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• **Supplementary File** •

Bipartite entanglement measures and entanglement constraints

Hui LI[1], Ting GAO[1*] & Fengli YAN[2*]

[1]*School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China*
[2]*College of Physics, Hebei Key Laboratory of Photophysics Research and Application, Hebei Normal University, Shijiazhuang 050024, China*

Appendix A G_q-concurrence

Appendix A.1 The comparison between G_q-concurrence and q-concurrence

Compared with q-concurrence [1], we elaborate on the merits of G_q-concurrence from the following two aspects:

(1) The G_q-concurrence exhibits better normalization. For instance, given the maximally entangled state $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and calculating its G_q-concurrence and q-concurrence, we find that $\mathscr{C}_q(|\phi\rangle)$ is closer to 1, as shown in Table A1.

Table A1 For the maximally entangled state $|\phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, its q-concurrence $C_q(|\phi\rangle)$ and G_q-concurrence $\mathscr{C}_q(|\phi\rangle)$ with some specific parameters q are given. Here four decimal places are retained.

q	$C_q(\|\phi\rangle)$	$\mathscr{C}_q(\|\phi\rangle)$
5.5	0.9558	0.9918
6.0	0.9688	0.9947
6.5	0.9779	0.9966
7.0	0.9844	0.9978
7.5	0.9890	0.9985
8.0	0.9922	0.9990
8.5	0.9945	0.9993
9.0	0.9961	0.9996

(2) The G_q-concurrence has a relatively high sensitivity, which can amplify minute differences in weakly entangled states and enable efficient discrimination of weakly entangled states. For example, consider the quantum state $|\phi\rangle = \sqrt{1-\varepsilon}|00\rangle + \sqrt{\varepsilon}|11\rangle$, $0 \leqslant \varepsilon \leqslant 0.05$, its G_q-concurrence and q-concurrence are plotted in Figure A1 when $q = \frac{3}{2}$. We observe that for the minor differences of weakly entangled states, the variation range of G_q-concurrence is larger and its sensitivity is stronger, which makes it more conducive to characterizing the entanglement characteristics of quantum states.

Appendix A.2 Faithfulness

Faithfulness is an essential property for entanglement quantifiers, which can clearly distinguish bipartite quantum states into two categories, entangled states and separable states. Next we demonstrate that G_q-concurrence is faithful.

Proposition A1. For any bipartite quantum state ρ_{AB}, we have $\mathscr{C}_q(\rho_{AB}) \geqslant 0$ for $q > 1$, the equality holds if and only if (iff) ρ_{AB} is a separable state.

Proof. It is obvious that $\mathscr{C}_q(\rho_{AB}) \geqslant 0$ since $\mathrm{Tr}(\rho_A^q) \leqslant 1$ for $q > 1$.

Next, we first prove the equality is true iff $|\phi\rangle_{AB}$ is a separable state. If a pure state $|\phi\rangle_{AB}$ is separable, then we can get $\mathrm{Tr}(\rho_A^q) = 1$, which leads $\mathscr{C}_q(|\phi\rangle_{AB}) = 0$. Conversely, let $|\phi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|i_A\rangle|i_B\rangle$, one has the reduced density operator $\rho_A = \sum_i \lambda_i |i_A\rangle\langle i_A|$. If $\mathscr{C}_q(|\phi\rangle_{AB}) = 0$, then the Schmidt number of $|\phi\rangle_{AB}$ must be one due to $0 \leqslant \lambda_i \leqslant 1$ and $q > 1$, i.e., $|\phi\rangle_{AB} = |i_A\rangle|i_B\rangle$, hence the pure state $|\phi\rangle_{AB}$ is separable.

For any separable mixed state ρ_{AB} with the pure decomposition $\{p_i, |\phi_i\rangle_{AB}\}$, $\mathscr{C}_q(\rho_{AB}) \leqslant \sum_i p_i \mathscr{C}_q(|\phi_i\rangle_{AB}) = 0$, owing to the nonnegativity of $\mathscr{C}_q(\rho_{AB})$, we have $\mathscr{C}_q(\rho_{AB}) = 0$. On the contrary, if $\mathscr{C}_q(\rho_{AB}) = 0$, according to the definition of G_q-concurrence, one has $\mathscr{C}_q(|\phi_i\rangle_{AB}) = 0$ for any i, which is equivalent to $|\phi_i\rangle_{AB}$ being separable for every i, so ρ_{AB} is separable.

To sum up, $\mathscr{C}_q(\rho_{AB}) > 0$ for all entangled states and $\mathscr{C}_q(\rho_{AB}) = 0$ for all separable states.

* Corresponding author (email: gaoting@hebtu.edu.cn, flyan@hebtu.edu.cn)

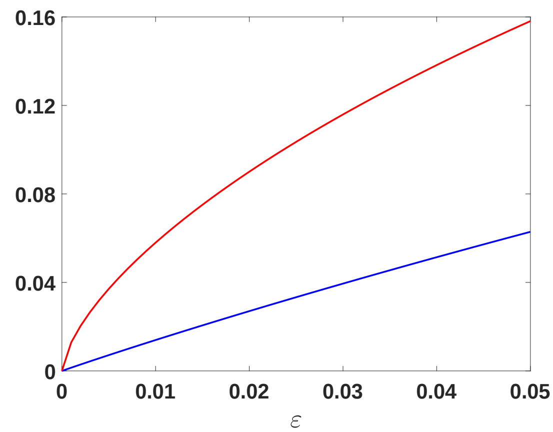

Figure A1 For the quantum state $|\phi\rangle=\sqrt{1-\varepsilon}|00\rangle+\sqrt{\varepsilon}|11\rangle$, $0\leqslant\varepsilon\leqslant 0.05$. Set $q=\frac{3}{2}$, the red (upper) line is the G_q-concurrence of $|\phi\rangle$, the blue (lower) line represents the q-concurrence of $|\phi\rangle$.

Appendix A.3 Monotonicity under LOCC and invariance under local unitary transformation

Vidal [2] put forward the only necessary condition on the entanglement measure should be that entanglement does not increase under local operation and classical communication (LOCC). We verify that G_q-concurrence obeys this requirement.

Proposition A2. Let ρ_{AB} be any bipartite state, the G_q-concurrence is non-increasing under LOCC operation $\Lambda_{\rm LOCC}$, i.e., $\mathscr{C}_q[\Lambda_{\rm LOCC}(\rho_{AB})]\leqslant\mathscr{C}_q(\rho_{AB})$.

Proof. The state $|\phi\rangle$ can be prepared from the state $|\varphi\rangle$ using only LOCC iff the vector $\vec{\lambda}_\phi$ majorizes $\vec{\lambda}_\varphi$ ($\vec{\lambda}_\varphi\prec\vec{\lambda}_\phi$), where $\vec{\lambda}_\varphi$ ($\vec{\lambda}_\phi$) is the Schmidt vector given by the squared Schmidt coefficients of the state $|\varphi\rangle$ ($|\phi\rangle$) and arranged in non-increasing order [3]. A function E is monotone on pure state iff it is Schur concave as a function of spectrum of subsystem, which is equivalent to the following two conditions [4]: (a) E is invariant under any permutation of two arguments; (b) any two components of $\vec{\lambda}$, λ_i and λ_j, satisfy $(\lambda_i-\lambda_j)(\frac{\partial E}{\partial\lambda_i}-\frac{\partial E}{\partial\lambda_j})\leqslant 0$.

