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Stability in distribution of nonlinear hybrid stochastic systems with non-differentiable time delays

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Abstract This paper delves into the analysis of a category of highly nonlinear hybrid stochastic systems with non-differentiable time delays, which has caught the attention of scholars in recent years due to its relevance to real-world applications. As an essential step in studying stochastic systems, a generalized Hasminskii-type theorem is initially formulated for the existence and uniqueness of the global solution. The central focus lies on addressing the crucial issue of stability in distribution. To this end, the paper presents several significant lemmas to examine the stability in distribution of this system and proposes some sufficient conditions. An example is provided to validate the accuracy and validity of the theoretical results.

Keywords stability in distribution, highly nonlinear, Hasminskii-type theorem, hybrid stochastic differential equation, non-differentiable time delay

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1 Introduction

Stochastic differential equations (SDEs) with Markov switching, also called hybrid SDEs (HSDEs), represent a class of dynamic models that capture abrupt changes in system structure and parameters, reflecting real-world phenomena encountered in diverse scientific and engineering domains [1–5]. These equations elegantly accommodate sudden shifts induced by environmental fluctuations, component malfunctions, data irregularities, and other stochastic influences. In numerous practical applications, ranging from biological systems to financial markets, scholars have increasingly focused on elucidating the behavior and properties of such systems.

Amidst this scholarly attention, the study of stability emerges as a pivotal pursuit, offering insights into the resilience and long-term dynamics of hybrid stochastic systems. Typically, analysis on stability revolves around the study of the trivial solution (equilibrium state) in terms of almost sure stability, moment stability and stability in probability [6–10]. Nonetheless, in many practical scenarios, the notion of stability as conventionally understood may prove too restrictive, especially considering the absence of deterministic steady states in numerous stochastic systems encountered in engineering and other fields. For instance, systems like multiple target tracking setups, fault-tolerant control systems, and flexible manufacturing systems often lack equilibrium states altogether, rendering discussions on trivial solution stability moot [11–13]. In such contexts, traditional notions of stability, focused on trivial solutions and deterministic equilibrium states, often fall short. Instead, there arises a need to assess whether the system's solution converges in distribution, a concept known as stability in distribution.

The exploration of stability in distribution extends beyond pure theoretical inquiry, bearing practical relevance. For instance, when studying population systems experiencing environmental fluctuations, the emphasis lies on identifying conditions under which the population can persist stochastically rather than face extinction. Hence, this research proves beneficial by shedding light on essential survival dynamics within ecological frameworks. The concept of stability in distribution, unlike conventional stability analysis of trivial solutions, poses a formidable challenge [14–17]. In order to address the challenges posed by stability in distribution, it is imperative to employ a number

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of sophisticated methods, including conditional expectation properties, indicator functions, probability measures, Cauchy sequence convergence, Chebyshev's inequality, as well as other related skills. Some preliminary results have been achieved through the efforts of researchers. For instance, the semi-linear HSDEs were investigated in [18], while HSDEs were explored in [14], and hybrid stochastic differential delay equations (HSDDEs) were examined in [15], marking significant contributions to the field. Subsequently, in 2014, the methodological improvements were made in [16] to the work of [15]. Notably, in 2019, contributions were made in [17] to the study of stochastic functional differential equations. However, it is important to note that these studies have largely overlooked scenarios where the drift and diffusion coefficients exhibit a highly nonlinear nature or where the delay functions within the system are non-differentiable. These complexities introduce additional challenges that warrant thorough investigation. Hence, in the following discussion, we will delve into a detailed exposition of these aspects.

On one hand, the adherence of diffusion and drift coefficients to linear growth conditions is often a prevailing assumption in the study of hybrid systems. However, many practical systems need to be accurately characterised by highly nonlinear HSDDEs, such as those found in financial and economic systems, population dynamics, and stochastic oscillators [19–21]. Consequently, extensive research has been devoted to studying the stability of HSDDEs with highly nonlinear coefficients in recent years [6, 7, 22–24]. However, there is a scarcity of research outcomes regarding the distributional stability of highly nonlinear systems.

On the other hand, regarding non-differentiable delay functions, delays are indispensable in real-world scenarios and often lead to adverse effects such as oscillations and instability within systems. Hence, investigating hybrid hyperdynamic equations with delays, namely HSDDEs, holds significant importance. However, many of the existing results share the common restriction that $\delta(t)$ is assumed to be differentiable and $\mathrm{d}\delta(t)/\mathrm{d}t < 1$, like in [6, 24]. This constraint is typically imposed due to the mathematical techniques employed, such as the method of time scaling. Yet, it may not reflect a realistic feature of HSDDE models. For instance, sawtooth delays or piecewise constant delays, often encountered in network-based control or sampled-data control, where the delay is frequently termed fast-varying delay, without the restriction on the delay derivatives. Also, data are usually buffered and sent through a network in packets traveling independently from each other, and the delay changes abruptly when processing proceeds from a packet to the subsequent one [10, 25]. A straightforward example is, for instance, the piecewise constant function, as follows:

$$\delta(t) = \sum_{m=0}^{\infty} K_m I_{[t_m, t_{m+1})}(t),$$

where $h_1 \leqslant K_m \leqslant h$. However, even such a straightforward function lacks differentiability. This clearly indicates the necessity of a weaker condition to substitute the differentiability assumption of $\delta(t)$ in the study of HSDDEs. Therefore, this paper conducts research on stochastic systems with non-differentiable delay functions.

In summary, our contributions are mainly focused on the following.

(1) An in-depth investigation into the stability in distribution of a category of highly nonlinear HSDDEs with non-differentiable time delays has been conducted, with sufficient criteria provided. New ideas and methods are offered for the study of many stochastic systems without deterministic steady states.

(2) The drift and diffusion coefficients of the systems under study exhibit highly nonlinear characteristics. The introduction of such highly nonlinear conditions weakens the reliance on linear growth conditions, greatly expanding the applicability range of the theorems. This innovation enables the coverage of a broader spectrum of practical systems and provides a more accurate description of their dynamic behaviors.

(3) The impact of delay on system dynamics has been considered. In practical applications, time delay is inevitable and crucially affects the speed of system response, accuracy of reaching targets, and timeliness of information transmission. However, existing literature often imposes conditions on the delay function being differentiable and its derivative being less than 1, which may not necessarily align with the characteristics of real systems. Thus, a class of HSDDEs with non-differentiable delays has been investigated, capturing the characteristics of real-world systems more accurately.

(4) An optimization of the sufficiency criterion has been performed, reducing the restrictions on the assumption conditions while maintaining applicability to a more generalized range of systems. This innovation enhances the practicality and universality of our research findings, providing more effective tools and methods for the analysis and application of real-world systems.

2 Preliminaries

Consider $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ to be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ that satisfies the usual conditions, and $B(t) = (B_1(t),\ \ldots, B_m(t))^{\mathrm{T}}$ an m-dimensional Brownian motion. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^+ = [0, \infty)$. Let $C(\mathbb{R}^n; \mathbb{R}^+)$ denote the family of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^+$. Let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions $\zeta : [-h, 0] \to \mathbb{R}^n$ with norm $\|\zeta\| = \sup_{-h\leqslant u\leqslant 0} |\zeta(u)|$. For $t \geqslant 0$, if $x(t)$ is an $\mathbb{R}^n$-value stochastic process, define $x_t = x_t(s) := \{x(t + s) : -h \leqslant s \leqslant 0\}$. For real numbers p and q are real numbers, then $p \wedge q = \min\{p, q\}$ and $p \vee q = \max\{p, q\}$. If A is a matrix, the trace norm is given by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. The indicator function I_A for a subset A of Ω is defined such that $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Consider $r(t)$, $t \geqslant 0$, a right-continuous Markov chain with values in a finite state space $S = \{1, 2, \ldots, N\}$ and a generator $Q = (\gamma_{ij})_{N\times N}$, where $\gamma_{ij} \geqslant 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. Let us assume that almost all of the sampled paths of $r(t)$ are right continuous.

Consider $C^2(\mathbb{R}^n \times S; \mathbb{R}^+)$ the set of nonnegative functions $V(x, i)$ defined on $\mathbb{R}^n \times S$ that are twice continuously differentiable in x. With respect to each $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}^+)$, it is useful to define

$$\mathcal{L}V(x, y, i) = \sum_{j=1}^{n} \gamma_{ij} V(x, j) + V_x(x, i) f(x, y, i) + \frac{1}{2}\mathrm{trace}\left\{g^{\mathrm{T}}(x, y, i) V_{xx}(x, i) g(x, y, i)\right\}$$

and

$$\begin{aligned} LV(x_1, x_2, \bar{x}_1, \bar{x}_2, i) =& \sum_{j=1}^{n} \gamma_{ij} V(x_1 - x_2, j) + V_x(x_1 - x_2, i) \left(f(x_1, \bar{x}_1, i) - f(x_2, \bar{x}_2, i)\right) \\ &+ \frac{1}{2}\mathrm{trace}\left[\left(g^{\mathrm{T}}(x_1, \bar{x}_1, i) - g^{\mathrm{T}}(x_2, \bar{x}_2, i)\right) V_{xx}(x - y, i) \left(g(x_1, \bar{x}_1, i) - g(x_2, \bar{x}_2, i)\right)\right], \end{aligned}$$

where

$$V_x(x, i) = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right), \quad V_{xx}(x, i) = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Consider a highly nonlinear HSDDE of the form

$$\mathrm{d}X(t) = f(X(t), X(t - \delta(t)), r(t))\mathrm{d}t + g(X(t), X(t - \delta(t)), r(t))\mathrm{d}B(t), \tag{1}$$

on $t \geqslant 0$ with the initial value

$$\begin{aligned} &\{X(t) : t \in [-h, 0]\} = \xi(t) \in C([-h, 0]; \mathbb{R}^n); \\ &r(0) = i \in S, \end{aligned} \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n\times m}$ are Borel measurable functions, and $h > 0$ represents the upper bound boundary of the delay.

