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Design of generalized zero-determinant strategies in time-variant game environment

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Abstract This paper investigates the control strategies in repeated games with time-variant environment, called generalized zero-determinant (GZD) strategies, which can enforce the linear relationship between collective welfare and environmental quality. First, we give the algebraic model for Markovian profile evolutionary dynamics and then integrate it with Markovian state evolutionary equation into an eco-evolutionary algebraic system. Based on this algebraic form, a simple algebraic formula is provided to design GZD strategies in a repeated three-player game with two different environments, and then some interesting results are presented. It is shown that when the two environments are indistinguishable, the GZD strategies will degenerate to the zero-determinant (ZD) strategies. Finally, to reduce the computation complexity and highlight the importance of coordination to succeed in large groups, the group-based method is proposed to design GZD strategies.

Keywords repeated game, generalized zero-determinant strategies, time-variant environment, group-based method, Semi-tensor product

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1 Introduction

Repeated games have provided a general and formal framework to describe the long-term behaviors and reciprocity of rational individuals [1]. The most famous model is the repeated prisoner's dilemma game and it is found that mutual cooperation would take place over the long term. Recently, finite-memory strategies in repeated games have been widely studied since the rationality of real players is bounded [2–4]. However, Press and Dyson [5] have proved that the long-memory strategies have no advantage over the memory-one strategies and proposed a new kind of memory-one strategies in the repeated prisoner's dilemma game, called zero-determinant (ZD) strategies. Since then the ZD strategies have made significant progress in many applications [6–9].

For repeated games with two players, ZD strategies enable one player to unilaterally set a linear relationship between his own payoff and that of the opponent, regardless of the opponent's strategies. Subsequently, ZD strategies have also been extended to repeated multi-player games [10, 11]. The authors have studied the ZD strategies in multi-player public goods games [10] and the existence of ZD strategies in finitely repeated multi-player games with a discount factor has been investigated in [11]. In addition, the evolutionary stability of ZD strategies has been substantially discussed [12, 13]. However, these previous studies have typically assumed that interacting individuals play games repeatedly in a constant social and natural environment. In contrast, in many practical applications, the state of the game environment is adaptive and the feasible payoffs of players can be influenced by random factors when the decision-making process is under dynamical environments.

Stochastic games [14] have been introduced to describe the repeated interactions between individuals in which the underlying state of the environment changes randomly and it often influences players' actions and their payoffs [15]. As a special class of stochastic games, Liu and Wu [16] have discussed the Markovian eco-evolutionary feedback model between environment and human behavior. Then a new extension of ZD strategies, called the welfare-time strategy, is proposed to build a linear relationship between collective welfare and the environmental quality. Compared with [16], we extend the game model from two-player two-action to multi-player multi-action case. In addition, people often naturally form different types of groups based on common characteristics, such as geographical

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location, cultural background, and social status. The group-based method thereby provides an effective way for analyzing and solving decision-making problems in game models with a large number of players [17]. Motivated by this, an algebraic formula is derived to design group-based ZD strategies in this paper.

As a novel matrix product, the semi-tensor product (STP) of matrices [18,19] has found successful applications in numerous fields, including biological systems [20–25], graph theory [26], game theory [27–29] and finite automata [30]. Using the algebraic representation developed by STP, a new formula is given to design ZD strategies for multi-player games with asymmetric strategies [31]. In [32], the ZD strategies for multi-player games with multiple memories are investigated. The authors have discussed the correlation between the expected payoffs and the structure of the payoff vector when players use ZD strategies in repeated symmetric or skew-symmetric games. In this paper, we also utilize STP to investigate the ZD strategies of repeated multi-player games. However, compared with the previous results in [31, 32], we consider the stochastic game model in which the environment can be changed in response to the actions of players. Besides, taking into account of the relationship between players' payoffs and environment, we give the definitions of collective welfare and environmental quality respectively, based on which the coupling correlations between them are clarified. The main contributions of this article include three aspects.

(1) Using the algebraic representation based on the STP method, an equivalent algebraic expression of Markovian eco-evolutionary dynamics is presented for repeated multi-player games with time-variant environment.

(2) Taking the dissemination of public opinion as the research background, the generalized zero-determinant (GZD) strategies are introduced to explore the linear relationship between payoffs for netizens involved in spreading information and the environmental quality. A simple formula is provided to design GZD strategies for a repeated three-player game with two different environments.

(3) To simplify the computational complexity, a group-based method for designing GZD strategies is derived. It is easy to verify that the total number of strategies is reduced from k^r to $\frac{(k+r-1)!}{(k-1)!r!}$ by aggregating r players into one group, where k is the number of strategies for each player. Some necessary and sufficient conditions are presented to ensure that the collective welfare and environmental quality are positively or negatively correlated.

This paper is organized as follows. Some necessary preparations are given in Section 2. Section 3 presents the matrix expression of eco-evolutionary dynamics. Section 4 explores GZD strategies in repeated three-player games with two environments. The group-based method is proposed to design GZD strategies in Section 5. An interesting discussion and a concise conclusion are shown in Section 6.

2 Preliminaries

For statement ease, we first give some notations.

- $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
- $\mathrm{Col}(M)$: the set of columns of M; $\mathrm{Col}_i(M)$ is the i-th column of M.
- M^*: the adjoint matrix of M.
- $\mathcal{D}_k := \{1, 2, \dots, k\},\ k \geqslant 2$.
- $\Delta_k := \left\{\delta_k^i | i = 1, \dots, k\right\}$, where δ_k^i is the i-th column of the identity matrix I_k.
- $L \in \mathcal{M}_{m\times n}$ is called a logical matrix, if and only if $\mathrm{Col}(L) \in \Delta_m$, and it can be simply expressed as $L = \delta_m[i_1, i_2, \dots, i_n]$.
- Υ^m is the set of m dimensional random vectors and $\Upsilon_{m\times n}$ is the set of $m \times n$ random matrices.
- $\mathbf{1}_k := (\underbrace{1, 1, \dots, 1}_{k})^{\mathrm{T}}$.
- $\mathrm{Span}(M)$: the subspace spanned by $\mathrm{Col}(M)$.

2.1 Semi-tensor product of matrices

STP is the fundamental tool in this paper. More details are described in [18, 19].

Definition 1. Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. Then STP of M and N, denoted by $M \ltimes N$, is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \tag{1}$$

where $\otimes$ is the Kronecker product.

Proposition 1. Let $X \in \mathbb{R}^n$ be a column and M be a matrix. Then $X \ltimes M = (I_n \otimes M)\,X$.

Theorem 1. Let $x_i \in \mathcal{D}_k$, $i = 1, \dots, n$, $f : \mathcal{D}_k^n \to \mathcal{D}_k$ (or $\mathcal{D}_k^n \to \mathbb{R}$) be a k-valued logical (or pseudo-logical) function. Then there exists a unique $M_f \in \mathcal{L}_{k \times k^n}$ (or $M_f \in \mathbb{R}^{k^n}$), such that

$$f(x_1, \dots, x_n) = M_f \ltimes_{i=1}^n x_i. \tag{2}$$

M_f is called the structure matrix (or structure vector) of f.

Definition 2. Let $M \in \mathcal{M}_{p \times m}$, $N \in \mathcal{M}_{q \times m}$. Then the Khatri-Rao Product is defined as

$$M * N = [\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \cdots \mathrm{Col}_m(M) \ltimes \mathrm{Col}_m(N)]. \tag{3}$$

2.2 Finite normal game and repeated game

Definition 3. Consider a finite normal game $G = (N, A, C)$, including three fundamental ingredients:

(i) Let $N = \{1, 2, \dots, n\}$ denote the set of players;

(ii) Let $A = \prod_{i=1}^n A_i$ denote the strategy profile, where $A_i = \{a_1, a_2, \dots, a_{k_i}\}$ is the set of strategies for player i, $i = 1, \dots, n$;

(iii) $C = (c_1, \dots, c_n) \in \mathbb{R}^n$, and define $c_i : A \to \mathbb{R}$ as the payoff function of player i.