We first show that the function $\mathscr{C}_q(|\phi\rangle)$ is non-increasing under LOCC with respect to any bipartite pure state $|\phi\rangle$ using the methods described above. Let $\mathscr{C}_q(|\phi\rangle)=f(\lambda_1,\lambda_2,\cdots,\lambda_d)=(1-\sum_{i=1}^d\lambda_i^q)^{\frac{1}{q}}$, where $\lambda_1,\lambda_2,\cdots,\lambda_d$ are the square of Schmidt coefficients of $|\phi\rangle$ and satisfy $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_d$. It is easy to get that $\mathscr{C}_q(|\phi\rangle)$ is invariant when any two arguments λ_i and λ_j of the vector $\vec{\lambda}_\phi$ permute, and

$$
\begin{aligned}
(\lambda_i-\lambda_j)(\frac{\partial\mathscr{C}_q}{\partial\lambda_i}-\frac{\partial\mathscr{C}_q}{\partial\lambda_j}) &= (\lambda_i-\lambda_j)[(1-\sum_{l=1}^d\lambda_l^q)^{\frac{1}{q}-1}\lambda_j^{q-1}-(1-\sum_{l=1}^d\lambda_l^q)^{\frac{1}{q}-1}\lambda_i^{q-1}]\\
&= (\lambda_i-\lambda_j)(1-\sum_{l=1}^d\lambda_l^q)^{\frac{1}{q}-1}(\lambda_j^{q-1}-\lambda_i^{q-1})\\
&\leqslant 0.
\end{aligned}
$$

Thus, we conclude that $\mathscr{C}_q[\Lambda_{\rm LOCC}(|\phi\rangle)]\leqslant\mathscr{C}_q(|\phi\rangle)$.

Adopting the convexity of $\mathscr{C}_q(\rho_{AB})$ and the monotonicity of $\mathscr{C}_q(|\phi\rangle)$, we can obtain the result that G_q-concurrence does not increase under LOCC for any bipartite mixed state.

It is known that local unitary transformations belong to the set of LOCC operations and are invertible [5]. From Proposition A2, we proceed directly to the following conclusion.

Proposition A3. For any bipartite quantum state ρ_{AB}, $\mathscr{C}_q(\rho_{AB})$ is invariant under any local unitary transformation, i.e., $\mathscr{C}_q(\rho_{AB})=\mathscr{C}_q(U_A\otimes U_B\rho_{AB}U_A^\dagger\otimes U_B^\dagger)$.

Appendix A.4 Entanglement monotone

Before proving the strong monotonicity of G_q-concurrence, let us present a lemma.

Lemma A1. The function

$$G_q(\rho)=(1-{\rm Tr}\rho^q)^{\frac{1}{q}},\tag{A1}$$

is concavity for any density operator ρ and $q>1$.

Proof. Let ρ, σ be two arbitrary density operators, we derive

$$\begin{aligned} G_q(\lambda\rho+\mu\sigma) =& \ [1-\mathrm{Tr}(\lambda\rho+\mu\sigma)^q]^{\frac{1}{q}} \\ \geqslant& \ \{1-[\lambda(\mathrm{Tr}\rho^q)^{\frac{1}{q}}+\mu(\mathrm{Tr}\sigma^q)^{\frac{1}{q}}]^q\}^{\frac{1}{q}} \\ \geqslant& \ [1-(\lambda\mathrm{Tr}\rho^q+\mu\mathrm{Tr}\sigma^q)]^{\frac{1}{q}} \\ =& \ [\lambda(1-\mathrm{Tr}\rho^q)+\mu(1-\mathrm{Tr}\sigma^q)]^{\frac{1}{q}} \\ \geqslant& \ \lambda(1-\mathrm{Tr}\rho^q)^{\frac{1}{q}}+\mu(1-\mathrm{Tr}\sigma^q)^{\frac{1}{q}} \\ =& \ \lambda G_q(\rho)+\mu G_q(\sigma), \end{aligned} \tag{A2}$$

where the first inequality can be obtained based on Minkowski's inequality $[\mathrm{Tr}(\rho+\sigma)^q]^{\frac{1}{q}} \leqslant (\mathrm{Tr}\rho^q)^{\frac{1}{q}} + (\mathrm{Tr}\sigma^q)^{\frac{1}{q}}$ with $q>1$, the second inequality holds because the function $y=x^q$ is convex for $q>1$, and the third inequality is due to the concavity of $y=x^\gamma$ for $0<\gamma<1$.

Proposition A4. For any bipartite state ρ_{AB}, the G_q-concurrence is an entanglement monotone, namely,

$$\mathscr{C}_q(\rho_{AB}) \geqslant \textstyle\sum_i p_i \mathscr{C}_q(\sigma_i), \tag{A3}$$

where the ensemble $\{p_i, \sigma_i\}$ is yielded after Λ_{LOCC} acting on ρ_{AB}.

Proof. Vidal [2] showed that an entanglement quantifier E obeys strong monotonicity if it satisfies the following two conditions: (c) $g(U\rho_A U^\dagger) = g(\rho_A)$ and g is a concave function, where $E(|\phi\rangle_{AB}) = g(\rho_A)$ and $\rho_A = \mathrm{Tr}_B(|\phi\rangle\langle\phi|)$; (d) E is given by convex roof extension for arbitrary mixed states. It is obvious that $\mathscr{C}_q(\rho_{AB})$ meets these conditions from Proposition A3, Lemma A1, and the definition of $\mathscr{C}_q(\rho_{AB})$. Therefore, the formula (A3) holds.

Appendix A.5 Convexity

According to the definition of $\mathscr{C}_q(\rho_{AB})$, the following result can be reached.

Proposition A5. The G_q-concurrence is convex on quantum state ρ_{AB}, that is, $\mathscr{C}_q(\rho_{AB}) \leqslant \sum_i p_i \mathscr{C}_q(\rho^i_{AB})$, where $\rho_{AB} = \sum_i p_i \rho^i_{AB}$, $\sum_i p_i = 1$, and $p_i > 0$.

Appendix A.6 Subadditivity

Proposition A6. The G_q-concurrence is subadditive, i.e., $\mathscr{C}_q(\rho_{AB} \otimes \sigma_{AB}) \leqslant \mathscr{C}_q(\rho_{AB}) + \mathscr{C}_q(\sigma_{AB})$.

Proof. Before proving subadditivity, we first show the inequality $(a+b)^\beta \leqslant a^\beta + b^\beta$ holds for $0 \leqslant a, b \leqslant 1$ and $0<\beta<1$, which is equivalent to

$$(\tfrac{a}{a+b})^\beta + (\tfrac{b}{a+b})^\beta \geqslant 1. \tag{A4}$$

When $0 \leqslant x_1, x_2 \leqslant 1$ and $x_1 + x_2 = 1$, we have $x_1^\beta \geqslant x_1$ and $x_2^\beta \geqslant x_2$, and this goes directly to inequality $x_1^\beta + x_2^\beta \geqslant 1$. Let $x_1 = \frac{a}{a+b}$ and $x_2 = \frac{b}{a+b}$, the formula (A4) can be obtained.

For any two pure states $|\phi\rangle_{AB}$ and $|\varphi\rangle_{AB}$, then we can see

$$\begin{aligned} \mathscr{C}_q(|\phi\rangle_{AB} \otimes |\varphi\rangle_{AB}) =& \ (1-\mathrm{Tr}\rho_A^q \mathrm{Tr}\delta_A^q)^{\frac{1}{q}} \\ \leqslant& \ (1-\mathrm{Tr}\rho_A^q + 1 - \mathrm{Tr}\delta_A^q)^{\frac{1}{q}} \\ \leqslant& \ (1-\mathrm{Tr}\rho_A^q)^{\frac{1}{q}} + (1-\mathrm{Tr}\delta_A^q)^{\frac{1}{q}} \\ =& \ \mathscr{C}_q(|\phi\rangle_{AB}) + \mathscr{C}_q(|\varphi\rangle_{AB}). \end{aligned} \tag{A5}$$

Here ρ_A and δ_A are respectively the reduced density matrices of $|\phi\rangle_{AB}$ and $|\varphi\rangle_{AB}$, the first inequality holds because $C_q(|\phi\rangle_{AB}) = 1 - \mathrm{Tr}(\rho_A^q)$ satisfies subadditivity [6], the second inequality can be obtained according to the relation $(a+b)^\beta \leqslant a^\beta + b^\beta$ for $0 \leqslant a, b \leqslant 1$ and $0<\beta<1$.

