varepsilon_{u_{m_2}^i}$, $\bar{\lambda}_{iu_1^i}$, $\bar{\lambda}_{iu_2^i}$, ..., $\bar{\lambda}_{iu_{m_2}^i}$ such that Eq. (9) and the following conditions hold for $j \in \ell_{uk}^i$:

$$\begin{bmatrix} \hat{\Omega}_{11i} & \Omega_{12i} & \Omega_{13i} \\ \star & -\Omega_{22i} & 0 \\ \star & \star & -\Omega_{22i} \end{bmatrix} < 0, \quad i \in \ell_k^i, \tag{18}$$

$$\begin{bmatrix} \Omega_{11i} & \Omega_{12i} & \Omega_{13i} & \bar{\varepsilon}_i I \\ \star & -\Omega_{22i} & 0 & 0 \\ \star & \star & -\Omega_{22i} & 0 \\ \star & \star & \star & -\bar{X}_i \end{bmatrix} < 0, \quad i \in \ell_{u_k}^i, \tag{19}$$

where

$$\hat{\Omega}_{11i} \triangleq -P_i + A_i^\top \mathcal{E}_{kj}\lambda_{ij}P_jA_i - \mathcal{E}_{uj}A_i^\top X_jA_i, \quad \Omega_{11i} \triangleq -X_i + A_i^\top \mathcal{E}_{kj}\lambda_{ij}P_jA_i - \mathcal{E}_{uj}A_i^\top X_jA_i,$$
$$\Omega_{12i} \triangleq [A_i^\top + \bar{\lambda}_{i1}A_i^\top, A_i^\top + \bar{\lambda}_{i2}A_i^\top, ..., A_i^\top + \bar{\lambda}_{im_2}A_i^\top], \quad \Omega_{13i} \triangleq [\varepsilon_1 A_i^\top, \varepsilon_2 A_i^\top, ..., \varepsilon_{m_2}A_i^\top],$$
$$\Omega_{22i} \triangleq diag\{\bar{X}_1, \bar{X}_2, ..., \bar{X}_{m_2}\}, \quad \bar{\varepsilon}_i \triangleq 1 - \frac{\varepsilon_i}{2}.$$

The unfixed transition probabilities are given by $\pi_{ij} = \frac{2}{\varepsilon_j}\bar{\pi}_{ij}$.

Proof. For any mode $i \in \ell \triangleq \ell_k^i \cup \ell_{u_k}^i$, one has from (6)

$$-P_i + A_i^\top \mathcal{E}_{kj}\lambda_{ij}P_jA_i + A_i^\top \mathcal{E}_{uj}\lambda_{ij}P_jA_i < 0. \tag{20}$$

Focusing on (20), two steps will be given for the proof.

Step 1: Inequality (20) holds for any mode $i \in \ell_k^i$ if and only if, for sufficiently small and mode-synchronous parameters $\varepsilon_j > 0$ with P_j, the following inequality holds:

$$A_i^\top \mathcal{E}_{kj}\lambda_{ij}P_jA_i + \mathcal{E}_{uj}\left(\frac{1}{2}\lambda_{ij}A_i^\top\right)P_jA_i + \mathcal{E}_{uj}A_i^\top P_j\left(\frac{1}{2}\lambda_{ij}A_i\right)$$
$$-P_i + \mathcal{E}_{uj}\varepsilon_j\left[\left(\frac{1}{2}\lambda_{ij}A_i\right)^\top P_j\left(\frac{1}{2}\lambda_{ij}A_i\right) + A_i^\top P_jA_i\right] < 0,$$

which can be rewritten as

$$\begin{bmatrix} \bar{\Omega}_{11i} & \bar{\Omega}_{12i} & \Omega_{13i} \\ \star & -\bar{\Omega}_{22i} & 0 \\ \star & \star & -\bar{\Omega}_{22i} \end{bmatrix} < 0, \tag{21}$$

where

$$\bar{\Omega}_{11i} \triangleq \mathcal{E}_{kj}\lambda_{ij}A_i^\top P_jA_i - \mathcal{E}_{uj}A_i^\top \varepsilon_j^{-1}P_jA_i - P_i, \bar{\Omega}_{12i} \triangleq \left[A_i^\top + \frac{1}{2}\varepsilon_1\lambda_{i1}A_i^\top, ..., A_i^\top + \frac{1}{2}\varepsilon_{m_2}\lambda_{im_2}A_i^\top\right],$$
$$\bar{\Omega}_{22i} \triangleq diag\{\varepsilon_1 P_1^{-1}, \varepsilon_2 P_2^{-1}, ..., \varepsilon_{m_2}P_{m_2}^{-1}\}.$$

Combining with (15), it yields (18).

Step 2: Inequality (20) holds for any mode $i \in \ell_{u_k}^i$ if and only if, for sufficiently small and mode-synchronous parameters $\varepsilon_j > 0$ with P_j and $\varepsilon_i > 0$ with P_i, the following inequality holds, where $i, j \in \ell_{u_k}^i$:

$$A_i^\top \mathcal{E}_{kj}\lambda_{ij}P_jA_i + \mathcal{E}_{uj}\left(\frac{1}{2}\lambda_{ij}A_i^\top\right)P_jA_i + \mathcal{E}_{uj}A_i^\top P_j\left(\frac{1}{2}\pi_{ij}A_i\right) + \frac{\varepsilon_i}{4}P_i - P_i$$

$$+\mathcal{E}_{uj}\varepsilon_j\left[\left(\frac{1}{2}\lambda_{ij}A_i\right)^\top P_j\left(\frac{1}{2}\lambda_{ij}A_i\right)+A_i^\top P_jA_i\right]<0.$$

With the similar procedure to (21), it yields (19). This completes the proof.