As previously stated, the non-differentiable nature of the time delay involved in the system is a key feature of our work. We will now explicitly state this as an assumption.

Assumption 1 (see [22])**.** The time-varying delay $\delta : \mathbb{R}^+ \to [h_1, h]$ is a Borel measurable function and meets $\bar{h} := \limsup_{\triangle\to 0^+} \left(\sup_{s\geqslant -h} \frac{\pi(M_{s,\triangle})}{\triangle}\right) < \infty$, where $0 < h_1 < h$, $M_{s,\triangle} = \{t \geqslant 0 : s \leqslant t - \delta(t) < s + \triangle\}$ and $\pi(\cdot)$ represents the Lebesgue measure.

It is important to note that the aforementioned assumption is less stringent than the conventional requirement for $\delta(t)$ to be differentiable, with $\mathrm{d}\delta(t)/\mathrm{d}t < 1$. Furthermore, numerous delay functions in practical scenarios do not fail to fulfill Assumption 1. Additionally, under the aforementioned assumption, it can be shown that $\bar{h} \geqslant 1$. Consequently, the subsequent lemma can be derived. For further details, please refer to [22, 23].

Lemma 1 (see [22])**.** Allow Assumption 1 to be satisfied. Consider $H > 0$ and continuous function $\phi : [-h, H - h_1] \to \mathbb{R}^+$. It yields

$$\int_0^H \phi(t - \delta(t))\mathrm{d}t \leqslant \bar{h} \int_{-h}^{H-h_1} \phi(t)\mathrm{d}t.$$

The transition probability of the time-homogeneous Markov process, denoted by $p(t, \xi, i; \mathrm{d}\zeta \times \{j\})$, is defined as the probability of transitioning from state i at time t to state j at time $t + \triangle t$, given that the process is in state ξ at time t. For convenience, the concept of stability in a distribution is introduced.

Definition 1 (see [15]). The system (1) is called stable in distribution if there is a probability measure $\mu(\cdot\times\cdot)$ on $C([-h,0];\mathbb{R}^n)\times S$ such that the transition probability $p(t,\xi,i;\mathrm{d}\zeta\times\{j\})$ of $(X_t,r(t))$ converges weakly to $\mu(\mathrm{d}\zeta\times\{j\})$ as $t\to\infty$ for every $(\xi,i)\in C([-h,0];\mathbb{R}^n)\times S$.

Clearly, the stability of the distribution of $(X_t,r(t))$ implies that there is a unique invariant probability measure for $(X_t,r(t))$.

3 Stability in distribution

There are several sufficient conditions for the existence and uniqueness of the global solution to system (1), in this section, and then we will introduce some key lemmas and provide a proof of the stability in distribution of system (1). To streamline and clarify the paper's content, we introduce some special symbols along with their corresponding descriptions. To underscore the significance of initial value (2), Markov chain begins with i at time 0 as $r^i(t)$ and solution $X(t)$ is defined as $X^{\xi,i}(t)$. Define $\mathcal{C}\subset C([-h,0];\mathbb{R}^n)$ to be any compact set.

In order to assess the existence and uniqueness of this solution, these next assumptions must be given.

Assumption 2. For any constant $l>0$, there exist constants $\omega_{1,l}$ and $\omega_{2,l}$ such that for any $x,\hat{x},y,\hat{y}\in\mathbb{R}^n$ with $|x|\vee|\hat{x}|\vee|y|\vee|\hat{y}|\leqslant l$,

$$|f(x,y,i)-f(\hat{x},\hat{y},i)|\leqslant\omega_{1,l}(|x-\hat{x}|+|y-\hat{y}|),$$
$$|g(x,y,i)-g(\hat{x},\hat{y},i)|\leqslant\omega_{2,l}(|x-\hat{x}|+|y-\hat{y}|)$$

hold, with $i\in S$.

Assumption 3. There are three nonnegative constants c_1, c_2 and c_3 and functions $V\in C^2(\mathbb{R}^n\times S;\mathbb{R}^+)$, U_1, $U_2\in C(\mathbb{R}^n;\mathbb{R}^+)$, such that for all $(x,y,i)\in\mathbb{R}^n\times\mathbb{R}^n\times S$,

$$\lim_{|x|\to\infty}U_1(x)=\infty,\quad U_1(x)\leqslant V(x,i),\quad \mathcal{L}V(x,y,i)\leqslant c_1-c_2U_2(x)+c_3U_2(y).$$

It is worth noting that here we do not require the condition $c_2>c_3\bar{h}$.

Theorem 1. Under Assumptions 1–3, the conclusion below can be obtained.

(i) With the initial value (2), system (1) has a unique global solution $X^{\xi,i}(t)$ for $t\in[-h,\infty)$.

(ii) There are $K=K(\mathcal{C})$ and $\bar{K}=\bar{K}(\mathcal{C})$ such that for all $T>0$, $(\xi,i)\in\mathcal{C}\times S$,

$$\sup_{0\leqslant t\leqslant T}EU_1\left(X^{\xi,i}(t)\right)\leqslant K<\infty,\qquad E\int_0^T U_2\left(X^{\xi,i}(t)\right)\mathrm{d}t\leqslant\bar{K}<\infty.$$

Proof. Given that the coefficients of the HSDDEs (1) are locally Lipschitz continuous, there exists a unique maximal local solution $X^{\xi,i}(t)$ for any given initial data (2) over $t\in[-h,e_\infty)$, where e_∞ denotes the explosion time (refer to [1]). To each integer $k\geqslant\|\xi\|$, the stopping time is defined as $\tau_k=\inf\left\{0\leqslant t<e_\infty:\left|X^{\xi,i}(t)\right|\geqslant k\right\}$, where $\inf\emptyset=\infty$ (with $\emptyset$ denoting the empty set). It is evident that τ_k increases as k approaches infinity. We set $\tau_\infty=\lim_{k\to\infty}\tau_k$, leading to $\tau_\infty\leqslant e_\infty$ almost surely. Demonstrating $\tau_\infty=\infty$ almost surely implies $e_\infty=\infty$ almost surely and validatesassertion (i). Subsequently, we will establish that $\tau_\infty=\infty$ almost surely.

When restricting t to $[0,h_1]$ and observing $-h\leqslant t-\delta(t)\leqslant 0$, it is evident that $X^{\xi,i}(t-\delta(t))=\xi(t-\delta(t))$. Utilizing the generalized Itô formula (refer to [1]) and Assumption 3, it is able to demonstrate that, for any $k\geqslant\|\xi\|$,

$$EU_1\left(X^{\xi,i}(t\wedge\tau_k)\right)-V(\xi(0),i)\leqslant E\int_0^{t\wedge\tau_k}c_1-c_2U_2\left(X^{\xi,i}(s)\right)+c_3U_2\left(X^{\xi,i}(s-\delta(s))\right)\mathrm{d}s. \tag{3}$$

This implies $EU_1\left(X^{\xi,i}(t\wedge\tau_k)\right)+c_2E\int_0^{t\wedge\tau_k}U_2\left(X^{\xi,i}(s)\right)\mathrm{d}s\leqslant K_1$, where $K_1=V(\xi(0),i)+c_1h_1+c_3E\int_0^{h_1}U_2(X^{\xi,i}(s-\delta(s)))\mathrm{d}s>0$. The rest of the proof procedure is very similar to that in [22, 23], and is omitted here for reasons of space.

Remark 1. It is crucial to emphasize that we do not require $c_2>c_3\bar{h}$, a significant departure from many existing papers, such as [9, 24]. Therefore, Theorem 1 encompasses a much broader class of HSDDEs.

In the theorem just demonstrated above, both the generalized Hasminskii-type theorem concerning the existence and uniqueness of the solution for system (1) is developed, and a valuable assertion (2) regarding the solutions of U_1 and U_2 is gained. In practical applications, however, the following assumption is often easier to satisfy compared to Assumption 3, as it eliminates the requirement to determine the functions U_1 and U_2.