The subadditivity of G_q-concurrence for any bipartite mixed state can be verified based on the convexity of $\mathscr{C}_q(\rho_{AB})$ and the relation derived in inequality (A5).

Appendix B Analytic formula

Before proving equation (6) in the letter, we first present two fundamental properties of $h_q(x)$.

Appendix B.1 Two fundamental properties of $h_q(x)$

Proposition B1. The function $h_q(x)$ is monotonically increasing with respect to x for $q>1$.

Proof. This proposition is true if the first derivative of $h_q(x)$ is nonnegative. Then we derive

$$\frac{dh_q(x)}{dx} = \frac{1}{2^q}[1-(\tfrac{1+\sqrt{1-x^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2}}{2})^q]^{\frac{1}{q}-1} \frac{x[(1+\sqrt{1-x^2})^{q-1} - (1-\sqrt{1-x^2})^{q-1}]}{\sqrt{1-x^2}}.$$

Obviously, there is $\frac{dh_q(x)}{dx} > 0$ for $0<x<1$ and $q>1$. Therefore, we can say $h_q(x)$ is monotonically increasing for $0 \leqslant x \leqslant 1$ owing to the fact that $h_q(x)$ is continuous. This makes that $h_q(0) = 0$ and $h_q(1) = (1-\frac{1}{2^{q-1}})^{\frac{1}{q}}$ correspond, respectively, to the minimum and maximum of $h_q(x)$ for the given parameter q.

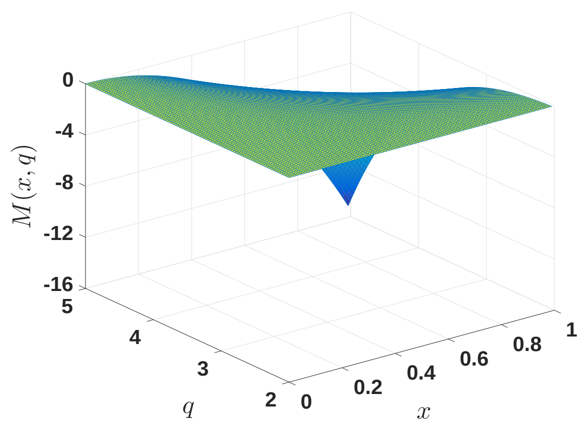

Figure B1 The function $M(x,q)$ is illustrated for $0 \leqslant x \leqslant 1$ and $2 < q < 5$.

Proposition B2. The function $h_q(x)$ is convex with respect to x for $1 < q \leqslant 2$.
Proof. This proposition is valid if the second derivative of $h_q(x)$ is nonnegativity. By derivation, we obtain

$$\frac{d^2h_q(x)}{dx^2} = \frac{1}{2^q}[1 - (\tfrac{1+\sqrt{1-x^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2}}{2})^q]^{\frac{1}{q}-2}M(x,q),$$

where $M(x,q) = \xi_1 + \xi_2(\xi_3 - \xi_4)$ with

$$\begin{aligned}
\xi_1 &= \tfrac{1-q}{2^q}(\tfrac{x[(1+\sqrt{1-x^2})^{q-1}-(1-\sqrt{1-x^2})^{q-1}]}{\sqrt{1-x^2}})^2,\\
\xi_2 &= 1 - (\tfrac{1+\sqrt{1-x^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2}}{2})^q,\\
\xi_3 &= \tfrac{(1+\sqrt{1-x^2})^{q-2}}{1-x^2}[\tfrac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}} - x^2(q-1)],\\
\xi_4 &= \tfrac{(1-\sqrt{1-x^2})^{q-2}}{1-x^2}[\tfrac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}} + x^2(q-1)].
\end{aligned}$$

We observe that judging the sign of $\frac{d^2h_q(x)}{dx^2}$ is actually equivalent to judging the sign of $M(x,q)$ because the term in front of $M(x,q)$ is positive for $0 < x < 1$. If the term $\frac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}} - x^2(q-1)$ in ξ_3 is non-positive, then there must be $M(x,q) < 0$ since $\xi_1 < 0$, $\xi_2 > 0$, $\xi_3 - \xi_4 < 0$ for $0 < x < 1$. By means of the result in Ref. [7], we deduce directly there is $x \in (0,1)$ such that $\frac{d^2h_q(x)}{dx^2} < 0$ for $q \geqslant 5$. In addition, when $2 < q < 5$, we can get that $h_q(x)$ is also not a convex function from Figure B1. Especially, when q takes 3 and 4, $h_3(x)$ and $h_4(x)$ are respectively

$$h_3(x) = \sqrt[3]{\tfrac{3}{4}}x^{\frac{2}{3}} \quad \text{and} \quad h_4(x) = \frac{\sqrt[4]{8x^2-x^4}}{\sqrt[4]{8}}.$$

Obviously, they are concave functions of x. Consequently, $h_q(x)$ is not a convex function for $q > 2$.

In particular, when $q = 2$, $h_2(x) = \frac{x}{\sqrt{2}}$ is a function that is both convex and concave.

Let us now show that $h_q(x)$ is a convex function on $x \in [0,1]$ for $1 < q < 2$, that is, prove that the minimum of $M(x,q)$ is nonnegative. It is acknowledged that the minimal value can only be generated at critical points or boundary points since $M(x,q)$ is continuous.

We first discuss whether there are critical points of $M(x,q)$ in the region $R = \{(x,q)|0 < x < 1, 1 < q < 2\}$. The gradient of $M(x,q)$ is

$$\nabla M(x,q) = (\tfrac{\partial M(x,q)}{\partial x}, \tfrac{\partial M(x,q)}{\partial q}),$$

where $\frac{\partial M(x,q)}{\partial x}$ and $\frac{\partial M(x,q)}{\partial q}$ are the first partial derivatives of $M(x,q)$ with respect to x and q, respectively.

The point (x_0, q_0) is a critical point if $\nabla M(x_0,q_0) = 0$. However, $\frac{\partial M(x,q)}{\partial x} = 0$ is unsolvable for $0 < x < 1$ and $1 < q < 2$, as shown in Figure B2. This suggests that there is no critical point of $M(x,q)$ in the region R and the maximal and minimal values of $M(x,q)$ can be respectively obtained at the boundary points $x = 1$ and $x = 0$ for given q. Through tedious calculations, we get

$$\begin{aligned}
&\lim_{x\to 0} M(x,q) = 0,\\
&\lim_{x\to 1} M(x,q) = \tfrac{-12(q^3-3q^2+3q-1)+(2^q-2)(-2q^3+12q^2-16q+6)}{3\times 2^q}.
\end{aligned}$$

The solutions of $\lim_{x\to 1} M(x,q) = 0$ only rise at $q = 1$ or 2 for $1 \leqslant q \leqslant 2$, $\lim_{x\to 1} M(x,q)$ is strictly positive for $1 < q < 2$, as shown in Figure B3. Therefore, one sees $M(x,q) > 0$ for $0 < x \leqslant 1$ and $1 < q < 2$.

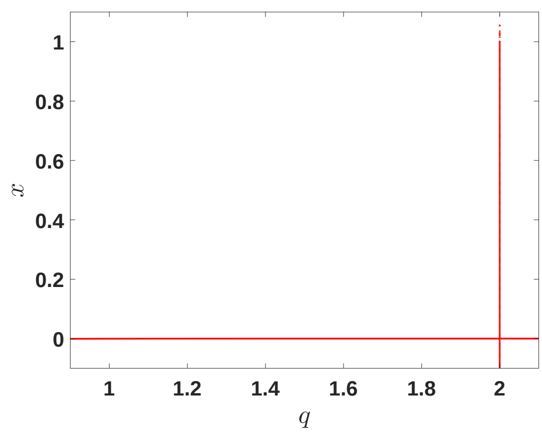

Figure B2 This diagram shows the solution for $\frac{\partial M(x,q)}{\partial x} = 0$.