Remark 2. The asynchronous mode-dependent parametric method, as proposed in literature (see [15,18]), introduces asynchronous mode-dependent parameter combinations, such as εP_j in [15] and $\varepsilon_i P_j$ in [18]. However, the solvability of the conditions obtained in [15,18] relies on the satisfaction of parameters ε and ε_i for all modes j. This means that if there exist modes $j\in\ell$ for which the parameters ε and ε_i cannot simultaneously satisfy the conditions from [15,18], the problem becomes infeasible. To address this issue, a synchronous mode-dependent parametric method is proposed, which introduces a set of mode-dependent parameter combinations, such as $\varepsilon_j P_j$, in the inequalities such as (13), (16). This approach effectively ensures the feasibility of the obtained conditions in this paper and improves the efficacy of finding feasible solutions compared to asynchronous ones (such as in [15,18]).

3.2 Hybrid design with static output feedback

In this subsection, our focus is on the co-design of a stabilizing static output feedback controller (7), (8) and the corresponding transition rates in Π and transition probabilities in Λ for Markovian jump systems. By applying controller (7) to system (1) and (8) to system (2), one has

$$\dot{x}(t)=(A_i+B_iK_iC_i)x(t), \tag{22}$$
$$x(k+1)=(A_i+B_iK_iC_i)x(k). \tag{23}$$

Next, we will address the problem of static output feedback stabilization with transition rate synthesis for system (22).

Theorem 3. For a given scalar β, the continuous-time Markovian jump system (22) is mean square stable if there exist positive-definite symmetric matrices $\bar{P}_{k_1^i}$, $\bar{P}_{k_2^i}$, ..., $\bar{P}_{k_{m_1}^i}$, $X_{u_1^i}$, $X_{u_2^i}$, ..., $X_{u_{m_2}^i}$, $\bar{X}_{u_1^i}$, $\bar{X}_{u_2^i}$, ..., $\bar{X}_{u_{m_2}^i}$, any matrices U_i, V_i, $\bar{K}_i$, a set of scalars $\varepsilon_{u_1^i}$, $\varepsilon_{u_2^i}$, ..., $\varepsilon_{u_{m_2}^i}$, $\bar{\pi}_{iu_1^i}$, $\bar{\pi}_{iu_2^i}$, ..., $\bar{\pi}_{iu_{m_2}^i}$ such that Eq. (9) and the following condition hold for $j\in\ell_{u_k}^i$:

$$\begin{bmatrix}\hat{\Upsilon}_{11i} & \Upsilon_{12i} & \Upsilon_{13i} & \Upsilon_{14i}\\ \star & -\Upsilon_{22i} & 0 & 0\\ \star & \star & -\Upsilon_{22i} & 0\\ \star & \star & \star & -\beta U_i-\beta U_i\end{bmatrix}<0,\quad i\in\ell_k^i, \tag{24}$$

$$\begin{bmatrix}\check{\Upsilon}_{11i} & \Upsilon_{12i} & \Upsilon_{13i} & \Upsilon_{14i}\\ \star & -\Upsilon_{22i} & 0 & 0\\ \star & \star & -\Upsilon_{22i} & 0\\ \star & \star & \star & -\beta U_i-\beta U_i\end{bmatrix}<0,\quad i\in\ell_{u_k}^i, \tag{25}$$

where

$$\check{\Upsilon}_{11i}\triangleq A_i^\top P_i+P_iA_i+\mathcal{E}_{kj}\pi_{ij}(P_j-P_i)-\mathcal{E}_{uj}X_j,\quad \Upsilon_{14i}\triangleq\beta P_iB_i+C_i^\top V_i^\top.$$

Then, the unfixed transition rates can be reconstructed by

$$\pi_{ij}=\begin{cases}\frac{2}{\varepsilon_j}\bar{\pi}_{ij}, & i\in\ell_k^i,\\ -\mathcal{E}_{kj}\pi_{ij}-\mathcal{E}_{uj}\pi_{ij}, & i\in\ell_{u_k}^i.\end{cases} \tag{26}$$

Meanwhile, the controller can be designed by

$$K_i=U_i^{-1}V_i. \tag{27}$$

Proof. Two cases will be discussed as follows.

Case 1: If $i\in\ell_k^i$, combining (12) and (22), it yields

$$(A_i+B_iK_iC_i)^\top P_i+P_i(A_i+B_iK_iC_i)+\mathcal{E}_{kj}\pi_{ij}P_j+\mathcal{E}_{uj}\pi_{ij}P_j<0. \tag{28}$$

Inequality (28) holds if and only if the following condition holds for sufficiently small scalars $\varepsilon_j > 0, j \in \ell^i_{u_k}$:

$$A_i^\top P_i + P_iA_i + (B_iK_iC_i)^\top P_i + P_iB_iK_iC_i + \mathcal{E}_{kj}\pi_{ij}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j$$
$$+\mathcal{E}_{uj}\varepsilon_j\left(\frac{\pi_{ij}}{2}A_i^\top P_j\frac{\pi_{ij}}{2}A_i + A_i^\top P_jA_i\right) < 0. \tag{29}$$

From (13) and Lemma 2, for any matrix U_i and any scalar $\beta \neq 0$, condition (29) can be rewritten as

$$\begin{bmatrix} \bar{\Upsilon}_{11i} & \bar{\Upsilon}_{12i} & \Upsilon_{13i} & \beta P_iB_i + C_i^\top K_i^\top U_i^\top \\ \star & -\bar{\Upsilon}_{22i} & 0 & 0 \\ \star & \star & -\bar{\Upsilon}_{22i} & 0 \\ \star & \star & \star & -\beta U_i - \beta U_i \end{bmatrix} < 0. \tag{30}$$

Defining $V_i \triangleq U_iK_i$, together with (15), it yields (24).