Assumption 4. For $x,\ y\in\mathbb{R}^n$, $i\in S$, there exist constants p, q with $p>2$, $q\geqslant 2$ and $a_i>0$ $(1\leqslant i\leqslant 3)$ such that

$$x^{\mathrm{T}}f(x,y,i)+\frac{q-1}{2}|g(x,y,i)|^2\leqslant a_1(|x|^2+|y|^2)-a_2|x|^p+a_3|y|^p.$$

Making $V(x,i)=|x|^q$ as well as applying Assumption 4, it is easy to demonstrate that

$$\begin{aligned}\mathcal{L}V(x,y,i)&\leqslant q|x|^{q-2}\left(a_1(|x|^2+|y|^2)-a_2|x|^p+a_3|y|^p\right)\\&\leqslant a_1q|x|^q+a_1q|x|^{q-2}|y|^2-a_2q|x|^{p+q-2}+a_3q|x|^{q-2}|y|^p.\end{aligned}$$

Based on Young's inequality, it can be further proven that

$$\begin{aligned}a_1|x|^q&\leqslant\frac{a_2}{4}|x|^{p+q-2}+\bar{a}_1,\\a_1|x|^{q-2}|y|^2&\leqslant\frac{a_2}{4}|x|^q+\bar{a}_2|y|^q\leqslant\frac{a_2}{4}\left(1+|x|^{p+q-2}\right)+\bar{a}_2\left(1+|y|^{p+q-2}\right),\\a_3|x|^{q-2}|y|^p&\leqslant\frac{a_2}{4}|x|^{p+q-2}+\bar{a}_3|y|^{p+q-2},\end{aligned}$$

where

$$\begin{aligned}\bar{a}_1&=\frac{q}{p+q-2}a_1^{\frac{p+q-2}{q}}\left(\frac{4(p-2)}{a_2(p+q-2)}\right)^{\frac{p-2}{p+q-2}},\\\bar{a}_2&=\frac{2}{q}a_1^{\frac{q}{2}}\left(\frac{4(q-2)}{a_2q}\right)^{\frac{q-2}{q}},\\\bar{a}_3&=\frac{p}{p+q-2}a_3^{\frac{p+q-2}{p}}\left(\frac{4(q-2)}{a_2(p+q-2)}\right)^{\frac{q-2}{p+q-2}}.\end{aligned}$$

Hence,

$$\mathcal{L}V(x,y,i)\leqslant\Phi-\frac{a_2q}{4}|x|^{p+q-2}+(\bar{a}_2+\bar{a}_3)q|y|^{p+q-2},$$

where $\Phi=\left(\frac{a_2}{4}+\bar{a}_2+\bar{a}_2\right)q<\infty$. At this point, let $U_1(x)=|x|^q$ and $U_2(x)=|x|^{p+q-2}$. Clearly, Assumption 3 holds. Then the next corollary logically follows from Theorem 1.

Corollary 1. Under Assumptions 1, 2, 4, system (1) with the initial value (2) has a unique global solution $X^{\xi,i}(t)$ on $[-h,\infty)$, and for all $t\geqslant 0$, $(\xi,i)\in\mathcal{C}\times S$,

$$E\left|X^{\xi,i}(t)\right|^q<\infty,\quad E\int_0^t\left|X^{\xi,i}(s)\right|^{p+q-2}\mathrm{d}s<\infty.$$

Moving forward, we will now delve into several lemmas crucial for establishing stability in distribution.

Lemma 2. Suppose that all conditions of Theorem 1 hold (i.e., satisfying Assumptions 1–3). Then for any $H>0$, $\varepsilon>0$, $t\in\mathbb{R}^+$ and $(\xi,i)\in\mathcal{C}\times S$, there exists a positive integer $R=R(\mathcal{C},H,\varepsilon)$ such that

$$P\left\{\|X_s^{\xi,i}\|<R,\ \forall s\in[t;\ t+H]\right\}\geqslant 1-\varepsilon.$$

Proof. For any $(\xi,i)\in\mathcal{C}\times S$ and $k\geqslant\|\xi\|$, define $\tau_k^t=\inf\left\{s\geqslant t:\|X_s^{\xi,i}(t)\|\geqslant k\right\}$. Analogous to Step 1 during the proof process of Theorem 1, we consider the interval $[t,(t+H)\wedge\tau_k^t]$ for $t\in[0,m_1h_1]$, where m_1 is any positive integer. Then according to (3), it yields

$$\begin{aligned}&EV\left(X^{\xi,i}((t+H)\wedge\tau_k^t),r^i((t+H)\wedge\tau_k^t)\right)\\&\leqslant EV\left(X^{\xi,i}(t),r^i(t)\right)+c_1H+c_3E\int_t^{(t+H)\wedge\tau_k^t}U_2\left(X^{\xi,i}(s-\delta(s))\right)\mathrm{d}s.\end{aligned}\tag{4}$$

Since $H>0$, it is surely possible to find a positive integer m_2 such that $(m_2-1)h_1<H\leqslant m_2h_1$. Again, still based on Lemma 1, the following derivation can be obtained:

$$E\int_t^{(t+H)\wedge\tau_k^t}U_2\left(X^{\xi,i}(s-\delta(s))\right)\mathrm{d}s\leqslant\bar{h}E\int_{t-h}^{(t+H)\wedge\tau_k^t-h_1}U_2\left(X^{\xi,i}(s)\right)\mathrm{d}s$$

$$\leqslant \bar{h}E\int_{-h}^{(m_1+m_2-1)h_1} U_2\left(X^{\xi,i}(s)\right)\mathrm{d}s$$
$$\leqslant \bar{h}\int_{-h}^{0} U_2(\xi(s))\mathrm{d}s + \bar{h}E\int_{0}^{(m_1+m_2-1)h_1} U_2\left(X^{\xi,i}(s)\right)\mathrm{d}s.$$

Combined with the above equation and assertion (2) in Theorem 1, Eq. (4) can be further estimated as

$$EV\left(X^{\xi,i}((t+H)\wedge\tau_k^t), r^i((t+H)\wedge\tau_k^t)\right) \leqslant K + c_1H + c_3\bar{h}\bar{K} + c_3\bar{h}\int_{-h}^{0} U_2(\xi(s))\mathrm{d}s := \bar{H} < \infty. \tag{5}$$

It is recalled $\lim_{|x|\to\infty} V(x,i)=\infty$. Therefore, one can define $R=R(\mathcal{C},H,\varepsilon)$ such that

$$\inf_{|y|\geqslant R, j\in S} V(y,j) \geqslant \frac{1}{\varepsilon}\bar{H}. \tag{6}$$

Applying (5) and (6), it yields

$$\begin{aligned}\left(\inf_{|y|\geqslant R, j\in S} V(y,j)\right)\cdot P\left\{\tau_k^t < t+H\right\} &\leqslant EV\left(X^{\xi,i}((t+H)\wedge\tau_k^t), r^i((t+H)\wedge\tau_k^t)\right) I_{\{\tau_k^t<t+H\}}\\ &\leqslant EV\left(X^{\xi,i}((t+H)\wedge\tau_k^t), r^i((t+H)\wedge\tau_k^t)\right)\\ &\leqslant \bar{H}.\end{aligned}$$

This means that $P\left\{\tau_k^t < t+H\right\} \leqslant \varepsilon$.

Lemma 3. Under Assumptions 1–3, the family of transition probabilities $\{p(t,\xi,i;d\zeta\times\{j\}) : t\geqslant 0\}$ is tight, for $\xi\in\mathcal{C}$, $i\in\times S$.

Proof. First, we show that for any ε_1, $\varepsilon_2>0$ and $(\xi,t,i)\in\mathcal{C}\times\mathbb{R}^+\times S$, there exists $\hbar=\hbar(\varepsilon_1,\varepsilon_2,\mathcal{C})>0$ such that

$$P\left\{\sup_{\substack{t-h\leqslant t_1<t_2\leqslant t\\ t_2-t_1\leqslant\hbar}} \left|X^{\xi,i}(t_2)-X^{\xi,i}(t_1)\right| \geqslant \varepsilon_1\right\} \leqslant \varepsilon_2. \tag{7}$$

By Lemma 2, there exists $R_1=R_1(\mathcal{C},h,\frac{\varepsilon_2}{2})$ such that the following statement holds:

$$P\left\{\sup_{t\leqslant s\leqslant t+h} \|X_s^{\xi,i}\| \leqslant R_1\right\} \geqslant 1-\frac{\varepsilon_2}{2}.$$

It is also assumed that $\|\xi\|\leqslant R_1$ for all $\xi\in\mathcal{C}$. For $i\in S$, let $R_2=\sup_{|x|\vee|y|\leqslant R_1}\{|f(x,y,i)|,|g(x,y,i)|\}$. For each $s\in\mathbb{R}^+$, define $\sigma_s=\inf\left\{u\geqslant s : \|X_u^{\xi,i}\|>R_1\right\}$. Let $h_0\in[0,h]$. By the Burkholder-Davis-Gundy inequality (see [1]), for any $t_1\in[t,t+h-h_0]$,

$$\begin{aligned}&E\left(\sup_{t_2\in[t_1,t_1+h_0]}\left|X^{\xi,i}(\sigma_{t_1}\wedge t_2)-X^{\xi,i}(t_1)\right|^4\right)\\ &\leqslant 8E\sup_{t_2\in[t_1,t_1+h_0]}\left|\int_{t_1}^{\sigma_{t_1}\wedge t_2} f(X^{\xi,i}(s),X^{\xi,i}(s-\delta(s)),r^i(s))\mathrm{d}s\right|^4\\ &\quad+8E\sup_{t_2\in[t_1,t_1+h_0]}\left|\int_{t_1}^{\sigma_{t_1}\wedge t_2} g(X^{\xi,i}(s),X^{\xi,i}(s-\delta(s)),r^i(s))\mathrm{d}B(s)\right|^4\\ &\leqslant 8E\left(\int_{t_1}^{t_1+h_0} I_{\{\sigma_{t_1}\geqslant s\}}\left|f(X^{\xi,i}(s),X^{\xi,i}(s-\delta(s)),r^i(s))\right|\mathrm{d}s\right)^4\\ &\quad+8C_pE\left(\int_{t_1}^{t_1+h_0} I_{\{\sigma_{t_1}\geqslant s\}}\left|g(X^{\xi,i}(s),X^{\xi,i}(s-\delta(s)),r^i(s))\right|^2\mathrm{d}s\right)^2\\ &\leqslant 8R_2^4h_0^4+8C_pR_2^4h_0^2\end{aligned}$$