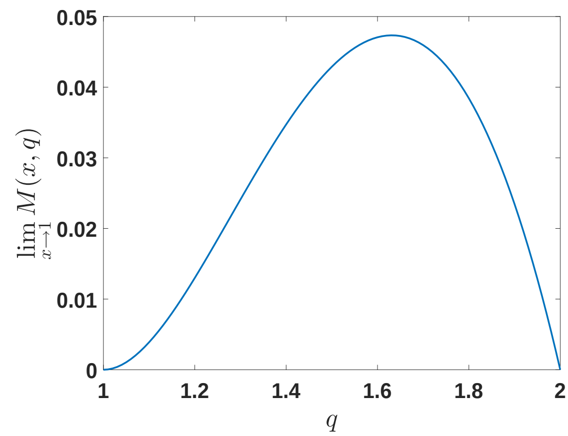

Figure B3 The function $\lim_{x\to 1} M(x,q)$ is plotted for $1 \leqslant q \leqslant 2$.

Based on the above analysis, we can directly obtain $\frac{d^2 h_q(x)}{dx^2} > 0$ in the region R. We proceed to discuss the second derivation $\frac{d^2 h_q(x)}{dx^2}$ at the points $x = 0$ and $x = 1$, and get

$$\begin{aligned}
\lim_{x\to 0} \frac{d^2 h_q(x)}{dx^2} &= +\infty, \\
\lim_{x\to 1} \frac{d^2 h_q(x)}{dx^2} &= \frac{(2^q-2)^{\frac{1-2q}{q}}}{2^{1-q}} \lim_{x\to 1} M(x,q).
\end{aligned}$$

This makes $\frac{d^2 h_q(x)}{dx^2} > 0$ for $x \in [0,1]$ and $q \in (1,2)$.

To sum up, the function $h_q(x)$ is convex for $1 < q \leqslant 2$ and $0 \leqslant x \leqslant 1$.

Based on propositions B1 and B2, we provide the proof of equation (6) below.

Appendix B.2 Proof of equation (6)

Let $|\phi\rangle_{AB}$ be a pure state on Hilbert space $\mathcal{H}^2 \otimes \mathcal{H}^d$ $(d \geqslant 2)$ (or especially be a two-qubit pure state with Schmidt decomposition $|\phi\rangle_{AB} = \sum_{i=1}^{2} \sqrt{\lambda_i}|ii\rangle_{AB}$), its G_q-concurrence is expressed as

$$\mathscr{C}_q(|\phi\rangle_{AB}) = (1 - \lambda_1^q - \lambda_2^q)^{\frac{1}{q}},$$

and its concurrence is

$$C(|\phi\rangle_{AB}) = 2\sqrt{\lambda_1\lambda_2}.$$

Moreover, it is easy to obtain that for any $2 \otimes d$ pure state there is an analytic function that relates G_q-concurrence $(q > 1)$ to concurrence, namely,

$$\mathscr{C}_q(|\phi\rangle_{AB}) = h_q[C(|\phi\rangle_{AB})], \tag{B1}$$

where $h_q(x)$ is

$$h_q(x) = [1 - (\tfrac{1+\sqrt{1-x^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2}}{2})^q]^{\frac{1}{q}},$$

for $0 \leqslant x \leqslant 1$.

For any $2 \otimes d$ mixed state ρ_{AB}, let $\{p_i, |\phi_i\rangle\}$ be the optimal pure state decomposition of $\mathscr{C}_q(\rho_{AB})$, then one has

$$\begin{aligned}
\mathscr{C}_q(\rho_{AB}) &= \textstyle\sum_i p_i \mathscr{C}_q(|\phi_i\rangle) \\
&= \textstyle\sum_i p_i h_q[C(|\phi_i\rangle)] \\
&\geqslant h_q[\textstyle\sum_i p_i C(|\phi_i\rangle)] \\
&\geqslant h_q[C(\rho_{AB})],
\end{aligned}$$

where the first inequality is due to the convexity of $h_q(x)$ for $1 < q \leqslant 2$ and the second inequality follows from the monotonicity of $h_q(x)$ for $q > 1$.

Hill and Wootters [8] pointed out that there is an optimal pure state decomposition $\{p_i, |\phi_i\rangle\}$ for any two-qubit mixed state ρ_{AB} such that the concurrence of each pure state is equal. Based on this assertion, one derives

$$\begin{aligned}
h_q[C(\rho_{AB})] &= h_q[\textstyle\sum_i p_i C(|\phi_i\rangle)] \\
&= \textstyle\sum_i p_i h_q[C(|\phi_i\rangle)] \\
&= \textstyle\sum_i p_i \mathscr{C}_q(|\phi_i\rangle) \\
&\geqslant \mathscr{C}_q(\rho_{AB}),
\end{aligned}$$

where the inequality holds according to the definition of $\mathscr{C}_q(\rho_{AB})$.

Therefore, the equation (6) in the letter is true for any two-qubit mixed state.

Appendix B.3 G_q-concurrence and concurrence are not equivalent for $q \neq 2$

We provide a concrete example to state that the pure state decomposition used to calculate the concurrence is not necessarily the same as the one used to compute the G_q-concurrence when $q \neq 2$.

Example 1. Consider a quantum state $\rho_{AB} = p|\Phi^+\rangle\langle\Phi^+| + (1-p)|01\rangle\langle 01|$, where $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, $0 < p < 1$. By calculation, we obtain $C(|\Phi^+\rangle) = 1$, $C(|01\rangle) = 0$, and $C(\rho_{AB}) = p$, so $\{p, |\Phi^+\rangle; 1-p, |01\rangle\}$ is the optimal pure state decomposition to calculate the concurrence. Let $q = \frac{3}{2}$, by Theorem 1, we have $\mathscr{C}_{\frac{3}{2}}(\rho_{AB}) = [1 - (\frac{1+\sqrt{1-p^2}}{2})^{\frac{3}{2}} - (\frac{1-\sqrt{1-p^2}}{2})^{\frac{3}{2}}]^{\frac{2}{3}}$, whereas $\mathscr{C}_{\frac{3}{2}}(\rho_{AB}) < p\mathscr{C}_{\frac{3}{2}}(|\Phi^+\rangle) + (1-p)\mathscr{C}_{\frac{3}{2}}(|01\rangle) = p[1 - 2\times(\frac{1}{2})^{\frac{3}{2}}]^{\frac{2}{3}}$, which means $\{p, |\Phi^+\rangle; 1-p, |01\rangle\}$ is not the optimal pure state decomposition to calculate the G_q-concurrence.

Appendix C Polygamy relation

To facilitate the proof of polygamy relation, we consider a function of two variables

$$H_q(x,y) = h_q(\sqrt{x^2+y^2}) - h_q(x) - h_q(y),$$

on the region $R' = \{(x,y)|0 \leqslant x, y, x^2+y^2 \leqslant 1\}$ for $1 < q \leqslant 2$.

Appendix C.1 Proof of the non-positivity of $H_q(x,y)$ on the domain

For the case $q = 2$, by simple calculation, we have $H_2(x,y) \leqslant 0$.