Case 2: If $i \in \ell^i_{u_k}$, combining (16) and (22), it yields

$$A_i^\top P_i + P_iA_i + \mathcal{E}_{kj}\pi_{ij}P_j + (B_iK_iC_i)^\top P_i + P_iB_iK_iC_i + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \pi_{ii}P_i$$
$$+\mathcal{E}_{uj}\varepsilon_j\left(\frac{\pi_{ij}}{2}A_i^\top P_j\frac{\pi_{ij}}{2}A_i + A_i^\top P_jA_i\right) < 0. \tag{31}$$

Due to $\pi_{ii} = -\mathcal{E}_{kj}\pi_{ij} - \mathcal{E}_{uj}\pi_{ij}$, if Eq. (31) holds, we have

$$A_i^\top P_i + P_iA_i + \mathcal{E}_{kj}\pi_{ij}P_j + (B_iK_iC_i)^\top P_i + P_iB_iK_iC_i + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j - \mathcal{E}_{kj}\pi_{ij}P_i$$
$$+\mathcal{E}_{uj}\varepsilon_j\left(\frac{\pi_{ij}}{2}A_i^\top P_j\frac{\pi_{ij}}{2}A_i + A_i^\top P_jA_i\right) < 0, \tag{32}$$

which can be rewritten as

$$\begin{bmatrix} \vec{\Upsilon}_{11i} & \bar{\Upsilon}_{12i} & \Upsilon_{13i} & \beta P_iB_i + C_i^\top K_i^\top U_i^\top \\ \star & -\bar{\Upsilon}_{22i} & 0 & 0 \\ \star & \star & -\bar{\Upsilon}_{22i} & 0 \\ \star & \star & \star & -\beta U_i - \beta U_i \end{bmatrix} < 0, \tag{33}$$

where

$$\vec{\Upsilon}_{11i} \triangleq A_i^\top P_i + P_iA_i + \mathcal{E}_{kj}\pi_{ij}(P_j - P_i) - \mathcal{E}_{uj}\varepsilon_j^{-1}P_j.$$

Defining $\bar{K}_i \triangleq \varepsilon_iK_i$, together with (15), with the similar procedure to (30), it yields (25). This completes the proof.

Remark 3. It has been observed in [15,18] that the asynchronous mode-dependent parametric method is utilized to address the coupling term $P_iB_iK_iC_i$ in continuous-time Markovian jump systems. However, this method introduces two types of decision variables, namely X_i and $\bar{X}_i$, leading to significant decision complexity if large switching rules are required. In this paper, we focus on the case where P_i is a fixed element that cannot be designed and needs to be preserved when $i \in \ell^i_k$. Consequently, it becomes necessary to retain P_i in the coupling term $P_iB_iK_iC_i$. As a result, the traditional asynchronous mode-dependent parametric method becomes unsuitable. To address this issue, we employ the matrix decoupled approach in Lemma 2 and the relation $\pi_{ij} = 0$ to separate P_i and K_i, effectively reducing the complexity associated with decision variables dependent on fixed transition rates. This allows us to focus solely on the variables dependent on fixed ones, such as $\bar{P}_{k^i_1}$, $\bar{P}_{k^i_{m_1}}$.

Now, we will deal with the static output feedback stabilization with transition probability synthesis for system (23).

Theorem 4. The discrete-time Markovian jump system (23) is mean square stable if there exist positive-definite symmetric matrices $\bar{P}_{k^i_1}, \bar{P}_{k^i_2}, \ldots, \bar{P}_{k^i_{m_1}}, X_{u^i_1}, X_{u^i_2}, \ldots, X_{u^i_{m_2}}, \bar{X}_{u^i_1}, \bar{X}_{u^i_2}, \ldots, \bar{X}_{u^i_{m_2}}$, any matrices $Q_{u^i_1}, Q_{u^i_2}, \ldots, Q_{u^i_{m_2}}$,

K_1, K_2, ..., K_N, a set of scalars $\varepsilon_{u_1^i}$, $\varepsilon_{u_2^i}$, ..., $\varepsilon_{u_{m_2}^i}$, $\bar{\lambda}_{iu_1^i}$, $\bar{\lambda}_{iu_2^i}$, ..., $\bar{\lambda}_{iu_{m_2}^i}$, $v_{iu_1^i}$, $v_{iu_2^i}$, ..., $v_{iu_{m_2}^i}$ such that Eq. (9) and the following conditions hold for $j \in \ell_{k_k}^i$:

$$\begin{bmatrix} \Gamma_{11i} & \Gamma_{12i} \\ \star & \Gamma_{22} \end{bmatrix} \leqslant 0, \tag{34}$$

$$\begin{bmatrix} \hat{\Psi}_{11i} & \hat{\Psi}_{12i} \\ \star & -\hat{\Psi}_{22i} \end{bmatrix} \leqslant 0, \quad i \in \ell_k^i, \tag{35}$$

$$\begin{bmatrix} \Psi_{11i} & \hat{\Psi}_{12i} & \bar{\varepsilon}_i I \\ \star & -\hat{\Psi}_{22i} & 0 \\ \star & \star & -\bar{X}_i \end{bmatrix} \leqslant 0, \quad i \in \ell_{u_k}^i, \tag{36}$$