Let $G \in \mathcal{G}_{[n;k_1,k_2,\dots,k_n]}$ denote such finite games. Then we give the definition of repeated multi-player games with time-variant environment.

Definition 4. A repeated multi-player game with time-variant environment can be denoted by $G = (N, S, A, C, Q)$, where

(1) $N = \{1, 2, \dots, n\}$ is the set of players;

(2) $S = \{s_1, s_2, \dots, s_m\}$ is the set of states, where s_1 is initial state;

(3) $A_i(s)$ is the set of strategies for player i in state $s \in S$. In this paper, we assume $A_i(s) = A_i$ for any $s \in S$;

(4) $C = (c_1, \dots, c_n)$, where $c_i : S \times A \to \mathbb{R}$ is called the payoff function of player i;

(5) $Q : S \times A \to \Delta(S)$ is the state transition function. $\Delta(S)$ represents the probability distribution over the set S.

3 Matrix expression for repeated multi-player games with time-variant environment

Consider a finite game $G \in \mathcal{G}_{[n;k_1,k_2,\dots,k_n]}$ played repetitively, all the players update their strategies based on historical information. This paper only considers repeated games with memory-one strategies. When the state of game environment is time-invariant, the strategy evolutionary dynamics can be described as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \dots, x_n(t)), \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \dots, x_n(t)), \end{cases} \tag{4}$$

where $x_i(t) \in \Upsilon_{k_i}$ is the strategy of player i at time t, and $f_i : \Upsilon_k \to \Upsilon_{k_i}$ denotes the stochastic rule, $k = \prod_{i=1}^n k_i$, $i = 1, \dots, n$.

Let $x(t) = \ltimes_{i=1}^n x_i(t)$, then Eq. (4) can be converted into

$$\mathbb{E}x_i(t+1) = L_i \mathbb{E}x(t), \ i = 1, 2, \dots, n, \tag{5}$$

where

$$L_i = \begin{bmatrix} p_{i,1}^1 & p_{i,1}^2 & \cdots & p_{i,1}^k \\ \vdots & \vdots & \ddots & \vdots \\ p_{i,k_i}^1 & p_{i,k_i}^2 & \cdots & p_{i,k_i}^k \end{bmatrix} \in \Upsilon_{k_i \times k}, \ i = 1, 2, \dots, n, \tag{6}$$

and $p_{i,j}^l = \Pr(x_i(t+1) = j | x(t) = l)$.

Multiply both sides of (5), we can obtain the matrix expression of Markovian profile dynamics

$$\mathbb{E}x(t+1) = L \mathbb{E}x(t), \tag{7}$$

Table 1 Payoff matrix of game 1.

$c\backslash a$	111	112	121	122	211	212	221	222
c_1	R_1	K_1	K_1	S_1	T_1	L_1	L_1	P_1
c_2	R_1	K_1	T_1	L_1	K_1	S_1	L_1	P_1
c_3	R_1	T_1	K_1	L_1	K_1	L_1	S_1	P_1

Table 2 Payoff matrix of game 2.

$c\backslash a$	111	112	121	122	211	212	221	222
c_1	R_2	K_2	K_2	S_2	T_2	L_2	L_2	P_2
c_2	R_2	K_2	T_2	L_2	K_2	S_2	L_2	P_2
c_3	R_2	T_2	K_2	L_2	K_2	L_2	S_2	P_2

where $L = L_1 * L_2 * \cdots * L_n \in \Upsilon_{k\times k}$ is called the transition matrix.

Next we consider a repeated multi-player game with time-variant environment, the transitions between different states can be deterministic or stochastic, state-dependent or state-independent. This paper only considers the probabilistic state-dependent case, then the strategy dynamics can be expressed as

$$\begin{cases} x_1(t+1) = f_1^{s(t)}(x_1(t), \cdots x_n(t)), \\ \quad \vdots \\ x_n(t+1) = f_n^{s(t)}(x_1(t), \cdots x_n(t)), \end{cases} \tag{8}$$

where $f_i^{s(t)} : \Upsilon_k \to \Upsilon_{k_i}$ is a state-dependent function.

Assume that the structure matrices of $f_i^{s(t)}$ is $L_i^{s(t)}$, then the matrix expression of Markovian profile dynamics is

$$\mathbb{E}x(t+1) = L^{s(t)}\mathbb{E}x(t) = L\mathbb{E}s(t)\mathbb{E}x(t), \tag{9}$$

where $L^{s(t)} = L_1^{s(t)} * L_2^{s(t)} * \cdots * L_n^{s(t)} \in \Upsilon_{k\times k}$, and $L = [L^1, L^2, \ldots, L^m] \in \Upsilon_{k\times mk}$.

Similarly, we have the matrix expression of Markovian state dynamics as

$$\mathbb{E}s(t+1) = Q\mathbb{E}s(t)\mathbb{E}x(t), \tag{10}$$

where $Q \in \Upsilon_{m\times mk}$. Set $z(t) = s(t)x(t)$, then the dynamics (9) and (10) can be integrated into an algebraic Markovian eco-evolutionary equation

$$\mathbb{E}z(t+1) = M\mathbb{E}z(t), \tag{11}$$

where $M = Q * L \in \Upsilon_{mk\times mk}$.

With the number of players and strategies increasing, as well as the effect of environmental stochasticity, the computational complexity becomes one of the main challenging issues. For simplicity, we consider a repeated three-player prisoner's dilemma game with two different environments in this paper.

In the background of sudden public events, online media platforms can adopt both active and passive control measures, where active control refers to the real-time control and review of information content by online media platforms, but passive control means allowing the dissemination of public opinion information to gain traffic and popularity when facing a sudden public event. For a certain topic of online public opinion, netizens usually have two different choices: spreading and not spreading. The benefits of netizens are closely related to the control behavior of online media platforms. First, we make the following assumptions: (1) there are only positive and negative regulation which can be characterized as two environment states (i.e., games 1 and 2); (2) the finite game is prisoner's dilemma, (3) the transition function is probabilistic state-dependent; (4) the action "not spread" means "cooperation" (C) and "spread" means "betrayal" (D). Then the payoff matrices are shown in Tables 1 and 2.

The game transition process can be shown in Figure 1, where f_{mn} is the probability of players playing game 1 in the next step, m represents the current state ($m = 1, 2$) and n ($n = 0, 1, 2$) is the number of cooperators at the current time step.

To ensure that the payoffs of the betrayers are higher than those of the cooperators [13], it needs to satisfy that $T_m > R_m > L_m > K_m > P_m > S_m$, $R_m > (T_m + K_m)/2$ and $K_m > (L_m + S_m)/2$, ($m = 1, 2$). According to

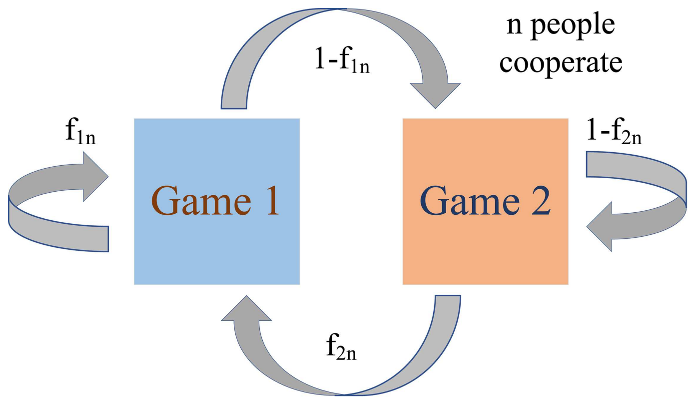

Figure 1 (Color online) Game transition diagram.

