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Bipartite entanglement measures and entanglement constraints

Hui LI[1], Ting GAO[1*] & Fengli YAN[2*]

[1]*School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China*
[2]*College of Physics, Hebei Key Laboratory of Photophysics Research and Application, Hebei Normal University, Shijiazhuang 050024, China*

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Quantum entanglement, as an essential physical resource, exhibits prominent advantages over classical theory in quantum information tasks. A fundamental and necessary task in entanglement resource theory is undoubtedly the development of a legitimate method for quantifying the entanglement of states. To gain a deeper insight into the essence of quantum entanglement, another imperative assignment is to explore the intrinsic properties of a measure, such as whether it satisfies monogamy of entanglement (MoE).

MoE means that there are some restrictions on shareability and distribution of entanglement [1]. The mathematical characterization of the monogamy relation was first introduced by Coffman et al. [2] based on the square of concurrence [3] in three-qubit systems. Subsequently, Osborne and Verstraete [4] extended it to multiqubit systems. However, the tangle generated from the MoE of the squared concurrence fails to detect the entanglement of some states, such as the W state [2]. To compensate for this deficiency, other novel monogamy relations beyond squared concurrence should be established. Therefore, we construct a class of bipartite entanglement measures G_q-concurrence $(q > 1)$ and verify their squares conform to the monogamy inequality with $1 < q < 2$. Most importantly, the series of indicators produced by MoE of the squared G_q-concurrence can identify all genuinely multiqubit entangled states.

G_q-concurrence. For any bipartite pure state $|\phi\rangle_{AB}$, the G_q-concurrence is defined as

$$\mathscr{C}_q(|\phi\rangle_{AB}) = [1 - \mathrm{Tr}(\rho_A^q)]^{\frac{1}{q}}, \tag{1}$$

where $q > 1$ and $\rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$.

A bipartite pure state $|\phi\rangle_{AB}$ can be expressed in the Schmidt decomposition form

$$|\phi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i}|i_A\rangle|i_B\rangle, \tag{2}$$

where λ_i is non-negative real number with $\sum_i \lambda_i = 1$. Then, $\mathscr{C}_q(|\phi\rangle_{AB})$ can be written as

$$\mathscr{C}_q(|\phi\rangle_{AB}) = \left(1 - \sum_{i=1}^{d} \lambda_i^q\right)^{\frac{1}{q}}, \; q > 1. \tag{3}$$

For any bipartite mixed state ρ_{AB}, its G_q-concurrence is defined as

$$\mathscr{C}_q(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i \mathscr{C}_q(|\phi_i\rangle), \tag{4}$$

where the minimum is taken over all possible ensemble decompositions $\{p_i, |\phi_i\rangle\}$ of ρ_{AB}.

In particular, G_q-concurrence $\mathscr{C}_q(\rho_{AB})$ and concurrence $C(\rho_{AB})$ [3] are equivalent when $q = 2$ for any bipartite quantum state ρ_{AB}, so the G_q-concurrence can be treated as a generalization of concurrence. Moreover, the relation between G_q-concurrence and q-concurrence [5] is $\mathscr{C}_q(|\phi\rangle_{AB}) = [C_q(|\phi\rangle_{AB})]^{1/q}$ for any bipartite pure state $|\phi\rangle_{AB}$. In addition, G_q-concurrence and q-concurrence are compared in Appendix A.1, and the merits of G_q-concurrence are illustrated in terms of normalization and sensitivity.

The G_q-concurrence of assistance (G_q-CoA), a dual quantity of G_q-concurrence, is given by

$$\mathscr{C}_q^a(\rho_{AB}) = \max_{\{p_i,|\phi_i\rangle\}} \sum_i p_i \mathscr{C}_q(|\phi_i\rangle), \tag{5}$$

where the maximum runs over all feasible ensemble decompositions of ρ_{AB}. If ρ_{AB} is a pure state, then there is $\mathscr{C}_q^a(\rho_{AB}) = \mathscr{C}_q(\rho_{AB})$.

Some studies suggest that a rational entanglement measure should satisfy several conditions [1], including (i) faithfulness, (ii) invariance under any local unitary transformation, and (iii) monotonicity under local operation and classical communication (LOCC).