where

$$\begin{aligned}
&\hat{\Psi}_{11i} \triangleq -2I + \bar{P}_i - \mathcal{E}_{uj} A_i^\top X_j A_i, \quad \Psi_{11i} \triangleq -X_i - \mathcal{E}_{uj} A_i^\top X_j A_i, \\
&\hat{\Psi}_{12i} \triangleq [\Psi_{12i}, \Psi_{13i}, \Psi_{14i}, \Psi_{15i}], \quad \hat{\Psi}_{22i} \triangleq diag\{\Psi_{22i}, \Psi_{33i}, \Psi_{33i}, \Psi_{33i}\}, \\
&\Psi_{22i} \triangleq diag\{\bar{P}_{k_1^i}, ..., \bar{P}_{k_{m_1}^i}\}, \quad \Psi_{33i} \triangleq diag\{\bar{X}_{u_1^i}, ..., \bar{X}_{u_{m_2}^i}\}, \quad K_{bi} = B_i K_i C_i, \\
&\Psi_{12i} \triangleq \left[\sqrt{\lambda_{ik_1^i}} (A_i + K_{bi})^\top, ..., \sqrt{\lambda_{ik_{m_1}^i}} (A_i + K_{bi})^\top\right], \quad \Psi_{13i} \triangleq [2\bar{\lambda}_{iu_1^i} A_i^\top + K_{bi}^\top, ..., 2\bar{\lambda}_{iu_{m_2}^i} A_i^\top + K_{bi}^\top], \\
&\Psi_{14i} \triangleq [\bar{\lambda}_{iu_1^i} A_i^\top + A_i^\top, ..., \bar{\lambda}_{iu_{m_2}^i} A_i^\top + A_i^\top], \quad \Psi_{15i} \triangleq [\varepsilon_{u_1^i} A_i^\top, \varepsilon_{u_2^i} A_i^\top, ..., \varepsilon_{u_{m_2}^i} A_i^\top], \\
&\Gamma_{11i} \triangleq -\mathcal{E}_{uj} X_j + \mathcal{E}_{uj}(Q_j^\top Q_j - Q_j^\top X_j - X_j^\top Q_j + v_{ij}^2 I - 2v_{ij}\bar{\lambda}_{ij} I), \\
&\Gamma_{12i} \triangleq [\bar{\lambda}_{iu_1^i} I + X_{u_1^i}, \bar{\lambda}_{iu_2^i} I + X_{u_2^i}, ..., \bar{\lambda}_{iu_{m_2}^i} I + X_{u_{m_2}^i}], \quad \Gamma_{22i} \triangleq diag\{I, I, ..., I\}.
\end{aligned}$$

Then, a stabilizing transition probability can be reconstructed by $\lambda_{ij} = \frac{2}{\varepsilon_j} \bar{\lambda}_{ij}$ and the controller can be design by K_i.

Proof. Three steps will be given for the proof.

Step 1: From (34), we have

$$\mathcal{E}_{uj}(Q_j^\top Q_j - Q_j^\top X_j - X_j^\top Q_j + v_{ij}^2 I - 2v_{ij}\bar{\lambda}_{ij} I) - \mathcal{E}_{uj} X_j + \mathcal{E}_{uj}(\bar{\lambda}_{ij} I + X_j)^\top (\bar{\lambda}_{ij} I + X_j) \leqslant 0, \tag{37}$$

which can be further rewritten as

$$2\mathcal{E}_{uj}\bar{\lambda}_{ij} X_j + \mathcal{E}_{uj}(Q_j - X_j)(Q_j - X_j) - \mathcal{E}_{uj} X_j + \mathcal{E}_{uj}(v_{ij} - \bar{\lambda}_{ij})(v_{ij} - \bar{\lambda}_{ij}) \leqslant 0. \tag{38}$$

From the following fact:

$$(Q_j - X_j)(Q_j - X_j) \geqslant 0, (v_{ij} - \bar{\lambda}_{jj})(v_{ij} - \bar{\lambda}_{jj}) \geqslant 0,$$

we have

$$-\mathcal{E}_{uj} X_j + 2\mathcal{E}_{uj}\bar{\lambda}_{ij} X_j \leqslant 0. \tag{39}$$

From (15), condition (39) implies

$$-\mathcal{E}_{uj}\varepsilon_j^{-1} P_j + \mathcal{E}_{uj}\lambda_{ij} P_j \leqslant 0. \tag{40}$$

Step 2: Following the fact

$$(I - P_i^{-1})^\top P_i (I - P_i^{-1}) \geqslant 0,$$

we have

$$-P_i \leqslant -2I + P_i^{-1}. \tag{41}$$

Step 3: The mean square stability of system (23), as defined by (6), is guaranteed if a set of positive definite matrices P_i fulfills the following inequality:

$$(A_i + B_iK_iC_i)^\top(\mathcal{E}_{kj}\lambda_{ij}P_j)(A_i + B_iK_iC_i) - P_i + (A_i + B_iK_iC_i)^\top(\mathcal{E}_{uj}\lambda_{ij}P_j)(A_i + B_iK_iC_i) < 0. \tag{42}$$

In the following, two cases on $i \in \ell_k^i$ and $i \in \ell_{u_k}^i$ will be respectively considered.

Case 1: If $i \in \ell_k^i$, one has from (40)–(42)

$$\begin{aligned} &P_i^{-1} - 2I + (A_i + B_iK_iC_i)^\top\lambda_{ij}P_j(A_i + B_iK_iC_i) + \mathcal{E}_{uj}A_i^\top P_j\left(\frac{\lambda_{ij}}{2}A_i\right) + \mathcal{E}_{uj}\left(\frac{\lambda_{ij}}{2}A_i\right)^\top P_jA_i \\ &+\mathcal{E}_{uj}(\lambda_{ij}A_i)^\top P_j(B_iK_iC_i) + \mathcal{E}_{uj}(B_iK_iC_i)^\top P_j(\lambda_{ij}A_i) + \mathcal{E}_{uj}(B_iK_iC_i)^\top\varepsilon_j^{-1}P_j(B_iK_iC_i) \\ &+\mathcal{E}_{uj}\varepsilon_j\left[\frac{5}{4}(\lambda_{ij}A_i)^\top P_j(\lambda_{ij}A_i) + A_i^\top P_jA_i\right] < 0, \end{aligned} \tag{43}$$