Theorem 1, we have the vectors of payoff functions

$$\begin{aligned} V_1^c &= (R_1, K_1, K_1, S_1, T_1, L_1, L_1, P_1, R_2, K_2, K_2, S_2, T_2, L_2, L_2, P_2), \\ V_2^c &= (R_1, K_1, T_1, L_1, K_1, S_1, L_1, P_1, R_2, K_2, T_2, L_2, K_2, S_2, L_2, P_2), \\ V_3^c &= (R_1, T_1, K_1, L_1, K_1, L_1, S_1, P_1, R_2, T_2, K_2, L_2, K_2, L_2, S_2, P_2). \end{aligned} \tag{12}$$

In (9), when $s(t) = 1$, we have

$$L_1^1 = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ 1-p_1 & 1-p_2 & 1-p_3 & 1-p_4 & 1-p_5 & 1-p_6 & 1-p_7 & 1-p_8 \end{bmatrix},$$

$$L_2^1 = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 & q_8 \\ 1-q_1 & 1-q_2 & 1-q_3 & 1-q_4 & 1-q_5 & 1-q_6 & 1-q_7 & 1-q_8 \end{bmatrix},$$

$$L_3^1 = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\ 1-r_1 & 1-r_2 & 1-r_3 & 1-r_4 & 1-r_5 & 1-r_6 & 1-r_7 & 1-r_8 \end{bmatrix},$$

and

$$L^1 = L_1^1 * L_2^1 * L_3^1 = \begin{bmatrix} p_1q_1r_1 & \cdots & p_8q_8r_8 \\ \vdots & \cdots & \vdots \\ (1-p_1)(1-q_1)(1-r_1) & \cdots & (1-p_8)(1-q_8)(1-r_8) \end{bmatrix},$$

where $p_j = \Pr(x_1(t+1) = 1|x(t) = j)$; $q_j = \Pr(x_2(t+1) = 1|x(t) = j)$ and $r_j = \Pr(x_3(t+1) = 1|x(t) = j)$, $j = 1, \ldots, 8$. Similarly, when $s(t) = 2$, we have

$$L^2 = L_1^2 * L_2^2 * L_3^2 = \begin{bmatrix} g_1h_1l_1 & \cdots & g_8h_8l_8 \\ \vdots & \cdots & \vdots \\ (1-g_1)(1-h_1)(1-l_1) & \cdots & (1-g_8)(1-h_8)(1-l_8) \end{bmatrix},$$

where $g_j = \Pr(x_1(t+1) = 1|x(t) = j)$; $h_j = \Pr(x_2(t+1) = 1|x(t) = j)$ and $l_j = \Pr(x_3(t+1) = 1|x(t) = j)$, $j = 1, \ldots, 8$.

Define $L = [L^1, L^2]$, and in (10), we have

$$Q = \begin{bmatrix} f_{13} & f_{12} & \cdots & f_{10} & f_{23} & f_{22} & \cdots & f_{20} \\ 1-f_{13} & 1-f_{12} & \cdots & 1-f_{10} & 1-f_{23} & 1-f_{22} & \cdots & 1-f_{20} \end{bmatrix} \in \Upsilon_{2\times 16}.$$

Then the matrix expression of Markovian eco-evolutionary equation (11) is obtained and we calculate

$$M = \begin{bmatrix} p_1q_1r_1f_{13} & \cdots & p_8q_8r_8f_{10} & g_1h_1l_1f_{23} & \cdots & g_8h_8l_8f_{20} \\ p_1q_1(1-r_1)f_{13} & \cdots & p_8q_8(1-r_8)f_{10} & \vdots & \cdots & \vdots \\ p_1(1-q_1)r_1f_{13} & \cdots & p_8(1-q_8)r_8f_{10} & (1-g_1)(1-h_1)(1-l_1)f_{23} & \cdots & (1-g_8)(1-h_8)(1-l_8)f_{20} \\ \vdots & \cdots & \vdots & g_1h_1l_1(1-f_{23}) & \cdots & g_8h_8l_8(1-f_{20}) \\ (1-p_1)(1-q_1)(1-r_1)f_{13} & \cdots & (1-p_8)(1-q_8)(1-r_8)f_{10} & \vdots & \vdots & \vdots \\ p_1q_1r_1(1-f_{13}) & \cdots & p_8q_8r_8(1-f_{10}) & (1-g_1)(1-h_1)(1-l_1)(1-f_{23}) & \cdots & (1-g_8)(1-h_8)(1-l_8)(1-f_{20}) \\ \vdots & \vdots & \vdots & & & \\ (1-p_1)(1-q_1)(1-r_1)(1-f_{13}) & \cdots & (1-p_8)(1-q_8)(1-r_8)(1-f_{10}) & & & \end{bmatrix} \in \Upsilon_{16\times 16}. \tag{13}$$

In this matrix, $p_1q_1(1-r_1)$ means the probability that profile is $\delta_8^2 \sim (1,1,2)$ at the next step while the current state is game 1 and the profile is $\delta_8^1 \sim (1,1,1)$. Then the probability that they play in game 2 at the next step is denoted as $p_1q_1(1-r_1)(1-f_{13})$ when all players cooperate at the current moment.

4 GZD strategies for repeated three-player games with two environments

In this section, we give an algebraic formula to design GZD Strategies, which is based on the properties of Markov transition matrix M in (11).

A Markovian chain is ergodic if it is irreducible and aperiodic, then the transition matrix M has a unique stationary distribution. Define $\lambda \in \Upsilon_{2k}$ as the stationary vector with respect to a unit eigenvalue, then we have $M\lambda = \lambda$, which is equivalent to

$$(M - I_{2k})\lambda = 0. \tag{14}$$

For a repeated three-player game, the stationary distribution is defined as

$$\lambda = (\sigma_1^1, \sigma_2^1, \ldots, \sigma_8^1, \sigma_1^2, \sigma_2^2 \ldots, \sigma_8^2)^{\mathrm{T}},$$

where the element σ_j^m represents the proportion of the time that the profile δ_8^j in game m across all the possible profiles and games, $j = 1, \ldots, 8$, $m = 1, 2$. Next, we explore the linear relationship between the collective welfare and the environmental quality, which are defined as follows.

Definition 5. Define N_m as the proportion of the time in game m $(m = 1, 2)$, then N_m is called environmental quality of game m. Let

$$f = (f_{13}, f_{12}, f_{12}, f_{11}, f_{12}, f_{11}, f_{11}, f_{10}, f_{23}, f_{22}, f_{22}, f_{21}, f_{22}, f_{21}, f_{21}, f_{20}),$$

then

$$\begin{aligned} N_1 &= \sum_{m=1}^{2} [f_{m3}\sigma_1^m + f_{m2}(\sigma_2^m + \sigma_3^m + \sigma_5^m) + f_{m1}(\sigma_4^m + \sigma_6^m + \sigma_7^m) + f_{m0}\sigma_8^m] \\ &= f \cdot \lambda, \end{aligned}$$

and $N_2 = 1 - N_1$.

Definition 6. The expected payoffs for three players are expressed as $\mathbb{E}c_i = V_i^c \lambda$, $i = 1, 2, 3$. Then

$$W = (V_1^c + V_2^c + V_3^c) \cdot \lambda = V \cdot \lambda = \sum_{i=1}^{3} \mathbb{E}c_i$$

is called the collective welfare, where $V = V_1^c + V_2^c + V_3^c$.

Lemma 1 ([31])**.** Consider a finite game $G \in \mathcal{G}_{[3;k_1,k_2,k_3]}$ played repetitively in two environments. The strategy extraction vector for player i with strategy j is denoted by $\Xi_{i,j} \in \mathbb{R}^{2k}$, $j = 1, \ldots, k_i$, $i = 1, 2, 3$, $k = k_1k_2k_3$, which is defined as

$$\Xi_{i,j} = \ltimes_{\mu=0}^{3} \kappa_\mu, \tag{15}$$

where

$$\kappa_\mu = \begin{cases} \mathbf{1}_2, & \mu = 0; \\ \delta_{k_i}^j, & \mu = i; \\ \mathbf{1}_{k_i}, & \mu \neq i. \end{cases}$$

Then we have

$$\Xi_{i,j}^{\mathrm{T}} M = \sum_{r \in \Phi_{i,j}} \mathrm{Row}_r(M), \tag{16}$$

where $\Phi_{i,j} = \{a \in A | a_i = j\} \subseteq A$ is the set of profiles that player i uses strategy j in profile $a \in A$.