$$\leqslant h_0^2 \left(8R_2^4h^2 + 8C_pR_2^4\right)$$
$$:= h_0^2\tilde{K}, \tag{8}$$

where C_p is the coefficient of the Burkholder-Davis-Gundy inequality. Consequently, together with Chebyshev inequality and (8), one gets

$$\frac{1}{h_0}P\left\{I_{\{\sigma_t\geqslant t+h\}}\sup_{t_1\leqslant t_2\leqslant t_1+h_0}\left|X^{\xi,i}(t_2)-X^{\xi,i}(t_1)\right|\geqslant\frac{\varepsilon_1}{3}\right\}$$
$$\leqslant\frac{81}{\varepsilon_1^4h_0}E\left(I_{\{\sigma_t\geqslant t+h\}}\sup_{t_1\leqslant t_2\leqslant t_1+h_0}\left|X^{\xi,i}(\sigma_{t_1}\wedge t_2)-X^{\xi,i}(t_1)\right|^4\right)$$
$$\leqslant\frac{81}{\varepsilon_1^4}\tilde{K}h_0. \tag{9}$$

Putting $h_0=\frac{\varepsilon_1^4\varepsilon_2}{162\tilde{K}}$, it follows from the Corollary on p. 83 of [26] that, for each $t\in\mathbb{R}^+$,

$$P\left\{\sup_{\substack{t\leqslant t_1<t_2\leqslant t+h\\ t_2-t_1\leqslant h_0}}\left|X^{\xi,i}(t_2)-X^{\xi,i}(t_1)\right|\geqslant\varepsilon_1\right\}$$
$$\leqslant P\left\{\sigma_t<t+h\right\}+P\left\{I_{\{\sigma_t\geqslant t+h\}}\sup_{\substack{t\leqslant t_1<t_2\leqslant t+h\\ t_2-t_1\leqslant h_0}}\left|X^{\xi,i}(t_2)-X^{\xi,i}(t_1)\right|\geqslant\varepsilon_1\right\}$$
$$\leqslant\varepsilon_2. \tag{10}$$

Since $\xi\in\mathcal{C}$, it follows from Arzelà-Ascoli theorem that there is $h'>0$ satisfying

$$\sup_{\substack{-h\leqslant t_1<t_2\leqslant 0\\ t_2-t_1\leqslant h'}}\left|X^{\xi,i}(t_2)-X^{\xi,i}(t_1)\right|=\sup_{\substack{-h\leqslant t_1<t_2\leqslant 0\\ t_2-t_1\leqslant h'}}|\xi(t_2)-\xi(t_1)|\leqslant\varepsilon_1. \tag{11}$$

Letting $\hbar=h_0\wedge h'$ and then Eq. (7) is given by (10) and (11).

Furthermore, one can derive with Lemma 2 that for $\varepsilon>0$, there is $R_3>0$ satisfying

$$P\left\{|X^{\xi,i}(t)|\geqslant R_3\right\}\leqslant\varepsilon, \tag{12}$$

for $t\in\mathbb{R}^+$.

According to (7) and (12), we can derive from Theorem 7.3 in [26] and $\{p(t,\xi,i;\mathrm{d}\zeta\times S):(t,\xi,i)\in\mathbb{R}^+\times\mathcal{C}\times S\}$ is tight. As the transition probabilities of $r^i(t)$ are by themselves tight, family $\{p(t,\xi,i;\mathrm{d}\zeta\times\{j\}):(t,\xi,i)\in\mathbb{R}^+\times\mathcal{C}\times S\}$ can be shown to be tight. This completes the proof.

Below, we must thoroughly investigate the discretization aspect concerning the two solutions of the system (1) that originate from distinct initial values. In order to proceed, it becomes imperative to introduce another pertinent assumption to ensure a comprehensive analysis.

Assumption 5. There exist functions $\overline{V}\in C^2(\mathbb{R}^n\times S;\mathbb{R}^+)$, $U_3\in C(\mathbb{R}^n;\mathbb{R}^+)$ and $c_4\in C(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^+)$, as well as a nonnegative constants c_5 such that $\overline{V}(\cdot,i)$, $U_3(\cdot)$ vanish only at 0 for $i\in S$, $c_4(x_1,x_2)>c_5\bar{h}$ with $x_1-x_2\neq 0$ and that

$$L\overline{V}(x_1,x_2,\bar{x}_1,\bar{x}_2,i)\leqslant -c_4(x_1,x_2)U_3(x_1-x_2)+c_5U_3(\bar{x}_1-\bar{x}_2).$$

Lemma 4. Under Assumptions 1, 2, 3, 5, for any $\varepsilon>0$ and ξ, $\eta\in\mathcal{C}$, there is $M=M(\varepsilon,\mathcal{C})$ such that

$$P\left\{\|X_t^{\xi,i}-X_t^{\eta,i}\|<\varepsilon\right\}\geqslant 1-\varepsilon, \tag{13}$$

for $i\in S$, $t>M$.

Proof. The proof is presented in two steps.

Step 1. Initially, it is shown that for any ρ, $l>0$,

$$\lim_{t\to\infty}P\left\{A_t^{\rho,l}\right\}=0\quad\text{uniformly in }\xi,\ \eta\in C,$$

where

$$A_t^{\rho,l} = \left\{\omega : \|X_t^{\xi,i}\| \vee \|X_t^{\eta,i}\| \leqslant l,\ |X^{\xi,i}(t) - X^{\eta,i}(t)| \geqslant \rho\right\}.$$

To simplify the notation, let

$$\begin{aligned} U_3^{\rho,l} &= \min\{U_3(x-y) : |x| \vee |y| \leqslant l,\ |x-y| \geqslant \rho\}, \\ c_4^{\rho,l} &= \min\{c_4(x,y) : |x| \vee |y| \leqslant l,\ |x-y| \geqslant \rho\} - c_5\bar{h}. \end{aligned}$$

By Assumption 5 and generalized Itô formula, it can be demonstrated

$$\begin{aligned} &E\overline{V}\left(X^{\xi,i}(t) - X^{\eta,i}(t), r^i(t)\right) \\ &\quad\leqslant \overline{V}\left(\xi(0) - \eta(0), i\right) + c_5\bar{h}\int_{-h}^{t} U_3\left(X^{\xi,i}(s) - X^{\eta,i}(s)\right) \mathrm{d}s \\ &\qquad - E\int_0^t c_4\left(X^{\xi,i}(s), X^{\eta,i}(s)\right) U_3\left(X^{\xi,i}(s) - X^{\eta,i}(s)\right) \mathrm{d}s \\ &\quad= \overline{V}\left(\xi(0) - \eta(0), i\right) + c_5\bar{h}\int_{-h}^{0} U_3\left(\xi(s) - \eta(s)\right) \mathrm{d}s \\ &\qquad - c_4^{\rho,l} E\int_0^t I_{\{A_s^{\rho,l}\}} U_3\left(X^{\xi,i}(s) - X^{\eta,i}(s)\right) \mathrm{d}s. \end{aligned} \tag{14}$$

Letting $t \to \infty$, it follows that for any $\rho,\ l > 0$,

$$\int_0^\infty P\left\{A_s^{\rho,l}\right\} \mathrm{d}s \leqslant \frac{1}{U_3^{\rho,l} c_4^{\rho,l}}\left(\overline{V}\left(\xi(0) - \eta(0), i\right) + c_5\bar{h}\int_{-h}^{0} U_3\left(\xi(s) - \eta(s)\right) \mathrm{d}s\right) < \infty. \tag{15}$$

We claim that $\lim_{t\to\infty} P\left\{A_t^{\rho,l}\right\} = 0$, for any $\rho,\ l > 0$. If not, there exist some $\rho_0,\ l_0 > 0$ so that $\lim_{t\to\infty} P\left\{A_t^{\rho_0,l_0}\right\} > 0$. Then there exist $m > 0$ and an increasing sequence t_n with $t_n \uparrow \infty$ satisfying

$$P\left\{A_{t_n}^{\rho_0,l_0}\right\} = P\left\{\|X_{t_n}^{\xi,i}\| \vee \|X_{t_n}^{\eta,i}\| \leqslant l_0,\ |X^{\xi,i}(t_n) - X^{\eta,i}(t_n)| \geqslant \rho_0\right\} \geqslant m, \quad \forall\, n \in \mathbb{N}. \tag{16}$$

There exist sufficiently small $\hbar > 0$ for $\varepsilon_1 = \rho_0/3$, $\varepsilon_2 = m/8$ application of (7) such that for $\zeta = \xi,\ \eta \in \mathcal{C}$,

$$P\left\{\sup_{t_n \leqslant s \leqslant t_n + \hbar} \left|X^{\zeta,i}(s) - X^{\zeta,i}(t_n)\right| \geqslant \frac{\rho_0}{3}\right\} \leqslant \frac{m}{8}. \tag{17}$$