which can be rewritten as

$$\begin{bmatrix} \bar{\Psi}_{11i} & \Psi_{12i} & \bar{\Psi}_{13i} & \bar{\Psi}_{14i} & \Psi_{15i} \\ \star & -\bar{\Psi}_{22i} & 0 & 0 & 0 \\ \star & \star & -\bar{\Psi}_{33i} & 0 & 0 \\ \star & \star & \star & -\bar{\Psi}_{33i} & 0 \\ \star & \star & \star & \star & -\bar{\Psi}_{33i}, \end{bmatrix} < 0, \tag{44}$$

where

$$\begin{aligned} &\bar{\Psi}_{11i} \triangleq -2I + P_i^{-1} - \mathcal{E}_{uj}\varepsilon_j^{-1}A_i^\top P_jA_i, \quad \Psi_{12i} \triangleq \left[\sqrt{\lambda_{ik_1^i}}(A_i + K_{bi})^\top, ..., \sqrt{\lambda_{ik_{m_1}^i}}(A_i + K_{bi})^\top\right], \\ &\bar{\Psi}_{13i} \triangleq [\varepsilon_{u_1^i}\lambda_{iu_1^i}A_i^\top + K_{bi}^\top, ..., \varepsilon_{u_{m_2}^i}\lambda_{iu_{m_2}^i}A_i^\top + K_{bi}^\top], \\ &\bar{\Psi}_{14i} \triangleq [\frac{\varepsilon_{u_1^i}}{2}\lambda_{iu_1^i}A_i^\top + A_i^\top, ..., \frac{\varepsilon_{u_{m_2}^i}}{2}\lambda_{iu_{m_2}^i}A_i^\top + A_i^\top], \quad \Psi_{15i} \triangleq [\varepsilon_{u_1^i}A_i^\top, \varepsilon_{u_2^i}A_i^\top, ..., \varepsilon_{u_{m_2}^i}A_i^\top], \\ &\bar{\Psi}_{22i} \triangleq diag\{P_{k_1^i}^{-1}, P_{k_2^i}^{-1}, ..., P_{k_{m_1}^i}^{-1}\}, quad\bar{\Psi}_{33i} \triangleq diag\{\varepsilon_{i1}P_{u_1^i}^{-1}, ..., \varepsilon_{im_2}P_{u_{m_2}^i}^{-1}\}. \end{aligned}$$

From (15), we can obtain (35).

Case 2: If $i \in \ell_{u_k}^i$, it further yields from (40) and (42)

$$\begin{aligned} &(A_i + B_iK_iC_i)^\top\lambda_{ij}P_j(A_i + B_iK_iC_i) - P_i + \mathcal{E}_{uj}A_i^\top P_j\left(\frac{\lambda_{ij}}{2}A_i\right) + \mathcal{E}_{uj}\left(\frac{\lambda_{ij}}{2}A_i\right)^\top P_jA_i \\ &+\mathcal{E}_{uj}(\lambda_{ij}A_i)^\top P_j(B_iK_iC_i) + \mathcal{E}_{uj}(B_iK_iC_i)^\top P_j(\lambda_{ij}A_i) + \mathcal{E}_{uj}(B_iK_iC_i)^\top\varepsilon_j^{-1}P_j(B_iK_iC_i) + \frac{\varepsilon_i}{4}P_i \\ &+\mathcal{E}_{uj}\varepsilon_j\left[\frac{5}{4}(\lambda_{ij}A_i)^\top P_j(\lambda_{ij}A_i) + A_i^\top P_jA_i\right] < 0, \end{aligned}$$

which can be rewritten as

$$\begin{bmatrix} \tilde{\Psi}_{11i} & \Psi_{12i} & \bar{\Psi}_{13i} & \bar{\Psi}_{14i} & \Psi_{15i} & \bar{\varepsilon}_iI \\ \star & -\bar{\Psi}_{22i} & 0 & 0 & 0 & 0 \\ \star & \star & -\bar{\Psi}_{33i} & 0 & & 0 \\ \star & \star & \star & -\bar{\Psi}_{33i} & 0 & 0 \\ \star & \star & \star & \star & -\bar{\Psi}_{33i} & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_iP_i^{-1} \end{bmatrix} < 0,$$

where $\tilde{\Psi}_{11i} \triangleq -\varepsilon_i^{-1}P_i - \mathcal{E}_{uj}\varepsilon_j^{-1}A_i^\top P_jA_i$. From the definition in (15), with the similar procedure to (43), we can obtain (36). This completes the proof.

Remark 4. It is seen from [18] that the 4-degree coupling terms $(B_iK_iC_i)^\top \sum_{j=1}^N \lambda_{ij} P_j(B_iK_iC_i)$ are addressed. The relation $\lambda_{ij} \leqslant 1$ is used to impose a constraint, leading to $(B_iK_iC_i)^\top \sum_{j=1}^N \lambda_{ij} P_j(B_iK_iC_i) \leqslant (B_iK_iC_i)^\top \sum_{j=1}^N P_j(B_iK_iC_i)$. However, since λ_{ij} are the parameters to be solved, eliminating them inevitably introduces conservatism. In this paper, we introduce condition (34) to retain the parameters λ_{ij} in Λ (refer to Step 1 in the proof of Theorem 4 for details), leading to less conservatism.

Remark 5. The hybrid control scheme in this paper distinguishes between the fixed transition probabilities-dependent matrices P_i and the unfixed transition probabilities-dependent matrices X_i and $\bar{X}_i$. This clear separation eliminates the complexity associated with a large number of decision variables and avoids the need for optimization algorithms. For instance, using the methods described in [15, 19], condition (13) can be expressed as

$$A_i^\top P_i + P_iA_i + \mathcal{E}_{kj}\pi_{ij}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\frac{\pi_{ij}}{2}P_j + \mathcal{E}_{uj}\varepsilon_j\frac{\pi_{ij}}{2}P_j\frac{\pi_{ij}}{2} + \varepsilon_i A_i^\top P_iA_i < 0.$$

This formulation will introduce additional combinations such as $\varepsilon_i X_i$, making the problem difficult to solve, which further demonstrates the advantages of the hybrid control scheme in the present paper.