Define z_i as the i-th row of matrix M, then $z_{i,j} = [\hat{z}_{i,j}^1, \hat{z}_{i,j}^2, \ldots, \hat{z}_{i,j}^8, \tilde{z}_{i,j}^1, \tilde{z}_{i,j}^2, \ldots, \tilde{z}_{i,j}^8], \forall i \in N, j \in A_i$, that is, in game 1, $\hat{z}_{i,j}^s = \Pr(x_i(t+1) = j | x(t) = s)$; and in game 2, $\tilde{z}_{i,j}^s = \Pr(x_i(t+1) = j | x(t) = s)$, $s = 1, \ldots, 8$. According to (14) and basic knowledge of linear algebra, for an arbitrary vector $e \in \mathbb{R}^{16}$, it is easy to calculate

$$e \cdot \lambda = \theta \det \begin{bmatrix} z_1 - (\delta_{16}^1)^{\mathrm{T}} \\ z_2 - (\delta_{16}^2)^{\mathrm{T}} \\ \vdots \\ z_{16} - (\delta_{16}^{15})^{\mathrm{T}} \\ e \end{bmatrix} = \det \begin{bmatrix} z_1 - (\delta_{16}^1)^{\mathrm{T}} \\ \vdots \\ z_{i,j} - \Xi_{i,j}^{\mathrm{T}} \\ \vdots \\ e \end{bmatrix}, \tag{17}$$

where θ is a non-zero constant, and the reason why the latter equal sign is valid is that we add all rows belonging to $\Phi_{i,j}$ to a certain row of $\Phi_{i,j}$. Obviously, if $z_{i,j} - \Xi_{i,j}^{\mathrm{T}}$ is proportional to e, it follows $e \cdot \lambda = 0$, that is,

$$z_{i,j} - \Xi_{i,j}^{\mathrm{T}} = ce,$$

where c is a non-zero adjustable parameter. Based on the above analysis, we have the following result.

Theorem 2. Consider an infinitely repeated game $G \in \mathcal{G}_{[3;2,2,2]}$ with two transformed environmental states. Assume player i aims to set the linear relationship between N_1 and W as

$$\alpha_1 + \alpha_2 N_1 + \alpha_3 W = 0. \tag{18}$$

Then the GZD strategies can be designed as

$$z_{i,j} = \eta_1 \mathbf{1}_{16}^{\mathrm{T}} + \eta_2 f + \eta_3 V + \Xi_{i,j}^{\mathrm{T}}, \quad i = 1, 2, 3, \quad j = 1, 2, \tag{19}$$

where the parameters α_r and $\eta_r = c\alpha_r$, $r = 1, 2, 3$, are determined unilaterally by player i.

Proof. Let

$$e = \alpha_1 \mathbf{1}_{16}^{\mathrm{T}} + \alpha_2 f + \alpha_3 (V_1^c + V_2^c + V_3^c).$$

If $z_{i,j}$ satisfies (19), that is

$$\begin{aligned} z_{i,j} =& \eta_1 \mathbf{1}_{16}^{\mathrm{T}} + \eta_2 f + \eta_3 V + \Xi_{i,j}^{\mathrm{T}} \\ =& c(\alpha_1 \mathbf{1}_{16}^{\mathrm{T}} + \alpha_2 f + \alpha_3 V) + \Xi_{i,j}^{\mathrm{T}} \\ =& ce + \Xi_{i,j}^{\mathrm{T}}. \end{aligned}$$

Bring it into (17), it is obvious that $e \cdot \lambda = 0$. On the other hand, we have

$$\begin{aligned} e \cdot \lambda &= (\alpha_1 \mathbf{1}_{16}^{\mathrm{T}} + \alpha_2 f + \alpha_3 V) \cdot \lambda \\ &= \alpha_1 \mathbf{1}_{16}^{\mathrm{T}} \cdot \lambda + \alpha_2 f \cdot \lambda + \alpha_3 V \cdot \lambda \\ &= \alpha_1 + \alpha_2 N_1 + \alpha_3 W, \end{aligned}$$

which leads to (18).

Remark 1. (i) The purpose of (16) is to select the set involving $z_{i,j}$ from matrix M, and such set is $\Phi_{i,j}$.

(ii) According to [33], if we define $T_i = [\Xi_i^{\mathrm{T}} M - \Xi_i^{\mathrm{T}}] \in \mathcal{M}_{2\times 16}$, $\mathbb{V} = [\mathbf{1}_{16}, (V_1^c + V_2^c + V_3^c)^{\mathrm{T}}, f^{\mathrm{T}}] \in \mathcal{M}_{16\times 3}$, where $\Xi_i = [\Xi_{i,1}, \Xi_{i,2}]$, then the GZD strategies can be existed if and only if

$$\mathrm{Span}({T_i}^{\mathrm{T}}) \cap \mathrm{Span}(\mathbb{V}) \neq \{\mathbf{0}_{16}\}.$$

It is shown that there exists at least one set of coefficients such that $l_1(z_{i,1} - \Xi_{i,1}^{\mathrm{T}}) + l_2(z_{i,2} - \Xi_{i,2}^{\mathrm{T}}) = \alpha_1 \mathbf{1}_{16}^{\mathrm{T}} + \alpha_2 f + \alpha_3 V$, which is essentially equivalent to (19).

(iii) Since $N_2 = 1 - N_1$, we can also set the linear relationship between N_2 and W, which is expressed as $\alpha_1 + \alpha_2(1 - N_1) + \alpha_3 W = 0$.

It is noted that Theorem 2 can also be extended to repeated games with multiple players and multiple strategies, but the corresponding complexity will be significantly high. Without loss of generality, we make the following assumptions.

(i) Let $\beta_1 = \frac{\alpha_3}{\alpha_2}$, $\beta_2 = \frac{\alpha_1}{\alpha_2}$, $\alpha_2 \neq 0$. Then we have

$$N_1 + \beta_1 W + \beta_2 = 0.$$

(ii) Using matrix M in (13) and strategy extraction vector (15), we have

$$\Xi_{1,1}^{\mathrm{T}} = [1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0].$$

It follows that

$$\begin{aligned} z_{1,1} =& \Xi_{1,1}^{\mathrm{T}} M = \sum_{r \in \Phi_{1,1}} \mathrm{Row}_r(M) \\ =& [\hat{z}_{1,1}^1, \hat{z}_{1,1}^2, \hat{z}_{1,1}^3, \hat{z}_{1,1}^4, \hat{z}_{1,1}^5, \hat{z}_{1,1}^6, \hat{z}_{1,1}^7, \hat{z}_{1,1}^8, \tilde{z}_{1,1}^1, \tilde{z}_{1,1}^2, \tilde{z}_{1,1}^3, \tilde{z}_{1,1}^4, \tilde{z}_{1,1}^5, \tilde{z}_{1,1}^6, \tilde{z}_{1,1}^7, \tilde{z}_{1,1}^8], \end{aligned}$$

which can be simply written as $z_{1,1} = [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8]$. Set $c = 1$, then we have

$$z_{1,1} = f + \beta_1 V + \beta_2 \mathbf{1}_{16}^{\mathrm{T}} + \Xi_{1,1}^{\mathrm{T}}. \tag{20}$$

(iii) In Tables 1 and 2, choose its payoffs as $R_m = a > 0$, $K_m = a - b_m$, $S_m = a - 2c_m$, $T_m = 2b_m > 0$, $L_m = c_m > 0$, $P_m = 0$ $(m = 1, 2)$ and $b_1 < b_2, c_1 < c_2$, that is, when online regulation is not as strict, players who choose to spread information will receive higher payoffs than those under strict regulation.