Since $H_q(x,y)$ is continuous on bounded closed set R', it can take maximal and minimal values for given q, which occur only at critical or boundary points. First, we determine whether there are critical points in the interior of R' by taking the first partial derivative of $H_q(x,y)$, its gradient is

$$\nabla H_q(x,y) = \left(\frac{\partial H_q(x,y)}{\partial x}, \frac{\partial H_q(x,y)}{\partial y}\right),$$

where

$$\begin{aligned}
\frac{\partial H_q(x,y)}{\partial x} =& \frac{x}{2^q}\{[1-(\tfrac{1+\sqrt{1-x^2-y^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2-y^2}}{2})^q]^{\frac{1}{q}-1}\frac{(1+\sqrt{1-x^2-y^2})^{q-1}-(1-\sqrt{1-x^2-y^2})^{q-1}}{\sqrt{1-x^2-y^2}}\\
&-[1-(\tfrac{1+\sqrt{1-x^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2}}{2})^q]^{\frac{1}{q}-1}\frac{(1+\sqrt{1-x^2})^{q-1}-(1-\sqrt{1-x^2})^{q-1}}{\sqrt{1-x^2}}\},\\
\frac{\partial H_q(x,y)}{\partial y} =& \frac{y}{2^q}\{[1-(\tfrac{1+\sqrt{1-x^2-y^2}}{2})^q - (\tfrac{1-\sqrt{1-x^2-y^2}}{2})^q]^{\frac{1}{q}-1}\frac{(1+\sqrt{1-x^2-y^2})^{q-1}-(1-\sqrt{1-x^2-y^2})^{q-1}}{\sqrt{1-x^2-y^2}}\\
&-[1-(\tfrac{1+\sqrt{1-y^2}}{2})^q - (\tfrac{1-\sqrt{1-y^2}}{2})^q]^{\frac{1}{q}-1}\frac{(1+\sqrt{1-y^2})^{q-1}-(1-\sqrt{1-y^2})^{q-1}}{\sqrt{1-y^2}}\}.
\end{aligned}$$

Assume that there is $(x_0,y_0) \in \{(x,y)|0 < x, y, x^2+y^2 < 1\}$ such that $\nabla H_q(x_0,y_0) = 0$, and we observe that $\nabla H_q(x_0,y_0) = 0$ is equivalent to

$$f_q(x_0) = f_q(y_0),$$

where the function $f_q(t)$ is

$$f_q(t) = [1-(\tfrac{1+\sqrt{1-t^2}}{2})^q - (\tfrac{1-\sqrt{1-t^2}}{2})^q]^{\frac{1}{q}-1}\frac{(1+\sqrt{1-t^2})^{q-1}-(1-\sqrt{1-t^2})^{q-1}}{\sqrt{1-t^2}}.$$

Then we evaluate the first derivative of $f_q(t)$,

$$\frac{df_q(t)}{dt} = [1-(\tfrac{1+\sqrt{1-t^2}}{2})^q - (\tfrac{1-\sqrt{1-t^2}}{2})^q]^{\frac{1}{q}-2}\widetilde{M}(t,q),$$

where

$$\begin{aligned}
\widetilde{M}(t,q) =& \frac{1-q}{2^q}\frac{t[(1+\sqrt{1-t^2})^{q-1}-(1-\sqrt{1-t^2})^{q-1}]^2}{1-t^2} + [1-(\tfrac{1+\sqrt{1-t^2}}{2})^q - (\tfrac{1-\sqrt{1-t^2}}{2})^q]\\
&\times\Big[\frac{(1+\sqrt{1-t^2})^{q-2}}{1-t^2}\Big(\frac{t(1+\sqrt{1-t^2})}{\sqrt{1-t^2}} - t(q-1)\Big) - \frac{(1-\sqrt{1-t^2})^{q-2}}{1-t^2}\Big(\frac{t(1-\sqrt{1-t^2})}{\sqrt{1-t^2}} + t(q-1)\Big)\Big].
\end{aligned}$$

Combining Figure C1, $\lim\limits_{t\to 0}\widetilde{M}(t,q) = 0$, and $\lim\limits_{q\to 1}\widetilde{M}(t,q) = 0$, we can see $\widetilde{M}(t,q) < 0$ for $1 < q < 2$ and $0 < t < 1$. This implies $\frac{df_q(t)}{dt} < 0$ for $1 < q < 2$ and $0 < t < 1$, namely, $f_q(t)$ is a strictly monotonically decreasing function with respect to t for given q, so $f_q(x_0) = f_q(y_0)$ means $x_0 = y_0$. If $\frac{\partial H_q(x,y)}{\partial x}|_{(x_0,y_0)} = 0$ and $x_0 = y_0 > 0$, then $f_q(\sqrt{2}x_0) = f_q(x_0)$, which contradicts to the strict monotonicity of $f_q(t)$. Hence $H_q(x,y)$ has no vanishing gradient in the interior of R'.

Next we discuss the boundary values of $H_q(x,y)$ in region R'. If $x = 0$ or $y = 0$, then $H_q(x,y) = 0$. If $x^2+y^2 = 1$, then $H_q(x,y)$ can be reduced to

$$l_q(x) = H_q(x,\sqrt{1-x^2}) = \tfrac{1}{2}\{(2^q-2)^{\frac{1}{q}} - [2^q - (1+\sqrt{1-x^2})^q - (1-\sqrt{1-x^2})^q]^{\frac{1}{q}} - [2^q - (1+x)^q - (1-x)^q]^{\frac{1}{q}}\}.$$

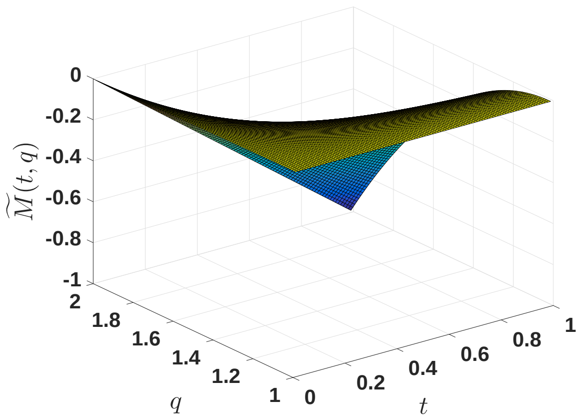

Figure C1 The $\widetilde{M}(t,q)$ is plotted as a function of t and q for $t \in (0,1)$ and $q \in (1,2)$.

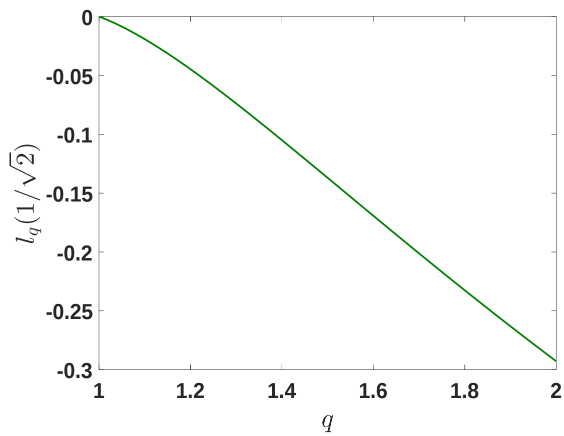

Figure C2 The function $l_q(1/\sqrt{2})$ is non-positive for $1 \leqslant q \leqslant 2$.

For the sake of judging the sign of $l_q(x)$, we calculate its first derivative

$$\begin{aligned}\frac{dl_q(x)}{dx} = & \frac{1}{2}\{-[2^q-(1+\sqrt{1-x^2})^q-(1-\sqrt{1-x^2})^q]^{\frac{1}{q}-1}\frac{x[(1+\sqrt{1-x^2})^{q-1}-(1-\sqrt{1-x^2})^{q-1}]}{\sqrt{1-x^2}}\\ & -[2^q-(1+x)^q-(1-x)^q]^{\frac{1}{q}-1}[(1-x)^{q-1}-(1+x)^{q-1}]\},\end{aligned}$$

where $\frac{dl_q(x)}{dx}=0$ corresponds to $x=\frac{1}{\sqrt{2}}$ on $0<x<1$. Since $l_q(0)=l_q(1)=0$, the sign of function $l_q(x)$ is determined by $l_q(\frac{1}{\sqrt{2}})=\frac{1}{2}\{(2^q-2)^{\frac{1}{q}}-2[2^q-(1+\frac{1}{\sqrt{2}})^q-(1-\frac{1}{\sqrt{2}})^q]^{\frac{1}{q}}\}$. We plot $l_q(\frac{1}{\sqrt{2}})$ in Figure C2 and obtain that $l_q(\frac{1}{\sqrt{2}})$ is always non-positive for $1<q<2$.

Therefore, we have

$$h_q(\sqrt{x^2+y^2}) \leqslant h_q(x)+h_q(y), \tag{C1}$$

for $1<q\leqslant 2$ and $(x,y)\in R'$.