From (16) and (17), it follows that for $s \in [t_n, t_n + \hbar]$,

$$P\left\{\left|X^{\xi,i}(s) - X^{\eta,i}(s)\right| \geqslant \frac{\rho_0}{3}\right\} \geqslant m - \frac{m}{8} - \frac{m}{8} = \frac{3m}{4}.$$

Considering Lemma 3, it can be observed that $R_4 = R_4(\mathcal{C}, m) \geqslant l_0$ satisfying $P\left\{\|X_t^{\xi,i}\| \leqslant R_4\right\} \geqslant 1 - \frac{m}{4}$, for arbitrary $\xi \in \mathcal{C}$ and $t \geqslant 0$. One derives that for $s \in [t_n, t_n + \hbar]$,

$$\begin{aligned} &P\left\{\|X_s^{\xi,i}\| \vee \|X_s^{\eta,i}\| \leqslant R_4,\ |X^{\xi,i}(s) - X^{\eta,i}(s)| \geqslant \frac{\rho_0}{3}\right\} \\ &\geqslant P\left\{|X^{\xi,i}(s) - X^{\eta,i}(s)| \geqslant \frac{\rho_0}{3}\right\} - P\left\{\|X_s^{\xi,i}\| \vee \|X_s^{\eta,i}\| \leqslant R_4\right\}^C \\ &\geqslant \frac{3m}{4} - 2\frac{m}{4} = \frac{m}{4}, \end{aligned}$$

where $\left\{\|X_s^{\xi,i}\| \vee \|X_s^{\eta,i}\| \leqslant R_4\right\}^C$ stands for the complement of $\left\{\|X_s^{\xi,i}\| \vee \|X_s^{\eta,i}\| \leqslant R_4\right\}$. This implies that for all $n \in \mathbb{N}$, $\int_{t_n}^{t_n+\hbar} P\left\{A_s^{\frac{\rho_0}{3},R_4}\right\} \mathrm{d}s \geqslant \frac{m\hbar}{4}$. Consequently, $\int_0^\infty P\left\{A_s^{\frac{\rho_0}{3},R_4}\right\} \mathrm{d}s = \infty$, which contradicts (15). We therefore conclude that

$$\lim_{t\to\infty} P\left\{A_t^{\rho,l}\right\} = 0. \tag{18}$$

Immediately, a proof of the consistency of the above $\xi,\eta\in\mathcal{C}$ is given, i.e., for any $\varepsilon,\rho,l>0$, there exists $T^{\rho,l}_{\varepsilon}=T^{\rho,l}_{\varepsilon}(\mathcal{C})>0$ such that for arbitrary $\xi,\eta\in\mathcal{C}$ and all $t>T^{\rho,l}_{\varepsilon}$ we have

$$P\left\{\|X^{\xi,i}_t\|\vee\|X^{\eta,i}_t\|\leqslant l,\ |X^{\xi,i}(t)-X^{\eta,i}(t)|\geqslant\rho\right\}<\varepsilon.$$

Considering Lemma 3, a constant $R_5=R_5(\mathcal{C},\varepsilon)\geqslant l$ can be found such that $P\left\{\|X^{\xi,i}_t\|>R_5\right\}<\frac{\varepsilon}{6}$, for arbitrary $\xi\in\mathcal{C}$. Put $V^{\rho,R_5}=\min\left\{\overline{V}(x-y,i):\ |x|\vee|y|\leqslant R_5,\ |x-y|\geqslant\rho\right\}$. Since $\overline{V}(0,i)=U_3(0)=0$ and $\overline{V}$, U_3 are continuous, for any $\varepsilon>0$, it is possible to choose $\hbar_0>0$ satisfying for ξ, $\eta\in\mathcal{C}$ with $\|\xi-\eta\|\leqslant\hbar_0$

$$\overline{V}\left(\xi(0)-\eta(0),i\right)+c_5\bar{h}\int_{-h}^{0}U_3\left(\xi(s)-\eta(s)\right)\mathrm{d}s<\frac{\varepsilon}{6}V^{\rho,R_5}. \tag{19}$$

Consequently, if $\|\xi-\eta\|\leqslant\hbar_0$, according to (14) and (19), it yields

$$\begin{aligned}
&P\left\{\|X^{\xi,i}_t\|\vee\|X^{\eta,i}_t\|\leqslant R_5,\ |X^{\xi,i}(t)-X^{\eta,i}(t)|\geqslant\rho\right\}\\
&\quad\leqslant P\left\{|X^{\xi,i}(t)|\vee|X^{\eta,i}(t)|\leqslant R_5,\ |X^{\xi,i}(t)-X^{\eta,i}(t)|\geqslant\rho\right\}\\
&\quad\leqslant\frac{1}{V^{\rho,R_5}}E\overline{V}\left(X^{\xi,i}(t)-X^{\eta,i}(t),r^i(t)\right)\\
&\quad\leqslant\frac{\varepsilon}{6}.
\end{aligned}$$

Due to the compactness of $\mathcal{C}$, there exist $\xi_i\in\mathcal{C}$, $i=1,2,\ldots,n$ such that for any $\xi\in\mathcal{C}$, ξ_i can be found such that $\|\xi-\xi_i\|\leqslant\hbar_0$. According to (18), there exists $T^{\rho,R_5}_{\varepsilon}>0$ such that

$$P\left\{\|X^{\xi_u,i}_t\|\vee\|X^{\xi_v,i}_t\|\leqslant R_5,\ |X^{\xi_u,i}(t)-X^{\xi_v,i}(t)|\geqslant\frac{\rho}{3}\right\}\leqslant\frac{\varepsilon}{6},$$

for all $1\leqslant u,v\leqslant n$ and $t\geqslant T^{\rho,R_5}_{\varepsilon}$. With respect to ξ, $\eta\in\mathcal{C}$, one can identify ξ_u, ξ_v so that $\|\xi-\xi_u\|\leqslant\hbar_0$ and $\|\eta-\xi_v\|\leqslant\hbar_0$. Then, for any $t\geqslant T^{\rho,R_5}_{\varepsilon}$,

$$\begin{aligned}
&P\left\{\|X^{\xi,i}_t\|\vee\|X^{\eta,i}_t\|\leqslant l,\ |X^{\xi,i}(t)-X^{\eta,i}(t)|\geqslant\rho\right\}\\
&\quad\leqslant P\left\{\|X^{\xi,i}_t\|\vee\|X^{\eta,i}_t\|\leqslant R_5,\ |X^{\xi,i}(t)-X^{\eta,i}(t)|\geqslant\rho\right\}\\
&\quad\leqslant P\left\{\|X^{\xi,i}_t\|\vee\|X^{\xi_u,i}_t\|\leqslant R_5,\ |X^{\xi,i}(t)-X^{\xi_u,i}(t)|\geqslant\frac{\rho}{3}\right\}\\
&\qquad+P\left\{\|X^{\xi_u,i}_t\|\vee\|X^{\xi_v,i}_t\|\leqslant R_5,\ |X^{\xi_u,i}(t)-X^{\xi_v,i}(t)|\geqslant\frac{\rho}{3}\right\}\\
&\qquad+P\left\{\|X^{\xi_v,i}_t\|\vee\|X^{\eta,i}_t\|\leqslant R_5,\ |X^{\xi_v,i}(t)-X^{\eta,i}(t)|\geqslant\frac{\rho}{3}\right\}\\
&\qquad+P\left\{\|X^{\xi_u,i}_t\|>R_5\right\}+P\left\{\|X^{\xi_v,i}_t\|>R_5\right\}\\
&\quad<\varepsilon,
\end{aligned}$$

as desired.

Step 2. Consider any $\varepsilon>0$. There exist $R_6=R_6(\mathcal{C},h,\varepsilon)$, according to Lemma 3, so that for any $\xi\in\mathcal{C}$ and $t\in\mathbb{R}^+$,

$$P\left\{\|X^{\xi,i}_s\|\leqslant R_6,\ s\in[t,t+h]\right\}\geqslant 1-\frac{\varepsilon}{16}. \tag{20}$$

Consider $\tau_t=\inf\{s\geqslant t:\|X^{\xi,i}_s\|\vee\|X^{\eta,i}_s\|>R_6\}$. Then, it yields from (20) that $P\left\{\tau_t<t+h\right\}=P\{\|X^{\xi,i}_s\|\vee\|X^{\eta,i}_s\|>R_6,\ s\in[t,t+h]\}\leqslant\frac{\varepsilon}{8}$. Making use of the similar methods of proof as in (9), for any $0<h_0<h$ and $t\leqslant s_1\leqslant s_1+h_0\leqslant t+h$, it follows that

$$\frac{1}{h_0}P\left\{I_{\{\tau_t\geqslant t+h\}}\sup_{s_1\leqslant s_2\leqslant s_1+h_0}\left|X^{\xi,i}(s_2)-X^{\xi,i}(s_1)\right|\geqslant\frac{\varepsilon}{3}\right\}\leqslant\frac{81}{\varepsilon^4}\hat{K}h_0, \tag{21}$$

where $\hat{K}$ is a constant associated with $\mathcal{C}$, R_6, and ε.