Based on the above assumptions, the GZD strategies in (20) can be rewritten as

$$\begin{cases} p_1 = f_{13} + 3a\beta_1 + \beta_2 + 1, \\ p_2 = p_3 = f_{12} + 2a\beta_1 + \beta_2 + 1, \\ p_4 = f_{11} + a\beta_1 + \beta_2 + 1, \\ p_5 = f_{12} + 2a\beta_1 + \beta_2, \\ p_6 = p_7 = f_{11} + a\beta_1 + \beta_2, \\ p_8 = f_{10} + \beta_2, \\ g_1 = f_{23} + 3a\beta_1 + \beta_2 + 1, \\ g_2 = g_3 = f_{22} + 2a\beta_1 + \beta_2 + 1, \\ g_4 = f_{21} + a\beta_1 + \beta_2 + 1, \\ g_5 = f_{22} + 2a\beta_1 + \beta_2, \\ g_6 = g_7 = f_{21} + a\beta_1 + \beta_2, \\ g_8 = f_{20} + \beta_2. \end{cases} \tag{21}$$

Obviously, $p_2 = p_3 = p_5 + 1$, $p_4 = p_6 + 1 = p_7 + 1$. Hence, to ensure the existence of GZD strategies, it is necessary to satisfy that

(i) $f_{m2} + 2a\beta_1 + \beta_2 = f_{m1} + a\beta_1 + \beta_2 = 0$, which leads to $\beta_2 = -f_{m1} - a\beta_1 = -f_{m2} - 2a\beta_1$, $m = 1, 2$;

(ii) $f_{11} = f_{21}, f_{12} = f_{22}$.

And when $\beta_2 = -f_{11} - a\beta_1$, we have

$$\begin{cases} p_1 = f_{13} + 2a\beta_1 - f_{11} + 1, \\ p_2 = p_3 = p_4 = 1, \\ p_5 = p_6 = p_7 = 0, \\ p_8 = f_{10} - a\beta_1 - f_{11}, \\ g_1 = f_{23} + 2a\beta_1 - f_{21} + 1, \\ g_2 = g_3 = g_4 = 1, \\ g_5 = g_6 = g_7 = 0, \\ g_8 = f_{20} - a\beta_1 - f_{21}, \end{cases} \tag{22}$$

where both β_1 and β_2 are solely determined by player 1. Then we have some interesting properties.

Proposition 2. Consider an infinitely repeated game $G \in \mathcal{G}_{[3;2,2,2]}$ with two environments.

(i) Environmental quality N_1 deteriorates as collective welfare W increases, if and only if $f_{m3} < f_{m2} < f_{m1} < f_{m0}$, $m = 1, 2$.

(ii) To ensure the existence of GZD strategies, the range of values for parameter β_1 is

$$\begin{aligned} &\max\left\{\frac{f_{m2} - f_{m3} - 1}{3R_m - 2K_m - T_m}, \frac{f_{m1} - f_{m3} - 1}{3R_m - 2L_m - S_m}, \frac{f_{m0} - f_{m1} - 1}{2K_m + L_m - 3P_m}, \frac{f_{m0} - f_{m1} - 1}{S_m + 2L_m - 3P_m}\right\} \leqslant \beta_1 \\ &\leqslant \min\left\{\frac{f_{m3} - f_{m2}}{2K_m + T_m - 3R_m}, \frac{f_{m3} - f_{m1}}{S_m + 2L_m - 3R_m}, \frac{f_{m0} - f_{m2}}{2K_m + T_m - 3P_m}, \frac{f_{m0} - f_{m1}}{S_m + 2L_m - 3P_m}\right\}, \end{aligned}$$

where $f_{m3} < f_{m2} < f_{m1} < f_{m0}$ or $f_{m3} > f_{m2} > f_{m1} > f_{m0}$ holds, $m = 1, 2$.

Proof. (i) According to equation $N_1 + \beta_1 W + \beta_2 = 0$, If $\beta_1 > 0$, then N_1 deteriorates with the increase of W. We first give the proof of necessity.

(Necessity) when $\beta_1 > 0$, due to $0 \leqslant p_j, g_j \leqslant 1$, $j = 1, 2, \ldots, 8$, it is necessary to ensure $f_{m3} - f_{m1} < 0$ and $f_{m0} - f_{m1} > 0$, $m = 1, 2$, in (22). That is, $f_{m3} < f_{m1} < f_{m0}$, $m = 1, 2$. Similarly, substitute $\beta_2 = -f_{m2} - 2a\beta_1$ into (21), we have $f_{m3} < f_{m2} < f_{m0}$, $m = 1, 2$. Finally, consider $\beta_2 = -f_{m1} - a\beta_1 = -f_{m2} - 2a\beta_1$, which is equivalent to $f_{m1} - f_{m2} = a\beta_1 > 0$, it yields $f_{m2} < f_{m1}$. In summary, we obtain that $f_{m3} < f_{m2} < f_{m1} < f_{m0}, m = 1, 2$.

The sufficiency is obvious. Hence, we can omit it here.

(ii) According to (20) and (22), we can obtain the following inequations for $s(t) = 1$:

$$\begin{cases} p_2 - p_1 = f_{12} - f_{13} + (2K_1 + T_1 - 3R_1)\beta_1 \geqslant 0, \\ p_4 - p_1 = f_{11} - f_{13} + (2L_1 + S_1 - 3R_1)\beta_1 \geqslant 0, \\ p_1 - p_5 = f_{13} - f_{12} - (2K_1 + T_1 - 3R_1)\beta_1 + 1 \geqslant 0, \\ p_1 - p_6 = f_{13} - f_{11} - (2L_1 + S_1 - 3R_1)\beta_1 + 1 \geqslant 0, \\ p_2 - p_8 = f_{11} - f_{10} + (2K_1 + T_1 - 3P_1)\beta_1 + 1 \geqslant 0, \\ p_4 - p_8 = f_{11} - f_{10} + (2L_1 + S_1 - 3P_1)\beta_1 + 1 \geqslant 0, \\ p_8 - p_5 = f_{10} - f_{12} - (2K_1 + T_1 - 3P_1)\beta_1 \geqslant 0, \\ p_8 - p_6 = f_{10} - f_{11} - (2L_1 + S_1 - 3P_1)\beta_1 \geqslant 0. \end{cases}$$

Similarly, for $s(t) = 2$, we have

$$\begin{cases} g_2 - g_1 = f_{22} - f_{23} + (2K_2 + T_2 - 3R_2)\beta_1 \geqslant 0, \\ g_4 - g_1 = f_{21} - f_{23} + (2L_2 + S_2 - 3R_2)\beta_1 \geqslant 0, \\ g_1 - g_5 = f_{23} - f_{22} - (2K_2 + T_2 - 3R_2)\beta_1 + 1 \geqslant 0, \\ g_1 - g_6 = f_{23} - f_{21} - (2L_2 + S_2 - 3R_2)\beta_1 + 1 \geqslant 0, \\ g_2 - g_8 = f_{21} - f_{20} + (2K_2 + T_2 - 3P_2)\beta_1 + 1 \geqslant 0, \\ g_4 - g_8 = f_{21} - f_{20} + (2L_2 + S_2 - 3P_2)\beta_1 + 1 \geqslant 0, \\ g_8 - g_5 = f_{20} - f_{22} - (2K_2 + T_2 - 3P_2)\beta_1 \geqslant 0, \\ g_8 - g_6 = f_{20} - f_{21} - (2L_2 + S_2 - 3P_2)\beta_1 \geqslant 0. \end{cases}$$

Table 3 Payoff matrix of game 1 with two groups.

$\Omega_1\backslash\Omega_2$	CC	$CD(DC)$	DD
C	$R_1^g, \tilde{R}_1^g$	$K_1^g, \tilde{L}_1^g$	$S_1^g, \tilde{T}_1^g$
D	$T_1^g, \tilde{S}_1^g$	$L_1^g, \tilde{K}_1^g$	$P_1^g, \tilde{P}_1^g$

Table 4 Payoff matrix of game 2 with two groups.

$\Omega_1\backslash\Omega_2$	CC	$CD(DC)$	DD
C	$R_2^g, \tilde{R}_2^g$	$K_2^g, \tilde{L}_2^g$	$S_2^g, \tilde{T}_2^g$
D	$T_2^g, \tilde{S}_2^g$	$L_2^g, \tilde{K}_2^g$	$P_2^g, \tilde{P}_2^g$

By solving the above two inequations, the range of values for β_1 can be provided.