Next, by virtue of the inequality (C1), we consider polygamy relation in multiqubit systems based on G_q-CoA.

Appendix C.2 The relation between G_q-CoA and CoA

We establish the relation between G_q-concurrence of assistance (G_q-CoA) and concurrence of assistance (CoA).

Lemma C1. For two-qubit state ρ_{AB}, there is

$$h_q[C^a(\rho_{AB})] \leqslant \mathscr{C}_q^a(\rho_{AB}), \tag{C2}$$

where $1<q\leqslant 2$.

Proof. Let $\{p_i,|\varphi_i\rangle\}$ be the pure state decomposition of ρ_{AB} such that $C^a(\rho_{AB})=\sum_i p_i C(|\varphi_i\rangle)$, then one sees

$$\begin{aligned}h_q[C^a(\rho_{AB})] &= h_q[\textstyle\sum_i p_i C(|\varphi_i\rangle)]\\ &\leqslant \textstyle\sum_i p_i h_q[C(|\varphi_i\rangle)]\\ &= \textstyle\sum_i p_i \mathscr{C}_q(|\varphi_i\rangle)\\ &\leqslant \mathscr{C}_q^a(\rho_{AB}),\end{aligned}$$

where the first inequality is due to the convexity of $h_q(x)$ for $1<q\leqslant 2$ and the second inequality is assured according to the definition of $\mathscr{C}_q^a(\rho_{AB})$.

Appendix C.3 Proof of inequality (8)

It is well-known that the square of CoA satisfies polygamy relation for n-qubit pure state $|\phi\rangle_{A_1A_2\cdots A_n}$, which reads [9]

$$C^2(|\phi\rangle_{A_1|A_2\cdots A_n}) \leqslant (C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_n})^2, \tag{C3}$$

where $C(|\phi\rangle_{A_1|A_2\cdots A_n})$ is the concurrence of $|\phi\rangle_{A_1A_2\cdots A_n}$ under bipartite splitting $A_1|A_2\cdots A_n$, and $C^a_{A_1A_j}$ is the CoA of reduced density operator $\rho_{A_1A_j}$, $j=2,\cdots,n$. Based on $H_q(x,y)\leqslant 0$ (see Appendix C.1), formulas (C2) and (C3), we will show the polygamy relation of n-qubit state in terms of G_q-CoA.

The detailed proof of inequality (8) is as follows:

On the one hand, if $(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_n})^2\leqslant 1$ for any multiqubit pure state $|\phi\rangle_{A_1\cdots A_n}$, then one has

$$\begin{aligned}\mathscr{C}_q(|\phi\rangle_{A_1|A_2\cdots A_n}) =&\ h_q[C(|\phi\rangle_{A_1|A_2\cdots A_n})]\\ \leqslant&\ h_q\big[\sqrt{(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_n})^2}\big]\\ \leqslant&\ h_q(C^a_{A_1A_2})+h_q\big[\sqrt{(C^a_{A_1A_3})^2+\cdots+(C^a_{A_1A_n})^2}\big]\\ \leqslant&\ \cdots\\ \leqslant&\ h_q(C^a_{A_1A_2})+\cdots+h_q(C^a_{A_1A_n})\\ \leqslant&\ \mathscr{C}^a_q(\rho_{A_1A_2})+\cdots+\mathscr{C}^a_q(\rho_{A_1A_n}).\end{aligned}\tag{C4}$$

Here the first inequality holds according to the formula (C3) and Proposition B1, the second inequality is true based on formula (C1), the penultimate inequality is obtained by iterating formula (C1), and the last inequality can be gotten from the formula (C2).

On the other hand, if $(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_n})^2>1$, then there is some j such that $(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_j})^2\leqslant 1$, whereas $(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_{j+1}})^2>1$, where $2\leqslant j\leqslant n$. Let

$$S=(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_{j+1}})^2-1,$$

then one reads

$$\begin{aligned}\mathscr{C}_q(|\phi\rangle_{A_1|A_2\cdots A_n}) =&\ h_q[C(|\phi\rangle_{A_1|A_2\cdots A_n})]\\ \leqslant&\ h_q(1)\\ =&\ h_q\big[\sqrt{(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_{j+1}})^2-S}\big]\\ \leqslant&\ h_q\big[\sqrt{(C^a_{A_1A_2})^2+\cdots+(C^a_{A_1A_j})^2}\big]+h_q\big[\sqrt{(C^a_{A_1A_{j+1}})^2-S}\big]\\ \leqslant&\ h_q(C^a_{A_1A_2})+\cdots+h_q(C^a_{A_1A_j})+h_q(C^a_{A_1A_{j+1}})\\ \leqslant&\ \mathscr{C}^a_q(\rho_{A_1A_2})+\cdots+\mathscr{C}^a_q(\rho_{A_1A_n}).\end{aligned}\tag{C5}$$

Here the ideas of proving these inequalities are consistent to that of proving the inequality (C4) above.

Let $\rho_{A_1A_2\cdots A_n}$ be a multiqubit mixed state and $\{p_i,|\phi_i\rangle_{A_1|A_2\cdots A_n}\}$ be the pure state decomposition of $\rho_{A_1A_2\cdots A_n}$ such that $\mathscr{C}^a_q(\rho_{A_1|A_2\cdots A_n})=\sum_i p_i\mathscr{C}_q(|\phi_i\rangle_{A_1|A_2\cdots A_n})$. Then one has

$$\begin{aligned}\mathscr{C}^a_q(\rho_{A_1|A_2\cdots A_n}) =&\ \sum_i p_i\mathscr{C}_q(|\phi_i\rangle_{A_1|A_2\cdots A_n})\\ \leqslant&\ \sum_i p_i[\mathscr{C}^a_q(\rho^i_{A_1A_2})+\cdots+\mathscr{C}^a_q(\rho^i_{A_1A_n})]\\ =&\ \sum_i p_i\mathscr{C}^a_q(\rho^i_{A_1A_2})+\cdots+\sum_i p_i\mathscr{C}^a_q(\rho^i_{A_1A_n})\\ \leqslant&\ \mathscr{C}^a_q(\rho_{A_1A_2})+\cdots+\mathscr{C}^a_q(\rho_{A_1A_n}).\end{aligned}\tag{C6}$$

Here $\rho^i_{A_1A_j}$ is the reduced density matrix of $|\phi_i\rangle_{A_1A_2\cdots A_n}$ with respect to A_1A_j, the first inequality can be derived by inequalities (C4) and (C5), and the second inequality follows the definition of G_q-CoA.

Combining inequalities (C4), (C5), and (C6), we get the polygamy inequality is valid for any n-qubit quantum state.

Appendix D Monogamy relation

In order to prove that the square of G_q-concurrence obeys monogamy relation, we first define a function

$$\widetilde{H}_q(x,y)=h^2_q(\sqrt{x^2+y^2})-h^2_q(x)-h^2_q(y),$$

on the region $R'=\{(x,y)|0\leqslant x,y,x^2+y^2\leqslant 1\}$ for $1<q\leqslant 2$.