In Appendixes A.2–A.6, we rigorously prove that G_q-concurrence not only satisfies the conditions (i)–(iii) listed above, but also fulfills the properties as follows: (iv) entanglement monotone (or strong monotonicity under LOCC); (v) convexity; (vi) subadditivity.

* Corresponding author (email: gaoting@hebtu.edu.cn, flyan@hebtu.edu.cn)

Analytic formula. An essential result of this work is to provide an analytical relation between G_q-concurrence and concurrence for any two-qubit quantum state ρ_{AB}, which is

$$\mathscr{C}_q(\rho_{AB}) = h_q[C(\rho_{AB})], \tag{6}$$

where $h_q(x) = [1 - (\frac{1+\sqrt{1-x^2}}{2})^q - (\frac{1-\sqrt{1-x^2}}{2})^q]^{\frac{1}{q}}$ and $1 < q \leqslant 2$. This relation bridges the gaps towards the in-depth discussion of any number of qubit systems. Furthermore, we analyze and demonstrate in Appendix B.1 that the function $h_q(x)$ possesses the following two properties: (B1) the function $h_q(x)$ is monotonically increasing with respect to x for $q > 1$; (B2) the function $h_q(x)$ is convex with respect to x for $1 < q \leqslant 2$. On the basis of the properties (B1) and (B2), we obtain that Eq. (6) is valid, and further details can be found in Appendix B.2. Meanwhile, we can straightforwardly arrive at the following conclusion. For any mixed state ρ_{AB} in $2 \otimes d$ systems, there is

$$\mathscr{C}_q(\rho_{AB}) \geqslant h_q[C(\rho_{AB})], \tag{7}$$

where $1 < q \leqslant 2$.

Here we particularly emphasize that when $q \neq 2$, the pure state decomposition used to calculate the concurrence is not necessarily the same as the one used to compute the G_q-concurrence. In Appendix B.3, we render a concrete example to illustrate this point. Then, based on (6), we next consider two questions: (Q1) does G_q-CoA obey the polygamy relation? and (Q2) does G_q-concurrence satisfy the monogamy relation?

Polygamy relation. We establish the polygamy relation of n-qubit quantum state $\rho_{A_1\cdots A_n}$ in terms of G_q-CoA, i.e.,

$$\mathscr{C}_q^a(\rho_{A_1|A_2\cdots A_n}) \leqslant \mathscr{C}_q^a(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^a(\rho_{A_1A_n}), \tag{8}$$

where $1 < q \leqslant 2$, $\mathscr{C}_q^a(\rho_{A_1|A_2\cdots A_n})$ denotes the G_q-CoA of $\rho_{A_1A_2\cdots A_n}$ in the partition $A_1|A_2\cdots A_n$, $\rho_{A_1A_j}$ is the reduced density matrix with respect to subsystem A_1A_j, $j = 2, 3, \cdots, n$. For the detailed proof of inequality (8), please see Appendix C.

Monogamy relation. Another key result of this work is to verify rigorously that the squared G_q-concurrence satisfies the monogamy relation for any n-qubit quantum state $\rho_{A_1A_2\cdots A_n}$,

$$\mathscr{C}_q^2(\rho_{A_1|A_2\cdots A_n}) \geqslant \mathscr{C}_q^2(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^2(\rho_{A_1A_n}), \tag{9}$$

where $1 < q \leqslant 2$, $\mathscr{C}_q(\rho_{A_1|A_2\cdots A_n})$ quantifies the entanglement in the partition $A_1|A_2\cdots A_n$.

As a by-product conclusion, we further obtain that for any n-qubit quantum state $\rho_{A_1\cdots A_n}$, the α-th ($\alpha \geqslant 2$) power of G_q-concurrence fulfills the monogamy relation,

$$\mathscr{C}_q^\alpha(\rho_{A_1|A_2\cdots A_n}) \geqslant \mathscr{C}_q^\alpha(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^\alpha(\rho_{A_1A_n}). \tag{10}$$

The detailed proofs of formulas (9) and (10) are provided in Appendix D.