Consider $m_0 \in \mathbb{N}$ such that $\frac{81}{\varepsilon^4}\hat{K}h_0 \leqslant \frac{\varepsilon}{8h}$ with $h_0 = \frac{h}{m_0}$. According to (21), for $k = 0, \dots, m_0 - 1$, it follows that

$$P\left\{\{\tau_t \geqslant t+h\} \cap \left\{\sup_{s\in[t+kh_0,t+(k+1)h_0]} \left|X^{\xi,i}(s) - X^{\xi,i}(t+kh_0)\right| \geqslant \frac{\varepsilon}{3}\right\}\right\} \leqslant \frac{\varepsilon}{8h}h_0.$$

Hence

$$P\left\{\{\tau_t \geqslant t+h\} \cap C_t^{\xi,i}\right\} \leqslant \frac{\varepsilon}{8h}h_0 m_0 = \frac{\varepsilon}{8},$$

where

$$C_t^{\xi,i} = \left\{\exists k \in \{0,\dots,m_0-1\} : \sup_{s\in[t+kh_0,t+(k+1)h_0]} \left|X^{\xi,i}(s) - X^{\xi,i}(t+kh_0)\right| \geqslant \frac{\varepsilon}{3}\right\}.$$

As a result,

$$P\left\{\{\tau_t \geqslant t+h\}\backslash C_t^{\xi,i}\right\} \geqslant 1 - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} = 1 - \frac{\varepsilon}{4}. \tag{22}$$

Analogously, it has

$$P\left\{\{\tau_t \geqslant t+h\}\backslash C_t^{\eta,i}\right\} \geqslant 1 - \frac{\varepsilon}{4}. \tag{23}$$

Due to the uniform convergence indicated in Step 1, it is possible to find $T_0 = T_0(\mathcal{C}, \varepsilon)$ so that for $t > T_0$, it holds that

$$\sum_{k=0}^{m_0-1} P\left\{\|X_{t+kh_0}^{\xi,i}\| \vee \|X_{t+kh_0}^{\eta,i}\| \leqslant R_6, \left|X^{\xi,i}(t+kh_0) - X^{\eta,i}(t+kh_0)\right| \geqslant \frac{\varepsilon}{3}\right\} \leqslant \frac{\varepsilon}{4},$$

which hints at the fact that $P\{D_t\} \leqslant \frac{\varepsilon}{4}$, where $\exists k \in \{0,\dots,m_0-1\}$,

$$D_t = \left\{k : \|X_{t+kh_0}^{\xi,i}\| \vee \|X_{t+kh_0}^{\eta,i}\| \leqslant R_6, \left|X^{\xi,i}(t+kh_0) - X^{\eta,i}(t+kh_0)\right| \geqslant \frac{\varepsilon}{3}\right\}.$$

Thus, for $t > T_0$,

$$P\left\{\{\tau_t \geqslant t+h\}\backslash D_t\right\} \geqslant 1 - \frac{\varepsilon}{8} - \frac{\varepsilon}{4} = 1 - \frac{3\varepsilon}{8}. \tag{24}$$

It is noticed that the event

$$\left\{\tau_t \geqslant t+h, \sup_{t\leqslant s\leqslant t+h} \left|X^{\xi,i}(s) - X^{\eta,i}(s)\right| < \varepsilon\right\}$$

will necessarily occur, if events $\{\tau_t \geqslant t+h\}\backslash C_t^{\xi,i}$, $\{\tau_t \geqslant t+h\}\backslash C_t^{\eta,i}$ and $\{\tau_t \geqslant t+h\}\backslash D_t$ occur simultaneously. Together with (22)–(24), it holds that for $t > T_0$,

$$P\left\{\tau \geqslant t+h, \sup_{t\leqslant s\leqslant t+h} \left|X^{\xi,i}(s) - X^{\eta,i}(s)\right| < \varepsilon\right\} \geqslant 1 - \frac{7\varepsilon}{8}. \tag{25}$$

By the definition of the norm $\|\cdot\|$, one has

$$P\left\{\|X_{t+h}^{\xi,i} - X_{t+h}^{\eta,i}\| < \varepsilon\right\} \geqslant P\left\{\tau_t \geqslant t+h, \sup_{t\leqslant s\leqslant t+h} \left|X^{\xi,i}(s) - X^{\eta,i}(s)\right| < \varepsilon\right\}. \tag{26}$$

So, according to (25) and (26), it follows that for all $t > T_0$, $P\left\{\|X_{t+h}^{\xi,i} - X_{t+h}^{\eta,i}\| < \varepsilon\right\} \geqslant 1 - \varepsilon$. The proof is completed.

Let $\mathbb{C}_h := C([-h,0];\mathbb{R}^n)$. Let $\mathcal{P}_h$ denote the family of probability measures on the measurable space $\mathbb{C}_h$. For $P_1, P_2 \in \mathcal{P}_h$, define metric $d_{\mathbb{L}}$ by

$$d_{\mathbb{L}}(P_1,P_2) = \sup_{\varphi\in\mathbb{L}} \left| \int_{\mathbb{C}_h} \varphi(\xi)P_1(\mathrm{d}\xi) - \int_{\mathbb{C}_h} \varphi(\xi)P_2(\mathrm{d}\xi) \right|,$$

where

$$\mathbb{L} = \{\varphi : \mathbb{C}_h \to \mathbb{R},\ |\varphi(\xi)-\varphi(\eta)| \leqslant \|\xi-\eta\| \text{ and } |\varphi(\xi)| \leqslant 1 \text{ for } \xi,\eta\in\mathbb{C}_h\}.$$

Lemma 5. Let Assumptions 1, 2, 3, 5 hold. Then,

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t,\xi,i;\cdot\times\cdot),\ p(t,\eta,j;\cdot\times\cdot)) = 0,$$

uniformly in $(\xi,\eta,i,j)\in\mathcal{C}\times\mathcal{C}\times S\times S$.

Proof. Consider $k_{ij} = \inf\left\{t\geqslant 0 : r^i(t) = r^j(t)\right\}$. Subsequently, it becomes evident that $k_{ij}<\infty$ almost surely due to the ergodic property of the Markov chain (refer to [27]). Consequently, there exists $T_1>0$, for any $\varepsilon>0$ and $i,\ j\in S$ such that $P\left\{k_{ij}\leqslant T_1\right\} > 1-\frac{\varepsilon}{6}$.

Let $\Omega_{\xi,i} = \left\{\omega\in\Omega : \sup_{-h\leqslant t\leqslant T_1}\left|X^{\xi,i}(t,\omega)\right| \leqslant R'\right\}$. Combined with Lemma 3, there is a sufficiently large $R'>0$ so that $P(\Omega_{\xi,i}) > 1-\frac{\varepsilon}{12}$. Let us now fix arbitrary $\xi,\ \eta\in\mathcal{C}$ and $i,j\in S$. Define $\Lambda = \{k_{ij}\leqslant T_1\}\cap\Omega_{\xi,i}\cap\Omega_{\eta,j}$. Consider $\varphi\in\mathbb{L}$ and $t\geqslant T_1$,

$$\begin{aligned}\left|E\varphi(X_t^{\xi,i}) - E\varphi(X_t^{\eta,j})\right| &\leqslant 2P\left\{k_{ij}>T_1\right\} + E\left(I_{\{k_{ij}\leqslant T_1\}}\left|\varphi(X_t^{\xi,i}) - \varphi(X_t^{\eta,j})\right|\right)\\ &\leqslant \frac{\varepsilon}{3} + 2P(\Omega-\Lambda) + E\left(I_\Lambda E\left|X^{\bar\xi,\lambda}(t-k_{ij}) - X^{\bar\eta,\lambda}(t-k_{ij})\right|\right),\end{aligned}$$

where $\bar\xi = X_{k_{ij}}^{\xi,i}$, $\bar\eta = X_{k_{ij}}^{\eta,j}$ and $\lambda = r^i(k_{ij}) = r^j(k_{ij})$. It is noted that $\|\xi\|\vee\|\eta\|\leqslant\lambda$ for any $\omega\in\Lambda$, it is possible to employ Lemma 4 to find that there exists $T_2>0$ so that for $T_1+T_2\leqslant t$, $E\left|X^{\bar\xi,\lambda}(t-k_{ij}) - X^{\bar\eta,\lambda}(t-k_{ij})\right| \leqslant \frac{\varepsilon}{3}$, whenever $\omega\in\Lambda$. Then, it is easy to obtain that for $t\geqslant T_1+T_2$, $\left|E\varphi(X_t^{\xi,i}) - E\varphi(X_t^{\eta,j})\right| \leqslant \varepsilon$. Since φ are arbitrary, we must have $\sup_{\varphi\in\mathbb{L}}\left|E\varphi(X_t^{\xi,i}) - E\varphi(X_t^{\eta,j})\right| \leqslant \varepsilon, \forall t\geqslant T_1+T_2$, which is $d_{\mathbb{L}}(p(t,\xi,i;\cdot\times\cdot),\ p(t,\eta,j;\cdot\times\cdot))\leqslant\varepsilon$, for $(\xi,\eta,i,j)\in\mathcal{C}\times\mathcal{C}\times S\times S$.

Lemma 6. Under Assumptions 1, 2, 3, 5, the family $\{p(t,\xi,i;d\zeta\times\{j\}) : t\geqslant 0\}$ is Cauchy for any $\xi\in\mathbb{C}_h$, $i\in S$ with metric $d_{\mathbb{L}}$, in the space $\mathcal{P}_h$.