Appendix D.1 Proof of the non-negativity of $\widetilde{H}_q(x,y)$ on the domain

For the special case $q=2$, it is easy to get $\widetilde{H}_2(x,y)=0$, namely, $h^2_2(\sqrt{x^2+y^2})=h^2_2(x)+h^2_2(y)$. For $1<q<2$, we compute the gradient of $\widetilde{H}_q(x,y)$, denoted $\nabla\widetilde{H}_q(x,y)=\big(\frac{\partial\widetilde{H}_q(x,y)}{\partial x},\frac{\partial\widetilde{H}_q(x,y)}{\partial y}\big)$. Then following similar procedures as derived in Appendix C.1, we find that $\nabla\widetilde{H}_q(x,y)$ does not disappear in the interior of R' for $1<q<2$. We proceed to consider the boundary values of $\widetilde{H}_q(x,y)$. If $x=0$ or $y=0$, then $\widetilde{H}_q(x,y)=0$. If $x^2+y^2=1$, then

$$\widetilde{l}_q(x)=\ \widetilde{H}_q(x,\sqrt{1-x^2})\tfrac{1}{4}\{(2^q-2)^{\frac{2}{q}}-[2^q-(1+\sqrt{1-x^2})^q-(1-\sqrt{1-x^2})^q]^{\frac{2}{q}}-[2^q-(1+x)^q-(1-x)^q]^{\frac{2}{q}}\}.$$

By taking the first derivative of $\widetilde{l}_q(x)$, we find that $\frac{d\widetilde{l}_q(x)}{dx}=0$ corresponds to $x=\frac{1}{\sqrt{2}}$ for $1<q<2$ and $0<x<1$. Due to $\widetilde{l}_q(0)=\widetilde{l}_q(1)=0$, $\widetilde{l}_q(\frac{1}{\sqrt{2}})$ determines the sign of function $\widetilde{l}_q(x)$. And we observe $\widetilde{l}_q(\frac{1}{\sqrt{2}})\geqslant 0$ from Figure D1. Therefore, we have

$$h^2_q(\sqrt{x^2+y^2})\geqslant h^2_q(x)+h^2_q(y),\tag{D1}$$

for $1<q\leqslant 2$. It is necessary to mention that inequality (D1) is strictly greater than if $1<q<2$, $0<x,y<1$, and $x^2+y^2\leqslant 1$.

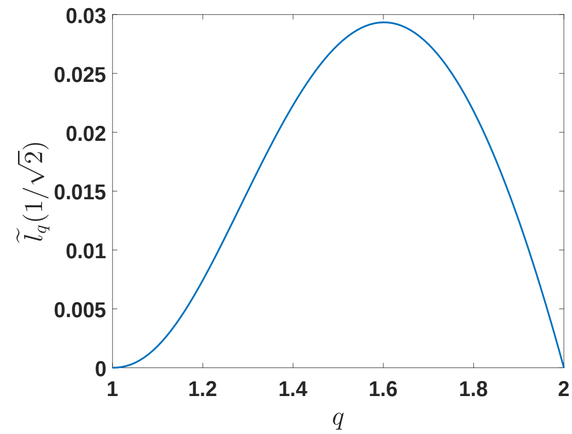

Figure D1 The function $\widetilde{l}_q(1/\sqrt{2})$ is nonnegative for $1 < q \leqslant 2$.

Appendix D.2 Proof of inequality (9)

Before proving the monogamy relation, we present a property of the function $h_q^2(x)$.

Proposition D1. The function $h_q^2(x)$ is monotonically increasing with respect to x for $0 \leqslant x \leqslant 1$ and $q > 1$.

This proposition can be obtained by utilizing the proof of similar approaches with Proposition B1.

Based on the non-negativity of $\widetilde{H}_q(x,y)$ on the domain and Proposition D1, we elaborate that the square of G_q-concurrence obeys monogamy relation for $1 < q \leqslant 2$.

The detailed proof of inequality (9) is as follows:

It is acknowledged that the square of concurrence satisfies monogamy relation and $\mathscr{C}_q(\rho_{A_1|A_2\cdots A_n}) = \frac{\sqrt{2}}{2}C(\rho_{A_1|A_2\cdots A_n})$ when $q = 2$, so G_2-concurrence obeys the monogamy relation naturally,

$$\mathscr{C}_2^2(\rho_{A_1|A_2\cdots A_n}) \geqslant \mathscr{C}_2^2(\rho_{A_1A_2}) + \cdots + \mathscr{C}_2^2(\rho_{A_1A_n}).$$

For any n-qubit pure state $|\phi\rangle_{A_1A_2\cdots A_n}$ and $1 < q < 2$, we derive

$$\begin{aligned}
\mathscr{C}_q^2(|\phi\rangle_{A_1|A_2\cdots A_n}) &= h_q^2[C(|\phi\rangle_{A_1|A_2\cdots A_n})] \\
&\geqslant h_q^2(\sqrt{C_{A_1A_2}^2 + \cdots + C_{A_1A_n}^2}) \\
&\geqslant h_q^2(C_{A_1A_2}) \\
&\quad + h_q^2(\sqrt{C_{A_1A_3}^2 + \cdots + C_{A_1A_n}^2}) \\
&\geqslant h_q^2(C_{A_1A_2}) + \cdots + h_q^2(C_{A_1A_n}) \\
&= \mathscr{C}_q^2(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^2(\rho_{A_1A_n}).
\end{aligned} \tag{D2}$$

Here the first inequality is because the function $h_q^2(x)$ is monotonically increasing with respect to x, the second and third inequalities can be gained by iterating formula (D1), the last equality follows from equation (6) in the letter.

Given a multiqubit mixed state $\rho_{A_1A_2\cdots A_n}$, we suppose that $\{p_i, |\phi_i\rangle_{A_1|A_2\cdots A_n}\}$ is the optimal pure state decomposition of $\mathscr{C}_q(\rho_{A_1|A_2\cdots A_n})$, that is, $\mathscr{C}_q(\rho_{A_1|A_2\cdots A_n}) = \sum_i p_i \mathscr{C}_q(|\phi_i\rangle_{A_1|A_2\cdots A_n})$. Then we see

$$\begin{aligned}
\mathscr{C}_q^2(\rho_{A_1|A_2\cdots A_n}) &= [\textstyle\sum_i p_i \mathscr{C}_q(|\phi_i\rangle_{A_1|A_2\cdots A_n})]^2 \\
&= \{\textstyle\sum_i p_i h_q[C(|\phi_i\rangle_{A_1|A_2\cdots A_n})]\}^2 \\
&\geqslant \{h_q[\textstyle\sum_i p_i C(|\phi_i\rangle_{A_1|A_2\cdots A_n})]\}^2 \\
&\geqslant h_q^2[C(\rho_{A_1|A_2\cdots A_n})] \\
&\geqslant h_q^2(\sqrt{C_{A_1A_2}^2 + \cdots + C_{A_1A_n}^2}) \\
&\geqslant \mathscr{C}_q^2(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^2(\rho_{A_1A_n}),
\end{aligned}$$

where the first inequality is assured owing to the convexity of $h_q(x)$ and the monotonicity of the function $y = x^2$ for $x > 0$, the second inequality is based on the fact that $h_q^2(x)$ is monotonically increasing and the definition of concurrence, the third inequality is valid because the square of concurrence satisfies monogamy relation for multiqubit quantum states [10], and the last inequality can be obtained by using the similar procedures with inequality (D2).

Appendix D.3 Proof of inequality (10)

Suppose that $\sum_{i=3}^{n} \mathscr{C}_q^2(\rho_{A_1A_i}) \geqslant \mathscr{C}_q^2(\rho_{A_1A_2})$, then one derives

$$\begin{aligned}
\mathscr{C}_q^{\alpha}(\rho_{A_1|A_2\cdots A_n}) &\geqslant [\mathscr{C}_q^2(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^2(\rho_{A_1A_n})]^{\frac{\alpha}{2}} \\
&= \Big(\sum_{i=3}^{n} \mathscr{C}_q^2(\rho_{A_1A_i})\Big)^{\frac{\alpha}{2}} \Big(1 + \frac{\mathscr{C}_q^2(\rho_{A_1A_2})}{\sum_{i=3}^{n} \mathscr{C}_q^2(\rho_{A_1A_i})}\Big)^{\frac{\alpha}{2}} \\
&\geqslant \Big(\sum_{i=3}^{n} \mathscr{C}_q^2(\rho_{A_1A_i})\Big)^{\frac{\alpha}{2}} \Big[1 + \Big(\frac{\mathscr{C}_q^2(\rho_{A_1A_2})}{\sum_{i=3}^{n} \mathscr{C}_q^2(\rho_{A_1A_i})}\Big)^{\frac{\alpha}{2}}\Big] \\
&= \mathscr{C}_q^{\alpha}(\rho_{A_1A_2}) + \Big(\sum_{i=3}^{n} \mathscr{C}_q^2(\rho_{A_1A_i})\Big)^{\frac{\alpha}{2}} \\
&\geqslant \mathscr{C}_q^{\alpha}(\rho_{A_1A_2}) + \cdots + \mathscr{C}_q^{\alpha}(\rho_{A_1A_n}).
\end{aligned}$$

Here the first inequality holds because $\mathscr{C}_q^2(\rho_{A_1|A_2\cdots A_n})$ obeys monogamy relation and $y = x^{\frac{\alpha}{2}}$ is a monotonically increasing function with respect to x for $0 \leqslant x \leqslant 1$ and $\alpha \geqslant 2$, the second inequality is according to the inequality $(1+x)^{\frac{\alpha}{2}} \geqslant 1 + x^{\frac{\alpha}{2}}$, and the last inequality is valid since the relation $(\sum_i x_i^2)^{\frac{\alpha}{2}} \geqslant \sum_i x_i^{\alpha}$ holds for $0 \leqslant x_i \leqslant 1$ and $\alpha \geqslant 2$.