Entanglement indicators. On account of the relation presented in inequality (9), we give a set of multipartite entanglement indicators,

$$\tau_q(\rho) = \min_{\{p_l,|\phi_l\rangle\}} \sum_l p_l \tau_q(|\phi_l\rangle_{A_1|A_2\cdots A_n}), \tag{11}$$

where the minimum is taken over all feasible ensemble decompositions, $\tau_q(|\phi_l\rangle_{A_1|A_2\cdots A_n}) = \mathscr{C}_q^2(|\phi_l\rangle_{A_1|A_2\cdots A_n}) - \sum_{j=2}^n \mathscr{C}_q^2(\rho^l_{A_1A_j})$, and $1 < q < 2$.

For any three-qubit quantum state ρ_{ABC}, we find that the tripartite entanglement indicator $\tau_q(\rho_{ABC})$ is zero for $1 < q < 2$ if and only if ρ_{ABC} is biseparable, i.e., $\rho_{ABC} = \sum_i p_i\rho^i_{AB} \otimes \rho^i_C + \sum_i q_i\rho^i_{AC} \otimes \rho^i_B + \sum_i r_i\rho^i_A \otimes \rho^i_{BC}$. This conclusion is elaborately demonstrated in Appendix E.1.

In fact, the inequality $\mathscr{C}_q^2(\rho_{A_i|\overline{A_i}}) \geqslant \sum_{j\neq i} \mathscr{C}_q^2(\rho_{A_iA_j})$ is true for any n-qubit quantum state. As a consequence, we construct a family of indicators

$$\tau_q^i(\rho) = \min_{\{p_l,|\phi_l\rangle\}} \sum_l p_l \tau_q^i(|\phi_l\rangle_{A_i|\overline{A_i}}), \tag{12}$$

where the minimum runs over all feasible pure decompositions, $\tau_q^i(|\phi_l\rangle_{A_i|\overline{A_i}}) = \mathscr{C}_q^2(|\phi_l\rangle_{A_i|\overline{A_i}}) - \sum_{j\neq i} \mathscr{C}_q^2(\rho^l_{A_iA_j})$, $i, j \in \{1, 2, \cdots, n\}$. The formula (11) is a special case of $\tau_q^i(\rho)$ corresponding to $i = 1$.

For any n-qubit quantum state ρ, we have that the multipartite entanglement indicator $\tau_q^i(\rho)$ is zero for $1 < q < 2$ if and only if the quantum state can be written in the form $\rho = \sum_k p_k\rho^k_{A_i} \otimes \rho^k_{\overline{A_i}} + \sum_{j\neq i}\sum_k q^j_k \rho^k_{A_iA_j} \otimes \rho^k_{\overline{A_iA_j}}$, where $i, j \in \{1, 2, \cdots, n\}$.

Such a series of entanglement indicators, $\tau_q^1(\rho)$ to $\tau_q^n(\rho)$, allows us to better understand the entanglement distribution of states and must be nonzero for any genuinely multiqubit entangled state. Therefore, these indicators we construct can compensate for the deficiency that the tangle fails to detect the entanglement of an n-qubit W state, as the example shows in Appendix E.2.

Conclusion. This work has proposed a type of one-parameter entanglement quantifiers, called G_q-concurrence ($q > 1$), as a generalization of concurrence, successfully addressing and compensating several deficiencies of concurrence. In addition, an analytic formula between G_q-concurrence and concurrence has been established for $1 < q \leqslant 2$ in two-qubit systems, which allows the easy calculation of G_q-concurrence for a pair of qubits. Using this analytic formula, we have provided the polygamy inequality based on G_q-CoA in multiqubit systems. Furthermore, a mathematical characterization of the monogamy relation in terms of the square of G_q-concurrence has been presented in multiqubit systems. From a practical standpoint, we have constructed a generalized set of entanglement indicators leveraging the MoE of G_q-concurrence, which can work well even when the tangle loses its efficacy. Moreover, a straightforward conclusion that the α-th ($\alpha \geqslant 2$) power of G_q-concurrence also satisfies the monogamy inequality can be reached. The calculation of the multipartite entanglement measures is an NP problem. Fortunately, the monogamy inequality in formula (9) has rendered a readily computable lower bound of G_q-concurrence.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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