Proof. Fix $\xi\in\mathbb{C}_h$, $i\in S$. Equivalently, it is necessary to prove that, there is $T_3>0$ for $\varepsilon>0$ such that $d_{\mathbb{L}}(p(u+v,\xi,i;\cdot\times\cdot),\ p(u,\xi,i;\cdot\times\cdot))\leqslant\varepsilon$, for $v>0$ and $u\geqslant T_3$. This is equivalent to

$$\sup_{\varphi\in\mathbb{L}}\left|E\varphi(X_{u+v}^{\xi,i}) - E\varphi(X_u^{\xi,i})\right| \leqslant \varepsilon. \tag{27}$$

By Lemma 3, there exists an $R_0>0$ so that $P\{\omega\in\Omega : \|X_v^{\xi,i}\|\leqslant R_0\} > 1-\frac{\varepsilon}{4}$, $\forall v>0$. As a result, for $u,\ v>0$ and $\varphi\in\mathbb{L}$, it yields

$$\begin{aligned}\left|E\varphi(X_{u+v}^{\xi,i}) - E\varphi(X_u^{\xi,i})\right| &= \left|E\left(E(\varphi(X_{u+v}^{\xi,i})\mid\mathcal{F}_v)\right) - E\varphi(X_u^{\xi,i})\right|\\ &\leqslant \frac{\varepsilon}{2} + \sum_{j\in S}\int_{Z_{R_0}}\left|E\varphi(X_u^{\eta,j}) - E\varphi(X_u^{\xi,i})\right| p(v,\xi,i;\mathrm{d}\eta\times\{j\}),\end{aligned}$$

where $Z_{R_0} = \{x\in\mathbb{C}_h : \|x\|\leqslant R_0\}$. Then by Lemma 5, there is a positive integer T_3 such that $\forall u\geqslant T_3$, $\sup_{\varphi\in\mathbb{L}}\left|E\varphi(X_u^{\xi,i}) - E\varphi(X_u^{\eta,j})\right| < \frac{\varepsilon}{2}$, whenever $(\xi,i)\in Z_{R_0}\times S$. We, therefore, obtain for $u\geqslant T_3$ and $v>0$, $\left|E\varphi(X_{v+u}^{\xi,i}) - E\varphi(X_u^{\xi,i})\right| \leqslant \varepsilon$. So Eq. (27) we asserted must hold, as it holds for any $\varphi\in\mathbb{L}$. Consequently, the family $\{p(t,\xi,i;\mathrm{d}\zeta\times\{j\}) : t\geqslant 0\}$ is a Cauchy sequence.

With the above lemmas prepared, it is time to describe the main results in this work.

Theorem 2. Under Assumptions 1, 2, 3, 5, system (1) with initial data (2) is stable in distribution.

Proof. In line with given definition, the objective is to establish the existence of a probability measure $\mu(\cdot\times\cdot)$, so that the transition probability $\{p(t,\xi,i;\cdot\times\cdot):t\geqslant 0\}$ weakly converges to $\mu(\cdot\times\cdot)\in\mathbb{C}_h\times S$ for any $(\xi,i)\in\mathbb{C}_h\times S$. Given the well-established understanding that weak convergence of a probability measure is a metric concept (as detailed in [28]), it is imperative to demonstrate that

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t,\xi,i;\cdot\times\cdot),\ \mu(\cdot\times\cdot))=0.$$

Then by Lemma 6, $\{p(t,\xi,i;\cdot\times\cdot):t\geqslant 0\}$ is Cauchy with metric $d_{\mathbb{L}}$ in the space $\mathcal{P}_h$. Thus there exists a unique $\mu(\cdot\times\cdot)\in\mathbb{C}_h\times S$ such that

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t,0,1;\cdot\times\cdot),\ \mu(\cdot\times\cdot))=0.$$

Then for $(\xi,i)\in\mathbb{C}_h\times S$, it follows from Lemma 5 that

$$\begin{aligned}\lim_{t\to\infty} d_{\mathbb{L}}(p(t,\xi,i;\cdot\times\cdot),\ \mu(\cdot\times\cdot)) &\leqslant \lim_{t\to\infty} d_{\mathbb{L}}(p(t,\xi,i;\cdot\times\cdot),\ p(t,0,1;\cdot\times\cdot))+\lim_{t\to\infty} d_{\mathbb{L}}(p(t,0,1;\cdot\times\cdot),\ \mu(\cdot\times\cdot))\\ &=0,\end{aligned}$$

as required.

In order to derive a practical corollary, we introduce a new assumption that can be more readily verified in real-world applications compared with Assumption 5.

Assumption 6. There are positive constants p, q, a_5 with $p>2$ and $q\geqslant 2$ as well as a function $a_4(\cdot,\cdot)\in C(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^+)$ such that $a_4(x_1,x_2)>a_5\bar{h}$ when $x_1-x_2\neq 0$ and

$$\begin{aligned}&(x_1-x_2)^{\rm T}\left(f(x_1,\bar{x}_1,i)-f(x_2,\bar{x}_2,i)\right)+\frac{q-1}{2}\left|g(x_1,\bar{x}_1,i)-g(x_2,\bar{x}_2,i)\right|^2\\ &\quad\leqslant -a_4(x_1,x_2)|x_1-x_2|^2+a_5|\bar{x}_1-\bar{x}_2|^2,\end{aligned}$$

for all x_1, x_2, $\bar{x}_1$, $\bar{x}_2\in\mathbb{R}^n$ and $i\in S$.

Letting $\overline{V}(x,i)=|x|^q$. The similarity in treatment to that of Assumption 4 is so pronounced that it has been omitted, leading to the direct presentation of the next corollary about Theorem 2.

Corollary 2. Under Assumptions 1, 2, 4, 6, the system (1) with initial value (2) is stable in distribution.

4 Example

While constraints on page space limit our discussion to one specific example, it is essential to note that the theoretical findings presented in this paper are thoroughly illustrated.

Consider the scalar HSDDE described by

$$\mathrm{d}X(t)=f(X(t),X(t-\delta(t)),r(t))\mathrm{d}t+g(X(t),X(t-\delta(t)),r(t))\mathrm{d}B(t),\tag{28}$$

on $t\geqslant 0$, with initial value implied but not explicitly stated. Here, f and g represent the coefficients defined as follows:

$$f(x,y,1)=-b_{11}x^3+b_{12}xy,\ \ g(x,y,1)=b_{13}x\cos(y),\ \ f(x,y,2)=-b_{21}x^3+b_{22}xy,\ \ g(x,y,2)=b_{23}x\sin(y),$$

for x, $y\in\mathbb{R}$, where b_{11}, b_{12} are arbitrary positive numbers, $S=\{1,2\}$. Furthermore, the time delay function $\delta(t)$ is defined as

$$\delta(t)=\sum_{m=0}^{\infty}\left\{(0.15+0.2(t-2m))I_{[2m,2m+1)}(t)+(0.3-0.2(t-2m-1))I_{[2m+1,2(m+1))}(t)\right\}.$$

This equation constitutes a simplified version of the HSDDE food chain model (see [29, 30]). Notably, the time delay function $\delta(t)$ meets Assumption 1 with $h_1=0.1$, $h=0.35$ and $\bar{h}\leqslant 1/(1-0.2)=1.25$. To obtain the existence and uniqueness result for the solution, let $V(x,i)=|x|^2$. By a simple calculation, it is possible to obtain $\mathcal{L}V(x,y,i)=-2b_{i1}|x|^4+2b_{i2}|x|^2|y|+b_{i3}^2|x|^2|y|^2\leqslant\frac{b_{i2}^2}{b_{i1}}-\frac{b_{i1}}{2}|x|^4+\frac{2b_{i2}^2+b_{i3}^4}{2b_{i1}}|y|^4$. And then it yields $\mathcal{L}V(x,y,i)\leqslant\beta_1-\beta_2|x|^4+\beta_3|y|^4$, where $\beta_1=\max\left\{\frac{b_{12}^2}{b_{11}},\ \frac{b_{22}^2}{b_{21}}\right\}$, $\beta_2=\min\left\{\frac{b_{11}}{2},\ \frac{b_{21}}{2}\right\}$, $\beta_3=\max\left\{\frac{2b_{12}^2+b_{13}^4}{2b_{11}},\frac{2b_{22}^2+b_{23}^4}{2b_{21}}\right\}$. It is easy

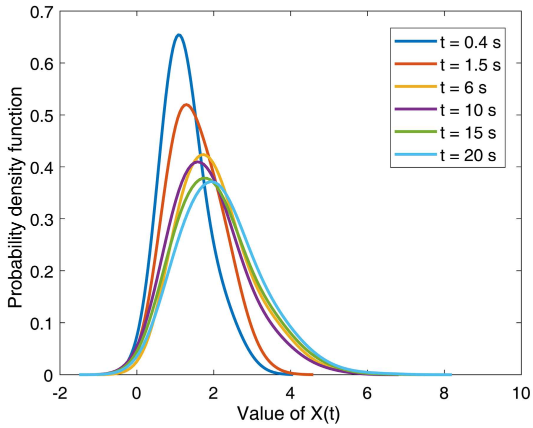

Figure 1 (Color online) Distribution of six different time points $t = 0.4$ s, $t = 1.5$ s, $t = 6$ s, $t = 10$ s, $t = 15$ s, $t = 20$ s.