Remark 2. (i) Similarly, environmental quality N_1 improves along with the increase of collective welfare W, if and only if $f_{m3} > f_{m2} > f_{m1} > f_{m0}, m = 1, 2$. That is to say, if more people do not spread rumours, then transforming into an environment with better environmental quality and stricter regulation is possible.

(ii) If the two environments are identical, then the GZD strategies will degenerate into the traditional ZD strategies.

(iii) In addition to the prisoner's dilemma, the GZD strategies remain valid for $2K_1 + T_1 = 2K_2 + T_2$ and $2L_1 + S_1 = 2L_2 + S_2$.

5 Group-based GZD strategies for repeated three-player games with two environments

In many situations, when players make decisions, they often choose to collaborate with others. Assume that $G \in \mathcal{G}_{[n;k]}$ is a symmetric game with n players. If each player acts individually without making any alliance, then the set of strategies for r $(r < n)$ players can be expressed as

$$\{(\underbrace{11\cdots1}_{r}), (\underbrace{11\cdots2}_{r}), \ldots, (\underbrace{kk\cdots k}_{r})\}.$$

Hence, the total number of strategies in this set is k^r. By contrast, when there are r players forming an alliance, we can ignore the order of players in this alliance, because $(\underbrace{11\cdots1}_{r-1}2)$ and $(2\underbrace{11\cdots1}_{r-1})$ are the same from the point of view for players outside this alliance. Then the total number of strategies is reduced from k^r to

$$\frac{(k+r-1)!}{(k-1)!r!}.$$

The computational complexity is obviously reduced by using the group-based method.

In this section, we only consider the group-based GZD strategies in repeated three-player games with two environments. Assume there are two groups, where $\Omega_1 = \{1\}$ is the first group and $\Omega_2 = \{2, 3\}$ is the second group, then the strategies for these two groups can be denoted as $A_{\Omega_1} = A_1^g = \{C, D\} \sim \{1, 2\}$, and $A_{\Omega_2} = A_2^g = \{CC, CD(DC), DD\} \sim \{1, 2, 3\}$. The payoff matrices of game m $(m = 1, 2)$ are shown in Tables 3 and 4, where $a = (a_1^g, a_2^g)$, $a_1^g \in A_1^g$, $a_2^g \in A_2^g$.

In the two tables, we assume $c_{\Omega_2} = c_2 + c_3$, that is, $\tilde{R}_m^g = 2R_m^g, \tilde{L}_m^g = K_m^g + T_m^g, \tilde{T}_m^g = 2L_m^g, \tilde{S}_m^g = 2K_m^g, \tilde{K}_m^g = L_m^g + S_m^g, \tilde{P}_m^g = 2P_m^g$.

To ensure rationality for maximizing each player's payoffs, according to [13], it should satisfy $T_m^g > R_m^g > L_m^g > K_m^g > P_m^g > S_m^g$; $R_m^g > (T_m^g + K_m^g)/2$; $K_m^g > (L_m^g + S_m^g)/2$. Then the payoffs for two groups can be expressed as

$$V_1^c = (R_1^g, K_1^g, S_1^g, T_1^g, L_1^g, P_1^g, R_2^g, K_2^g, S_2^g, T_2^g, L_2^g, P_2^g),$$
$$V_{\Omega_2}^c = (\tilde{R}_1^g, \tilde{L}_1^g, \tilde{T}_1^g, \tilde{S}_1^g, \tilde{K}_1^g, \tilde{P}_1^g, \tilde{R}_2^g, \tilde{L}_2^g, \tilde{T}_2^g, \tilde{S}_2^g, \tilde{K}_2^g, \tilde{P}_2^g).$$

Let $x_1^{mg} \in A_1^g$ and $x_2^{mg} \in A_2^g$, $m = 1, 2$. When $m = 1$, set

$$\begin{aligned}
p_j &= \Pr(x_1^{1g}(t+1) = 1 | x(t) = j);\\
q_j &= \Pr(x_2^{1g}(t+1) = 1 | x(t) = j);\\
r_j &= \Pr(x_2^{1g}(t+1) = 2 | x(t) = j);\\
1 - q_j - r_j &= \Pr(x_2^{1g}(t+1) = 3 | x(t) = j), \quad j = 1, \ldots, 6.
\end{aligned}$$

The strategy dynamics can be described as

$$\begin{aligned}\mathbb{E}x_1^{1g}(t+1) = L_1^{1g}\mathbb{E}x(t),\\ \mathbb{E}x_2^{1g}(t+1) = L_2^{1g}\mathbb{E}x(t),\end{aligned}$$

where

$$L_1^{1g} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ 1-p_1 & 1-p_2 & 1-p_3 & 1-p_4 & 1-p_5 & 1-p_6 \end{bmatrix},$$

and

$$L_2^{1g} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\ 1-q_1-r_1 & 1-q_2-r_2 & 1-q_3-r_3 & 1-q_4-r_4 & 1-q_5-r_5 & 1-q_6-r_6 \end{bmatrix}.$$

Then the strategy profile dynamics can be expressed as

$$\mathbb{E}x(t+1) = L^{1g}\mathbb{E}x(t),$$

where

$$L^{1g} = L_1^{1g} * L_2^{1g} = \begin{bmatrix} p_1q_1 & p_2q_2 & \cdots & p_6q_6 \\ p_1r_1 & p_2r_2 & \cdots & p_6r_6 \\ p_1(1-q_1-r_1) & p_2(1-q_2-r_2) & \cdots & p_6(1-q_6-r_6) \\ \vdots & \vdots & & \vdots \\ (1-p_1)(1-q_1-r_1) & (1-p_2)(1-q_2-r_2) & \cdots & (1-p_6)(1-q_6-r_6) \end{bmatrix}.$$

Similarly, we calculate the strategy profile dynamics for $m = 2$ as

$$\mathbb{E}x(t+1) = L^{2g}\mathbb{E}x(t),$$

where

$$L^{2g} = L_1^{2g} * L_2^{2g} = \begin{bmatrix} g_1l_1 & g_2l_2 & \cdots & g_6l_6 \\ \vdots & \vdots & & \vdots \\ (1-g_1)(1-l_1-h_1) & (1-g_2)(1-l_2-h_2) & \cdots & (1-g_6)(1-l_6-h_6) \end{bmatrix},$$

and

$$\begin{aligned}&g_j = \Pr(x_1^{2g}(t+1) = 1 | x(t) = j);\\ &l_j = \Pr(x_2^{2g}(t+1) = 1 | x(t) = j);\\ &h_j = \Pr(x_2^{2g}(t+1) = 2 | x(t) = j);\\ &1 - l_j - h_j = \Pr(x_2^{2g}(t+1) = 3 | x(t) = j), \quad j = 1, \ldots, 6.\end{aligned}$$

Define $L^g = [L^{1g}, L^{2g}]$. According to (10), set

$$Q^g = \begin{bmatrix} f_{13} & f_{12} & \cdots & f_{10} & f_{23} & f_{22} & \cdots & f_{20} \\ 1-f_{13} & 1-f_{12} & \cdots & 1-f_{10} & 1-f_{23} & 1-f_{22} & \cdots & 1-f_{20} \end{bmatrix} \in \Upsilon_{2\times 12}.$$