Remark 6. Compared to traditional methods that solely rely on feedback controllers [7, 12], this paper proposes a hybrid control strategy that stabilizes the closed-loop system through two main aspects. First, it optimizes the transition rates/probabilities to improve the internal structure of the Markov jump system, a capability that traditional feedback control does not possess. Second, it further adjusts and supplements the optimized structure through the feedback controller. Therefore, the hybrid controller possesses the ability for internal self-optimization, which can further enhance the system performance.

Remark 7. It is important to emphasize that this study does not aim to extend or refine the existing literature on partially transition probabilities/rates [12]. Instead, it approaches the problem from a novel perspective by investigating the designability of non-fixed transition probabilities. As such, the proposed framework is fundamentally distinct and should not be interpreted as a direct generalization of existing work on general uncertain transition probabilities/rates [24]. Building on this foundation, future research could explore how to more effectively design non-fixed transition matrices or identify conditions under which certain non-fixed transition structures remain inherently non-designable.

4 Examples

Example 1. Consider the system (1) borrowed from [15], where the parameters are defined as follows:

$$A_1 = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

It is evident that A_2 is stable, but A_1 is not. On this basis, we will consider the following two cases.

(1) If $\Pi = \begin{bmatrix} \circ & \circ \\ 3 & -3 \end{bmatrix}$, we can stabilize the system through transition rate design and hybrid control design.

(i) Using Theorem 1, we can find the unfixed elements as

$$\Pi_0 = [-9.4773, 9.4773].$$

(ii) Using Theorem 3, we obtain the hybrid unfixed elements and static feedback controller as

$$(\Pi, K) = ([-4.1536, 4.1536], [-0.6787, -1.6101]). \tag{45}$$

Under the initial condition $x(0) = [-5, 5]^\top$, the corresponding stabilization results are shown in Figures 3–5. Therein, Figure 3 demonstrates the stable state response achieved through designing the transition rate matrix Π_0. Figure 4 indicates the stable state response achieved through co-designing transition rates and a static feedback controller (Π, K). Furthermore, Figure 5 shows that the convergence rate of the system under the hybrid control strategy is faster than that of the design considering only transition rates, where a smaller convergence domain implies better system performance.

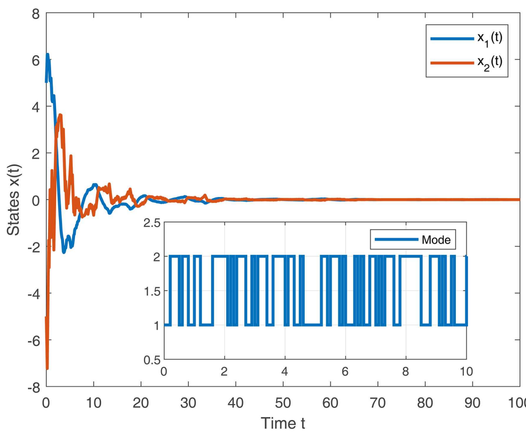

Figure 3 (Color online) Stabilization via the designed transition rates.

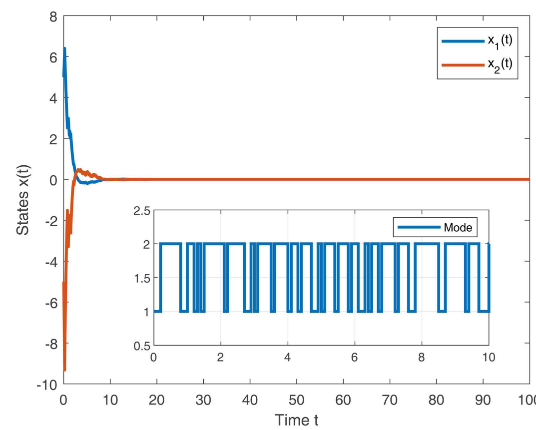

Figure 4 (Color online) Stabilization via the co-designed transition rates and static feedback controller in (45).

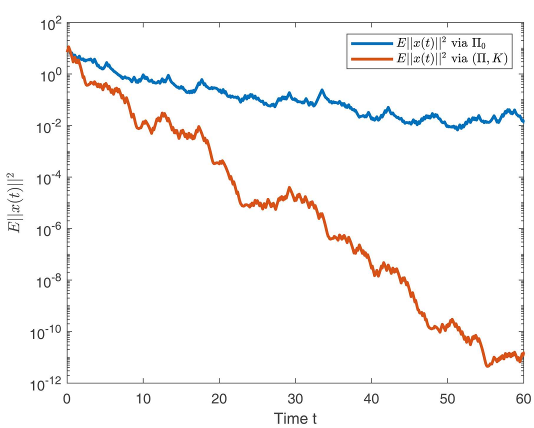

Figure 5 (Color online) The convergence tendencies via Π_0 and (Π, K).

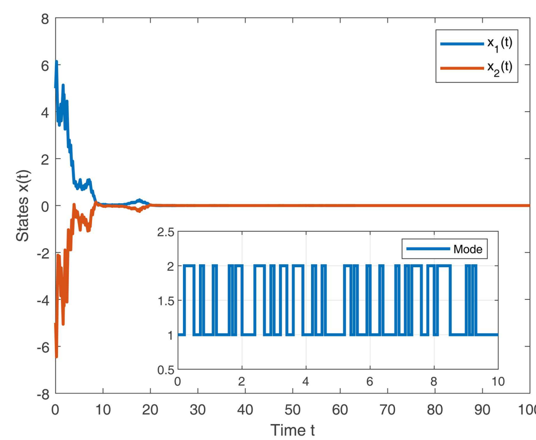

Figure 6 (Color online) Stabilization via the co-designed transition rates and static feedback controller in (46).