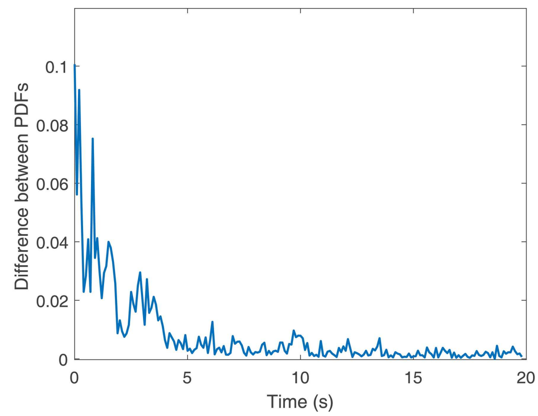

Figure 2 (Color online) Difference between PDFs.

to note that for $U_1(x) = |x|^2$ and $U_2(x) = |x|^4$, Assumption 3 is satisfied, which implies that Theorem 1 holds. In order to employ Theorem 2, for all x_1, x_2, $\bar{x}_1$, $\bar{x}_2 \in \mathbb{R}^n$ and $i \in S$, consider $\overline{V}(x,i) = |x|^2$. Based on Assumption 2,

$$\begin{aligned} LV(x_1,x_2,\bar{x}_1,\bar{x}_2,i) &= (x_1-x_2)^{\mathrm{T}}\left(f(x_1,\bar{x}_1,i) - f(x_2,\bar{x}_2,i)\right) + |g(x_1,\bar{x}_1,i) - g(x_2,\bar{x}_2,i)|^2 \\ &\leqslant (x_1-x_2)^{\mathrm{T}}\omega_{1,l}\left(|x_1-x_2| + |\bar{x}_1-\bar{x}_2|\right) + \omega_{2,l}^2\left(|x_1-x_2| + |\bar{x}_1-\bar{x}_2|\right)^2, \end{aligned}$$

where $x_1 = X^{\xi,i}(t)$, $x_2 = X^{\eta,i}(t)$, $\bar{x}_1 = X^{\xi,i}(t-\delta(t))$, and $\bar{x}_2 = X^{\eta,i}(t-\delta(t))$. Assuming $X^{\xi,i}(t) < X^{\eta,i}(t)$, in other words, $x_1 - x_2 < 0$, which yields $LV(x_1,x_2,\bar{x}_1,\bar{x}_2,i) \leqslant -(\omega_{1,l} - 2\omega_{2,l}^2)|x_1-x_2|^2 + 2\omega_{2,l}^2|\bar{x}_1-\bar{x}_2|^2$. Now, we need to impose a certain requirement on these parameters to ensure that Assumption 5 is valid, which is $\omega_{1,l} - 2\omega_{2,l}^2 > 2\omega_{2,l}^2\bar{h}$, i.e. $\omega_{1,l} > 4.5\omega_{2,l}^2$. Thus by Theorem 2, with any given initial value $\{X(t) : -h \leqslant t \leqslant 0\}$ (where $h = 0.35$) and $r(0) = 1$ or 2, system (28) is stable in distribution.

Finally, we verify the results with simulation, using the Euler-Maruyama method to simulate the sample, where $X(0) = 4$, $r(0) = 1$, the time step $\mathrm{d}t = 10^{-3}$ s and sample size of 10^3. And, simply choose $b_{11} = b_{21} = 3$, $b_{12} = b_{13} = b_{22} = b_{23} = 2$, $\gamma_1 = 5$ and $\gamma_2 = 1$. Then, from 10^3 probability density functions, we pick six time points $t = 0.4$ s, $t = 1.5$ s, $t = 6$ s, $t = 10$ s, $t = 15$ s, $t = 20$ s and produce the bar graph of $X(0.4)$, $X(1.5)$, $X(6)$, $X(10)$, $X(15)$, $X(20)$ in Figure 1, respectively. It is clear that the graphs are getting closer. Next, the difference between probability distribution functions (PDFs) is measured using the Kolmogorov-Smirnov test (K-S test). The difference between PDFs for the system (28) is shown in Figure 2, where it can be seen that the difference gradually decreases, which indicates that PDFs at quite distant time points follow the same distribution.

5 Conclusion

The stability in distribution of a class of highly nonlinear HSDDEs has been thoroughly investigated in this study. One of the major advancements lies in relaxing the requirement for the system's time delay functions to be differentiable, eliminating the necessity that its derivative is less than 1. Another significant progress is the allowance for the coefficients of the system to exhibit high nonlinearity, thereby lifting the constraints imposed by classical linear growth conditions and only necessitating compliance with the generalized Hasminskii-type conditions for Lyapunov functions. Crucially, a novel theorem concerning the stability in distribution of the system has been established, providing explicitly the sufficient criterion, which holds substantial value for the study of numerous practical systems.

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References

1 Mao X, Yuan C. Stochastic Differential Equations with Markovian Switching. London: Imperial College Press, 2006
2 Cui R H, Xie X J. Finite-time stochastic integral input-to-state stability and its applications. Automatica, 2023, 158: 111311
3 Zhao G H, Liu S J. Finite-time stabilization of nonlocal Lipschitzian stochastic time-varying nonlinear systems with Markovian switching. Sci China Inf Sci, 2022, 65: 212204

4 Wu Z H, Zhou H C, Guo B Z, et al. On the convergence of tracking differentiator with multiple stochastic disturbances. Sci China Inf Sci, 2024, 67: 122203
5 Fu L, Peng S G, Deng F Q, et al. Mean-square exponential stability of stochastic Volterra systems in infinite dimensions. Sci China Inf Sci, 2024, 67: 202206
6 Wu A, Yu H, Zeng Z. Stabilization of highly nonlinear hybrid neutral stochastic neural networks with time-varying delays by variable-delay feedback control. Syst Control Lett, 2023, 172: 105434
7 Wu A, Yu H, Zeng Z. Variable-delay feedback control for stabilisation of highly nonlinear hybrid stochastic neural networks with time-varying delays. Int J Control, 2024, 97: 744–755
8 Cui R H, Xie X J. Exponential stabilization of stochastic feedforward nonlinear systems: a dynamic gain approach. IEEE Trans Automat Contr, 2024, 69: 2605–2612
9 Hu L, Mao X, Shen Y. Stability and boundedness of nonlinear hybrid stochastic differential delay equations. Syst Control Lett, 2013, 62: 178–187
10 Zhang J, Fridman E. Dynamic event-triggered control of networked stochastic systems with scheduling protocols. IEEE Trans Automat Contr, 2021, 66: 6139–6147
11 Nguyen D H, Nguyen D, Nguyen S L. Stability in distribution of path-dependent hybrid diffusion. SIAM J Control Optim, 2021, 59: 434–463
12 Athans M. Command and control (C2) theory: a challenge to control science. IEEE Trans Automat Contr, 1987, 32: 286–293
13 Sworder D, Robinson V. Feedback regulators for jump parameter systems with state and control dependent transition rates. IEEE Trans Automat Contr, 1973, 18: 355–360
14 Yuan C, Mao X. Asymptotic stability in distribution of stochastic differential equations with Markovian switching. Stochastic Processes Their Appl, 2003, 103: 277–291
15 Yuan C, Zou J, Mao X. Stability in distribution of stochastic differential delay equations with Markovian switching. Syst Control Lett, 2003, 50: 195–207
16 Du N H, Dang N H, Dieu N T. On stability in distribution of stochastic differential delay equations with Markovian switching. Syst Control Lett, 2014, 65: 43–49
17 Wang Y, Wu F, Mao X. Stability in distribution of stochastic functional differential equations. Syst Control Lett, 2019, 132: 104513
18 Basak G K, Bisi A, Ghosh M K. Stability of a random diffusion with linear drift. J Math Anal Appl, 1996, 202: 604–622
19 Bao J, Böttcher B, Mao X, et al. Convergence rate of numerical solutions to SFDEs with jumps. J Comput Appl Math, 2011, 236: 119–131
20 Mao X, Yuan C, Zou J. Stochastic differential delay equations of population dynamics. J Math Anal Appl, 2005, 304: 296–320
21 Song G, Wang Y, Li T, et al. Quantized feedback stabilization for nonlinear hybrid stochastic time-delay systems with discrete-time observation. IEEE Trans Cybern, 2022, 52: 13373–13382
22 Dong H, Mao X. Advances in stabilization of highly nonlinear hybrid delay systems. Automatica, 2022, 136: 110086
23 Hu J, Mao W, Mao X. Advances in nonlinear hybrid stochastic differential delay equations: existence, boundedness and stability. Automatica, 2023, 147: 110682
24 Li X, Mao X. Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control. Automatica, 2020, 112: 108657
25 Fridman E. Introduction to Time-Delay Systems: Analysis and Control. Basel: Birkhauser, 2014
26 Billingsley P. Convergence of Probability Measures. 2nd ed. Hoboken: John Wiley & Sons, Inc., 1999
27 Anderson W. Continuous-Time Markov Chains. New York: Springer, 1991
28 Ikeda N, Watanabe S. Stochastic Differential Equations and Diffusion Processes. Amsterdam: North-Holland, 1981
29 Bahar A, Mao X. Stochastic delay population dynamics. Int J Pure Appl Math, 2004, 11: 377–399
30 Mao X. Stochastic Differential Equations and Their Applications. Amsterdam: Elsevier, 2007