Then the Markovian eco-evolutionary dynamics for the group-based three-player games is

$$\mathbb{E}z(t+1) = M^g\mathbb{E}z(t), \tag{23}$$

where $M^g = Q^g * L^g \in \Upsilon_{12\times 12}$ is written as

$$\left[\begin{array}{cccc} p_1q_1f_{13} & p_2q_2f_{12} & \cdots & p_6q_6f_{10} \\ p_1r_1f_{13} & p_2r_2f_{12} & \cdots & p_6r_6f_{10} \\ p_1(1-q_1-r_1)f_{13} & p_2(1-q_2-r_2)f_{12} & \cdots & p_6(1-q_6-r_6)f_{10} \\ \vdots & \vdots & \cdots & \vdots \\ (1-p_1)(1-q_1-r_1)f_{13} & (1-p_2)(1-q_2-r_2)f_{12} & \cdots & (1-p_6)(1-q_6-r_6)f_{10} \\ p_1q_1(1-f_{13}) & p_2q_2(1-f_{12}) & \cdots & p_6q_6(1-f_{10}) \\ p_1r_1(1-f_{13}) & p_2r_2(1-f_{12}) & \cdots & p_6r_6(1-f_{10}) \\ p_1(1-q_1-r_1)(1-f_{13}) & p_2(1-q_2-r_2)(1-f_{12}) & \cdots & p_6(1-q_6-r_6)(1-f_{10}) \\ \vdots & \vdots & \vdots & \vdots \\ (1-p_1)(1-q_1-r_1)(1-f_{13}) & (1-p_2)(1-q_2-r_2)(1-f_{12}) & \cdots & (1-p_6)(1-q_6-r_6)(1-f_{10}) \end{array}\right.$$

$$\left.\begin{array}{cccc} g_1l_1f_{23} & g_2l_2f_{22} & \cdots & g_6l_6f_{20} \\ g_1h_1f_{23} & g_2h_2f_{22} & \cdots & g_6h_6f_{20} \\ g_1(1-l_1-h_1)f_{23} & g_2(1-l_2-h_2)f_{22} & \cdots & g_6(1-l_6-h_6)f_{20} \\ \vdots & \vdots & \cdots & \vdots \\ (1-g_1)(1-l_1-h_1)f_{23} & (1-g_2)(1-l_2-h_2)f_{22} & \cdots & (1-g_6)(1-l_6-h_6)f_{20} \\ g_1l_1(1-f_{23}) & g_2l_2(1-f_{22}) & \cdots & g_6l_6(1-f_{20}) \\ \vdots & \vdots & \vdots & \vdots \\ (1-g_1)(1-l_1-h_1)(1-f_{23}) & (1-g_2)(1-l_2-h_2)(1-f_{22}) & \cdots & (1-g_6)(1-l_6-h_6)(1-f_{20}) \end{array}\right].$$

Based on the algebraic form (23), we derive the group-based GZD strategies in three player model. Define the stationary distribution as

$$\lambda^g = (\sigma_1^1, \sigma_2^1, \ldots, \sigma_6^1, \sigma_1^2, \sigma_2^2 \ldots, \sigma_6^2)^{\mathrm{T}}.$$

Similar to Definitions 5 and 6, for the group-based three-player games, the environmental quality N_1^g is defined as

$$N_1^g = \sum_{m=1}^{2} [f_{m3}\sigma_1^m + f_{m2}(\sigma_2^m + \sigma_4^m) + f_{m1}(\sigma_3^m + \sigma_5^m) + f_{m0}\sigma_6^m] = f^g \cdot \lambda^g,$$

where $f^g = (f_{13}, f_{12}, f_{11}, f_{12}, f_{11}, f_{10}, f_{23}, f_{22}, f_{21}, f_{22}, f_{21}, f_{20})$.

The expected payoffs for player 1 and group Ω_2 are expressed as

$$\mathbb{E}c_1 = V_1^c \lambda^g,$$
$$\mathbb{E}c_{\Omega_2} = V_{\Omega_2}^c \lambda^g.$$

Then the collective welfare W^g for the group-based three-player games is defined as

$$W^g = (V_1^c + V_{\Omega_2}^c) \cdot \lambda^g = V^g \cdot \lambda^g = \mathbb{E}c_1 + \mathbb{E}c_{\Omega_2},$$

where $V^g = V_1^c + V_{\Omega_2}^c$.

In Lemma 1, let $i = 1, 2,\ k_1 = 2, k_2 = 3$, we can obtain the corresponding strategy extraction vector $\Xi_{i,j}^g \in \mathbb{R}^{12}$, and set

$$e = \alpha_1 \mathbf{1}_{12}^{\mathrm{T}} + \alpha_2 f^g + \alpha_3 (V_1^c + V_{\Omega_2}^c).$$

Similar to Theorem 2, we have the following result.

Theorem 3. Consider a group-based repeated three-player game with two environments. Assume $\Omega_1 = \{1\}$ is the first group and $\Omega_2 = \{2, 3\}$ is the second group. Group i aims to set the linear relationship between N_1^g and W^g as

$$\alpha_1 + \alpha_2 N_1^g + \alpha_3 W^g = 0.$$

Table 5 Payoff matrix of game m for two groups.

$\Omega_1\backslash\Omega_2$	CC	$CD(DC)$	DD
C	$a, 2a$	$a-b_m, a+b_m$	$a-2c_m, 2c_m$
D	$2b_m, 2a-2b_m$	$c_m, a-c_m$	$0, 0$

Then the group-based GZD strategies can be designed as

$$z^g_{i,j} = \eta_1 \mathbf{1}^{\mathrm{T}}_{12} + \eta_2 f^g + \eta_3 V^g + (\Xi^g_{i,j})^{\mathrm{T}}, \quad i = 1, 2,\ j \in A^g_i, \tag{24}$$

where the parameters α_r and $\eta_r = c\alpha_r$, $r = 1, 2, 3$, are determined unilaterally by group i.

For simplicity, let $\beta_1 = \frac{\alpha_3}{\alpha_2}$, $\beta_2 = \frac{\alpha_1}{\alpha_2}$, $\alpha_2 \neq 0$, it follows that

$$N^g_1 + \beta_1 W^g + \beta_2 = 0.$$

According to the algebraic equation (23) and the construction of strategy extraction vector (15), we have

$$(\Xi^g_{1,1})^{\mathrm{T}} = [1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0].$$

It yields that

$$\begin{aligned} z^g_{1,1} &= (\Xi^g_{1,1})^{\mathrm{T}} M^g \\ &= \textstyle\sum_{r\in\Phi_{1,1}} \mathrm{Row}_r(M^g) \\ &= [\hat{z}^1_{1,1}, \hat{z}^2_{1,1}, \hat{z}^3_{1,1}, \hat{z}^4_{1,1}, \hat{z}^5_{1,1}, \hat{z}^6_{1,1}, \tilde{z}^1_{1,1}, \tilde{z}^2_{1,1}, \tilde{z}^3_{1,1}, \tilde{z}^4_{1,1}, \tilde{z}^5_{1,1}, \tilde{z}^6_{1,1}], \end{aligned}$$

which can be simply expressed as $[p_1, p_2, p_3, p_4, p_5, p_6, g_1, g_2, g_3, g_4, g_5, g_6]$. Set $c = 1$, and then we have

$$z^g_{1,1} = f^g + \beta_1 V^g + \beta_2 \mathbf{1}^{\mathrm{T}}_{12} + (\Xi^g_{1,1})^{\mathrm{T}}. \tag{25}$$

In Table 3, set $R^g_m = a > 0$, $K^g_m = a - b_m$, $S^g_m = a - 2c_m$, $T^g_m = 2b_m > 0$, $L^g_m = c_m > 0$, $P^g_m = 0$, $(m = 1, 2)$ and $b_1 < b_2, c_1 < c_2$. Then we have $\tilde{R}^g_m = 2a, \tilde{L}^g_m = a + b_m, \tilde{T}^g_m = 2c_m, \tilde{S}^g_m = 2a - 2b_m, \tilde{K}^g_m = a - c_m, \tilde{P}^g_m = 0$, which is shown in Table 5.

To ensure the rationality of GZD strategies, it is necessary to meet that

(i) $\beta_2 = -f_{m1} - a\beta_1 = -f_{m2} - 2a\beta_1$, $m = 1, 2$;

(ii) $f_{11} = f_{21}, f_{12} = f_{22}$.

If $\beta_2 = -f_{11} - a\beta_1$, the GZD strategies (25) for group 1 can be rewritten as

$$\begin{cases} p_1 = f_{13} + 2a\beta_1 - f_{11} + 1, \\ p_2 = p_3 = 1, \\ p_4 = p_5 = 0, \\ p_6 = f_{10} - a\beta_1 - f_{11}, \\ g_1 = f_{23} + 2a\beta_1 - f_{21} + 1, \\ g_2 = g_3 = 1, \\ g_4 = g_5 = 0, \\ g_6 = f_{20} - a\beta_1 - f_{21}, \end{cases} \tag{26}$$

where both β_1 and β_2 are solely determined by group 1. Then the following properties can be obtained.