(2) If $\Pi = \begin{bmatrix} -3 & 3 \\ \circ & \circ \end{bmatrix}$, it can be seen that this system can not be stable via designing transition probabilities only. Thus, using Theorem 3, it yields the hybrid unfixed elements and static feedback controller as

$$(\Pi, K) = ([3.3292, -3.3292], [-1.4744, -0.6937]). \tag{46}$$

Under the hybrid control scheme, we can see from Figure 6 that the system is stable, which shows the hybrid control design method is efficient.

In the above example, we have demonstrated the proposed method to continuous-time Markovian jump systems. Now, let us provide an example to illustrate the benefits of the proposed method for discrete-time Markovian jump systems.

Example 2. Consider two discrete-time linear subsystems as follows:

$$A_1 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & -0.9 & 0 \\ 0 & 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0.1 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}.$$

It is clear that A_1 is asymptotically stable but not A_2. On this basis, two cases will be considered as follows.

(1) If $\Lambda = \begin{bmatrix} 0.5 & 0.5 \\ \circ & \circ \end{bmatrix}$, we can stabilize the system via transition probability design and hybrid control design.

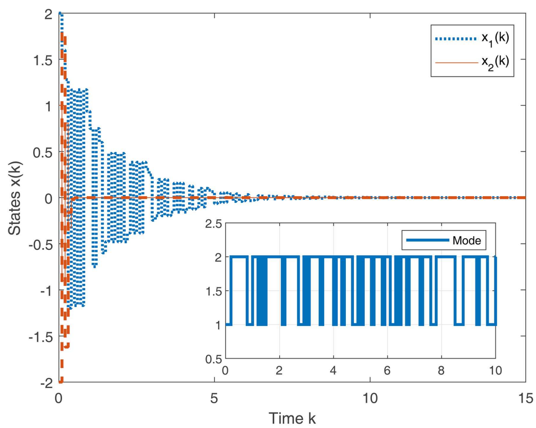

Figure 7 (Color online) Stabilization via the designed transition probabilities.

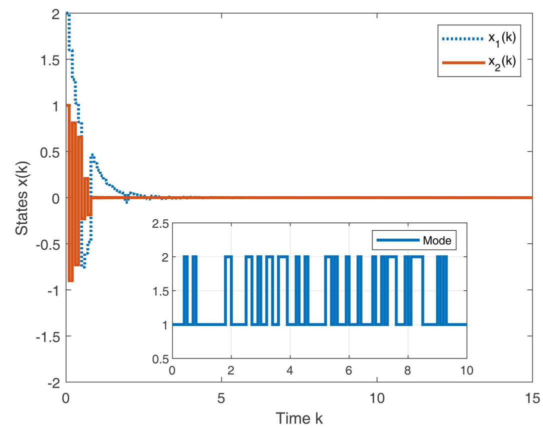

Figure 8 (Color online) Stabilization via the co-designed transition probabilities and static feedback controller in (47).

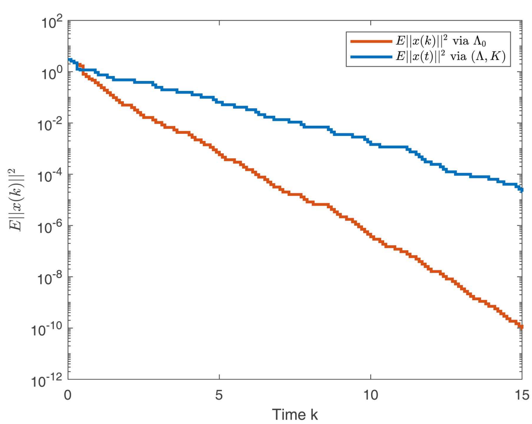

Figure 9 (Color online) The convergence tendencies via Λ_0 and (Λ, K).

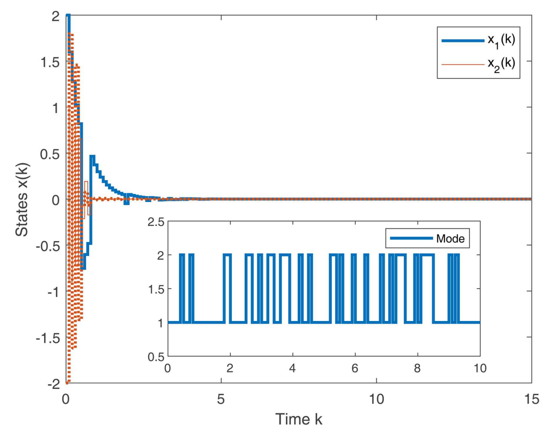

Figure 10 (Color online) Stabilization via the co-designed transition probabilities and static feedback controller in (48).

(i) Using Theorem 2, it yields the unfixed elements as $\Lambda_0 = [0.3555, 0.6445]$.

(ii) Using Theorem 4, it yields the hybrid unfixed elements and static feedback controller as

$$(\Lambda, K) = ([0.7096, 0.2914], [0, -0.0485]). \tag{47}$$

Under the initial condition $x(0) = [2, 1, -2]^\top$, it can be seen from Figures 7–9 that the state response is stable via designing the transition probability matrix Λ_0. From Figure 8, it can be concluded that the state response is stable via co-designing transition probabilities and static feedback controller (Λ, K). Moreover, it can be further summarized from Figure 9 that the convergence rate of the system under the hybrid control scheme is faster than that of the design transition probability only.