Appendix E Entanglement indicators

Appendix E.1 Proof of $\tau_q(\rho_{ABC}) = 0$ if and only if ρ_{ABC} is biseparable

If a three-qubit pure state $|\phi\rangle_{ABC}$ is biseparable, then its forms might be

$$\begin{aligned}
&|\phi\rangle_{ABC} = |\phi\rangle_{AB} \otimes |\phi\rangle_C, \\
&|\phi\rangle_{ABC} = |\phi\rangle_{AC} \otimes |\phi\rangle_B, \\
&|\phi\rangle_{ABC} = |\phi\rangle_A \otimes |\phi\rangle_{BC}, \\
&|\phi\rangle_{ABC} = |\phi\rangle_A \otimes |\phi\rangle_B \otimes |\phi\rangle_C.
\end{aligned}$$

We have $\tau_q(|\phi\rangle_{A|BC}) = 0$ for these states.

Next, the sufficiency is proven. We will illustrate the fact that there is at most one nonzero two-qubit concurrence for three-qubit pure state if $\tau_q(|\phi\rangle_{A|BC}) = 0$.

If $\mathscr{C}_q(\rho_{AB}) > 0$ and $\mathscr{C}_q(\rho_{AC}) > 0$, then we derive

$$\begin{aligned}
\mathscr{C}_q^2(|\phi\rangle_{A|BC}) - \mathscr{C}_q^2(\rho_{AB}) - \mathscr{C}_q^2(\rho_{AC}) &= h_q^2[C(|\phi\rangle_{A|BC})] - h_q^2[C(\rho_{AB})] - h_q^2[C(\rho_{AC})] \\
&\geqslant h_q^2[\sqrt{C^2(\rho_{AB}) + C^2(\rho_{AC})}] - h_q^2[C(\rho_{AB})] - h_q^2[C(\rho_{AC})] \\
&> 0,
\end{aligned}$$

which is contradictory to the precondition $\tau_q(|\phi\rangle_{A|BC}) = 0$, where the second inequality is attained due to $h_q^2(\sqrt{x^2+y^2}) > h_q^2(x) + h_q^2(y)$ for $1 < q < 2$, $0 < x, y < 1$, and $x^2 + y^2 \leqslant 1$.

If $\mathscr{C}_q(\rho_{AB}) = \mathscr{C}_q(\rho_{AC}) = \mathscr{C}_q(|\phi\rangle_{A|BC}) = 0$, then the state may be in the forms $|\phi\rangle_{ABC} = |\phi\rangle_A \otimes |\phi\rangle_{BC}$ or $|\phi\rangle_{ABC} = |\phi\rangle_A \otimes |\phi\rangle_B \otimes |\phi\rangle_C$.

According to the strict concavity of Tsallis entropy and $y = x^{\gamma}$ $(0 < \gamma < 1)$, it is not difficult to show that $G_q(\rho)$ is also a strictly concave function.

If there is only one nonzero two-qubit concurrence, $\mathscr{C}_q(\rho_{AB}) > 0$ or $\mathscr{C}_q(\rho_{AC}) > 0$, based on the strict concavity of $G_q(\rho)$ and by means of similar procedures to that in Ref. [11], we get that the corresponding forms of states are $|\phi\rangle_{ABC} = |\phi\rangle_{AB} \otimes |\phi\rangle_C$ and $|\phi\rangle_{ABC} = |\phi\rangle_{AC} \otimes |\phi\rangle_B$, respectively.

Based on the above discussion, we can derive easily $\tau_q(\rho_{ABC}) = 0$ for any three-qubit mixed state ρ_{ABC} iff ρ_{ABC} can be expressed in the form $\rho_{ABC} = \sum_i p_i \rho_{AB}^i \otimes \rho_C^i + \sum_i q_i \rho_{AC}^i \otimes \rho_B^i + \sum_i r_i \rho_A^i \otimes \rho_{BC}^i$.

For any n-qubit quantum state ρ, the result that $\tau_q^i(\rho) = 0$ for $1 < q < 2$ iff $\rho = \sum\limits_k p_k \rho_{A_i}^k \otimes \rho_{\overline{A_i}}^k + \sum\limits_{j\neq i} \sum\limits_k q_k^j \rho_{A_iA_j}^k \otimes \rho_{\overline{A_iA_j}}^k$ can be proved by using analogous procedures to lemmas b and c in supplementary material of Ref. [11].

Appendix E.2 Detection of W state entanglement via τ_q

There exists a right-neighborhood of one $(1, 1+\delta_1)$ and a left-neighborhood of two $(2-\delta_2, 2)$ such that $\tau_q(\rho)$ is strictly greater than zero but very close to zero. For clarity, we discuss in the interval $[1+\delta_1, 2-\delta_2]$.

Example 2. For n-qubit W state $|W_n\rangle_{A_1\cdots A_n} = \frac{|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |00\cdots1\rangle}{\sqrt{n}}$, its concurrence between subsystems A_1 and $A_2\cdots A_n$ is $C(|W_n\rangle_{A_1|A_2\cdots A_n}) = \frac{2\sqrt{n-1}}{n}$, $C(\rho_{A_1A_j}) = \frac{2}{n}$, $j = 2, \cdots, n$, then we have

$$\tau_q(|W_n\rangle) = \Big[1 - \Big(\frac{1+\frac{n-2}{n}}{2}\Big)^q - \Big(\frac{1-\frac{n-2}{n}}{2}\Big)^q\Big]^{\frac{2}{q}} - (n-1)\Big[1 - \Big(\frac{1+\frac{\sqrt{n^2-4}}{n}}{2}\Big)^q - \Big(\frac{1-\frac{\sqrt{n^2-4}}{n}}{2}\Big)^q\Big]^{\frac{2}{q}}.$$

Taking $n = 3, 6, 9$ and plotting them in Figure E1, we observe $\tau_q(|W_n\rangle)$ is greater than zero obviously for $q \in [1.05, 1.95]$.

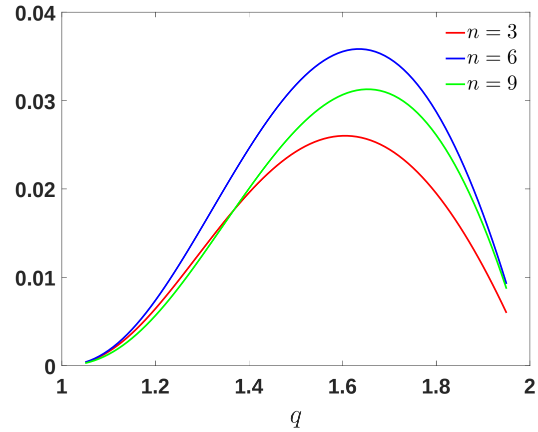

Figure E1 The red line, blue line, and green line correspond to $\tau_q(|W_n\rangle)$ with $n = 3, 6, 9$, where $q \in [1.05, 1.95]$.

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