Proposition 3. Consider a group-based three-player repeated game with two environments.

(i) Environmental quality N^g_1 decreases as collective welfare W^g increases, if and only if $f_{m3} < f_{m2} < f_{m1} < f_{m0}$, $m = 1, 2$.

(ii) To ensure the existence of GZD strategies, the range of values for parameter β_1 is

$$\begin{aligned} &\max\left\{\frac{f_{m2} - f_{m3} - 1}{R^g_m + \tilde{R}^g_m - T^g_m - \tilde{S}^g_m}, \frac{f_{m0} - f_{m3} - 1}{R^g_m + \tilde{R}^g_m - L^g_m - \tilde{K}^g_m}, \frac{f_{m0} - f_{m2} - 1}{K^g_m + \tilde{L}^g_m - P^g_m - \tilde{P}^g_m}, \frac{f_{m0} - f_{m1} - 1}{S^g_m + \tilde{T}^g_m - P^g_m - \tilde{P}^g_m}\right\} \leqslant \beta_1 \\ &\leqslant \min\left\{\frac{f_{m3} - f_{m2}}{K^g_m + \tilde{L}^g_m - R^g_m - \tilde{R}^g_m}, \frac{f_{m3} - f_{m1}}{S^g_m + \tilde{T}^g_m - R^g_m - \tilde{R}^g_m}, \frac{f_{m2} - f_{m0}}{P^g_m + \tilde{P}^g_m - T^g_m - \tilde{S}^g_m}, \frac{f_{m1} - f_{m0}}{P^g_m + \tilde{P}^g_m - L^g_m - \tilde{K}^g_m}\right\}, \end{aligned}$$

where $f_{m3} < f_{m2} < f_{m1} < f_{m0}$ or $f_{m3} > f_{m2} > f_{m1} > f_{m0}$ holds, $m = 1, 2$. The proof process is similar to Proposition 2, and we omit it here.

Table 6 Payoff matrix of example 1 with $m = 1$.

$\Omega_1\backslash\Omega_2$	CC	$CD(DC)$	DD
C	$5, 10$	$2, 8$	$-1, 6$
D	$6, 4$	$3, 2$	$0, 0$

Table 7 Payoff matrix of example 1 with $m = 2$.

$\Omega_1\backslash\Omega_2$	CC	$CD(DC)$	DD
C	$5, 10$	$1.5, 8.5$	$-3, 8$
D	$7, 3$	$4, 1$	$0, 0$

Remark 3. In (26), $p_2 = p_3 = 1$ and $g_2 = g_3 = 1$, which means that if player 1 cooperates and at least one player in group 2 defects at the current step, regardless of the environments, player 1 will continue to cooperate at the next step. That is, player 1 forgives group 2 under the GZD strategies. Similarly, $p_4 = p_5 = 0$ and $g_4 = g_5 = 0$, that is, player 1 in each environment will continue to defect at the next step if he decides to defect and at least one player in group 2 cooperates at the current step. Finally, with the increase of β_1, player 1 who adopts the GZD strategy is more willing to cooperate at the next step when two groups decide to cooperate at present, and he will probably choose to defect at the next step when two groups decide to defect at the current step.

A numerical example for the group-based repeated three-player game with two environments is provided in the following.

Example 1. Consider a group-based repeated three-player game with two environments, its payoff matrix is described in Table 5. Set $a = 5, b_1 = 3, c_1 = 3, b_2 = 3.5, c_2 = 4$, then the payoff matrices are shown in Tables 6 and 7.

Set parameters $f_{13} = 0.85, f_{12} = 0.8, f_{11} = 0.2, f_{10} = 0.1, f_{23} = 0.9, f_{22} = 0.8, f_{21} = 0.2, f_{20} = 0.15$, then according to Proposition 3, environmental quality N_1^g increases as collective welfare W^g increases for $f_{m3} > f_{m2} > f_{m1} > f_{m0}$, $m = 1, 2$, and the boundary of β_1 is $-0.165 \leqslant \beta_1 \leqslant -0.07$. Choose $\beta_1 = -0.12$, then $\beta_2 = -f_{11} - a\beta_1 = 0.4$, group 1 can set the linear relationship between N_1^g and W^g as

$$N_1^g - 0.12W^g + 0.4 = 0,$$

where the GZD strategies (26) for group 1 will be $z_{1,1}^g = [0.45, 1, 1, 0, 0, 0.5, 0.5, 1, 1, 0, 0, 0.55]$.

6 Discussion and conclusion

For the ZD strategies, one player is able to unilaterally set the opponent's payoffs. For the welfare-time strategy in two-player games proposed in [16], there also exists a strategy that allows one player to control the value of the population welfare, which is called the determined-welfare (D-W) strategy. So, is there a D-W strategy like this in the case of three players?

Recall (18) and (19), let $\alpha_2 = 0$, and then the strategy for player 1 is $z_{1,1} = \alpha_1 \mathbf{1}_{16}^{\mathrm{T}} + \alpha_3 V + \Xi_{1,1}^{\mathrm{T}}$, where $c = 1$. That is, $\alpha_1 \mathbf{1}_{16}^{\mathrm{T}} + \alpha_3 V = 0$. Multiply both sides of this equation by λ simultaneously, which leads to $\alpha_1 + \alpha_3 W = 0$. Then W can be set as $-\frac{\alpha_1}{\alpha_3}$. And the D-W strategy can be designed as

$$\begin{cases} p_1 = 3R_1\alpha_3 + \alpha_1 + 1, \\ p_2 = p_3 = (2K_1 + T_1)\alpha_3 + \alpha_1 + 1, \\ p_4 = (S_1 + 2L_1)\alpha_3 + \alpha_1 + 1, \\ p_5 = (2K_1 + T_1)\alpha_3 + \alpha_1, \\ p_6 = p_7 = (S_1 + 2L_1)\alpha_3 + \alpha_1, \\ p_8 = 3P_1\alpha_3 + \alpha_1, \\ g_1 = 3R_2\alpha_3 + \alpha_1 + 1, \\ g_2 = g_3 = (2K_2 + T_2)\alpha_3 + \alpha_1 + 1, \\ g_4 = (S_2 + 2L_2)\alpha_3 + \alpha_1 + 1, \\ g_5 = (2K_2 + T_2)\alpha_3 + \alpha_1, \\ g_6 = g_7 = (S_2 + 2L_2)\alpha_3 + \alpha_1, \\ g_8 = 3P_2\alpha_3 + \alpha_1. \end{cases} \tag{27}$$

To ensure the existence of D-W strategy in (27), it is necessary to satisfy $2K_m + T_m = S_m + 2L_m, m = 1, 2$. However, the payoffs cannot satisfy $T_m > R_m > L_m > K_m > P_m > S_m$ and $K_m > (L_m + S_m)/2$ simultaneously. Hence, the D-W strategy in (27) is invalid for the GZD strategies with three players. Similar results are also obtained in the group-based model.

This paper investigates the linear relationship between collective welfare and environmental quality in repeated finite games with time-variant environment. According to the profile and state dynamics, an equivalent eco-evolutionary algebraic model is provided. Then the designing formulas of GZD strategies are obtained for repeated three-player games with two environments.

It is remarkable that there are many interesting issues to explore GZD strategies in the future. In particular, as the number of players increases, the control ability for a single player continuously decreases. Hence, the control ability of ZD strategies can be improved through different types of alliance mechanisms, where players not only need to consider how to optimize their own payoffs through ZD strategies, but also how to maximize collective interests in an alliance. From this point of view, the existence of GZD strategy alliances has been further investigated. Moreover, since game-theoretic approaches can be used to study several cooperative control problems of multiagent systems [34–36], we hope that our designed method may be applied to consensus problems and synchronization analysis of multiagent systems in further investigation [37, 38].

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