(2) If $\Lambda = \begin{bmatrix} \circ & \circ \\ 0.5 & 0.5 \end{bmatrix}$, it can be seen that this system cannot be stable via designing transition probabilities only. Thus, using Theorem 4, it yields the hybrid unfixed elements and static feedback controller as

$$(\Lambda, K) = ([0.2935, 0.7065], [0, -0.0748]). \tag{48}$$

Under the hybrid control scheme, we can see from Figure 10 that the system is stable, which shows the hybrid control design method is efficient.

5 Conclusion

This paper has explored the problem of hybrid control schemes for continuous-time and discrete-time Markovian jump systems using a synchronous mode-dependent parametric method. The objective is the co-design of partly transition rates/probabilities and output feedback control. First, we establish novel necessary and sufficient conditions to characterize the switching rates/probabilities that ensure mean square stability in both continuous-time and discrete-time Markovian jump linear systems. Subsequently, we derive stabilization conditions for the hybrid design. It is crucial to strictly separate the decision matrices for fixed and unfixed transition rates/probabilities, which reduces complexity requirements and eliminates the need to solve complex parameters. Finally, we demonstrate the effectiveness of the proposed methods through two numerical examples.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 62403084, U22A20101, 62173051) and Fundamental Research Funds for the Central Universities (Grant No. 2025CDJZKZCQ-08).

References

1 Wu X T, Tang Y, Mao S, et al. Stability analysis and stabilization of semi-Markov jump linear systems with unavailable sojourn-time information. Sci China Inf Sci, 2024, 67: 172201
2 Wang L, Jiang X, Zhao D, et al. Multiplayer Pareto optimal control with H_∞ constraint for nonlinear stochastic system via online synchronous reinforcement learning. J Auto Intell, 2025, 4: 207–216
3 Tian Y, Su X, Shen C, et al. Exponentially extended dissipativity-based filtering of switched neural networks. Automatica, 2023, 161: 111465
4 Ahmad I, Ge X, Han Q. Stochastic sampled-data multi-objective control of active suspension systems for in-wheel motor driven electric vehicles. J Auto Intell, 2024, 3: 2–18
5 Li F, Li X, Zhang X, et al. Asynchronous filtering for delayed Markovian jump systems via homogeneous polynomial approach. IEEE Trans Automat Contr, 2020, 65: 2163–2170
6 Li F, Xu S, Shen H, et al. Passivity-based control for hidden Markov jump systems with singular perturbations and partially unknown probabilities. IEEE Trans Automat Contr, 2020, 65: 3701–3706
7 Tian Y, Yan H, Zhang H, et al. Resilient static output feedback control of linear semi-Markov jump systems with incomplete semi-Markov kernel. IEEE Trans Automat Contr, 2021, 66: 4274–4281
8 Bueno J N A D, Marcos L B, Rocha K D T, et al. Regulation of Markov jump linear systems subject to polytopic uncertainties. IEEE Trans Automat Contr, 2022, 67: 6279–6286
9 Ning Z, Zhang L, Feng G, et al. Observation for Markov jump piecewise-affine systems with admissible region-switching paths. IEEE Trans Automat Contr, 2021, 66: 4319–4326
10 Wu X, Shi P, Tang Y, et al. Stability analysis of semi-Markov jump stochastic nonlinear systems. IEEE Trans Automat Contr, 2022, 67: 2084–2091
11 Shen H, Zhang Y, Wang J, et al. Observer-based control for discrete-time hidden semi-Markov jump systems. IEEE Trans Automat Contr, 2022, 68: 6255–6261
12 Zhang L, Boukas E K. Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities. Automatica, 2009, 45: 463–468
13 Kao Y, Han Y, Zhu Y, et al. Stability analysis of delayed discrete singular piecewise homogeneous Markovian jump systems with unknown transition probabilities via sliding-mode approach. IEEE Trans Automat Contr, 2024, 69: 315–322
14 Sehgal V. Markovian models based stochastic communication in networks-in-package. IEEE Trans Parallel Distrib Syst, 2015, 26: 2806–2821
15 Feng J-E, Lam J, Shu Z. Stabilization of Markovian systems via probability rate synthesis and output feedback. IEEE Trans Automat Contr, 2010, 55: 773–777
16 Ma S, Xiong J. Stabilization of time-delay Markovian jump systems via probability rate synthesis and state feedback. IFAC Proc Vol, 2014, 47: 10415–10420
17 Zhang P, Cao J, Wang G. Mode-independent guaranteed cost control of singular Markovian delay jump systems with switching probability rate design. Int J Innov Comput Inf Control, 2014, 10: 1291–1303
18 Song J, Niu Y, Lam J, et al. A hybrid design approach for output feedback exponential stabilization of Markovian jump systems. IEEE Trans Automat Contr, 2018, 63: 1404–1417
19 Cao Z, Niu Y, Lam J, et al. A hybrid sliding mode control scheme of Markovian jump systems via transition rates optimal design. IEEE Trans Syst Man Cybern Syst, 2020, 51: 7752–7763
20 Zhang X, Wang H, Song J, et al. Co-design of adaptive event generator and asynchronous fault detection filter for Markov jump systems via genetic algorithm. IEEE Trans Cybern, 2022, 53: 5059–5068
21 Yang Y, Niu Y. Co-design of scheduling protocol and fuzzy sliding mode controller under communication constraints. Automatica, 2023, 147: 110698
22 Boukas E. Stochastic Switching Systems: Analysis and Design. Berlin: Birkhauser, 2005
23 Chang X H, Park J H, Zhou J. Robust static output feedback H_∞ control design for linear systems with polytopic uncertainties. Syst Control Lett, 2015, 85: 23–32
24 Guo Y, Wang Z. Stability of Markovian jump systems with generally uncertain transition rates. J Franklin Inst, 2013, 350: 2826–2836