SCIENCE CHINA
Information Sciences

• LETTER •

December 2025, Vol. 68, Iss. 12, 229202:1–229202:2
https://doi.org/10.1007/s11432-023-4560-2

A new optimization scheme for uncertain problems: a globally robust solution

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Received 16 October 2023/Revised 13 October 2024/Accepted 5 August 2025/Published online 23 October 2025

Citation Zhu Y M, Li W Y, Song E B, et al. A new optimization scheme for uncertain problems: a globally robust solution. Sci China Inf Sci, 2025, 68(12): 229202, https://doi.org/10.1007/s11432-023-4560-2

Robust methods for uncertain problems have drawn extensive attention across diverse fields, including science, social science, and military affairs. For uncertain problems, one usually considers the least favorable case and chooses the most reliable scheme—the well-known minimax criterion, which has been extensively studied [1]. In terms of this criterion, a minimum cost solution for the least favorable case under uncertain assumptions provides a minimum cost upper bound, regardless of which assumption holds true. Due to this advantage, there are numerous applications in estimation, filtering, decision, detection, statistics, control, and optimization. However, this traditional criterion is criticized because of its conservativeness [2]. Therefore, a lot of research is dedicated to improving the traditional minimax.

An uncertain problem may involve various types of prior knowledge, such as uncertainty levels and probability distributions, along with differing practical requirements. For example, some scenarios require a minimal worst-case cost deviation upper bound between the robust solution and all potential actual solutions, while others may favor minimizing average cost deviation or total cost deviation to tolerate uncertainty. In this study, we will develop a new scheme that is more globally robust and applicable for dealing with uncertain problems. Furthermore, the corresponding globally robust solution (GRS) will also be discussed. In our new scheme, the GRS can leverage a priori problem knowledge and meet practical requirements. Therefore, it can address a wider range of uncertain problems than previous robust methods, leading to better performance. We emphasize that the advantages of the GRS compared to traditional minimax methods stem from a crucial technical change. The GRS relies more on the entire optimal solution set, rather than solely depending on the least favorable case.

To visually clarify the above statement, Appendix A introduces the main idea of our scheme through well-known Bayesian binary hypothesis testing with unknown a priori probabilities. We then generalize the GRS and highlight its advantages over minimax solutions, followed by the introduction of specific GRSs based on various criteria. In Appendix B, we present a signal detection example with uncertain noise to demonstrate how to obtain the GRS and further verify the analyses presented above. Finally, we will provide some concluding remarks.

General formulation and several criteria for the GRS. In an uncertain problem, an assumption $A \in \mathcal{A}$ is not precisely known, and the optimal solution $S(A) \in \mathcal{S}(\mathcal{A})$ corresponding to A is determined based on a specific criterion. Here, $\mathcal{A}$ represents the set of all possible actual assumptions, and $\mathcal{S}$ is the set of corresponding optimal solutions.

A natural idea is to consider the difference between the robust solution $S(A^*)$ and the optimal solution $S(A)$, i.e., $\|S(A^*) - S(A)\|$ for a specific norm $\|\cdot\|$. When solving this difference is difficult, we can use $C(A, S(A^*))$ (e.g., the mean squared error (MSE) in estimation problems or the error probability in hypothesis testing problems) to represent the loss of the robust solution $S(A^*)$ when the true hypothesis is A. For a fixed $A^* \in \mathcal{A}$, the cost can be denoted simply as $C_{A^*}(A)$. Clearly, when A^* is the same as the actual assumption A, the cost function $C_A(A)$ depends only on A and can be denoted as $C(A)$. This represents the minimal cost function for each actual assumption A according to the given criterion. Using the above notations, we can formulate the general GRS as follows:

$$A^G = \arg\min_{A^*} \|C_{A^*}(\cdot) - C(\cdot)\|_c \tag{1}$$

with the optimal solution $S(A^G)$ and the cost function $C_{A^G}(\cdot)$. That is to say, in the new scheme, the GRS A^G determines the cost function $C_{A^G}(A)$ which is an optimal approximation to the minimal cost function $C(A)$ of every actual assumption A in terms of the given criterion $\|\cdot\|_c$.

Remark 1. The above framework has broad applications. For instance, in the uncertain estimator problem, where the observation y is used to estimate the random signal x, assumption A can be viewed as the joint density $p_A(x, y)$. The solution $S(A)$ represents the optimal linear estimator associated with $p_A(x, y)$. Furthermore, the cost $C(A, S(A^*))$ is the MSE between the estimator corresponding to the assumption $p_{A^*}(x, y)$ and the random signal x. The goal of the optimal robust estimator in our scheme is to find the op-

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timal density $p_{A^G}(x,y)$ based on certain criteria, resulting in the global optimal estimator.

Remark 2. The GRS in the new uncertain scheme is essentially to derive an optimal approximation to the optimal solution cost function from a feasible solution cost function set in terms of a given criterion. Clearly, this is an infinite-dimensional optimization problem.

Advantages of the GRS. Based on the discussion above, we can conclude the advantages of the GRS in addressing the shortcomings of the minimax criterion solution.

Advantage 1. Global robustness. Compared to the conservativeness of minimax solutions, $\mathcal{S}(A^G)$ is a globally robust solution that minimum $\|C_{A^G}(\cdot) - C(\cdot)\|_c$ since $C(\cdot)$ is the cost functional of the entire solutions set of the uncertain problem. In other words, the GRS globally suppresses the deviations across the entire minimum cost curve $C(\cdot)$.

Advantage 2. Utilizing more problem knowledge. In contrast to minimax solutions, which focus only on the least favorable case, $C_{A^G}(A)$ depends strongly on the entire problem knowledge, making it sensitive to even slight changes. Furthermore, in the following section, we will present specific criteria for the GRS to effectively use available knowledge and meet practical requirements.

Advantage 3. Improve the performance. The GRS generally outperforms traditional minimax solutions, as confirmed by the experiments in Appendix B. Notably, the GRS avoids trivial solutions. Even in Bayesian binary hypothesis testing, where the two families of uncertain conditional densities intersect, as noted in [3], the GRS remains nontrivial.

Remark 3. We acknowledge that the GRS does not offer a worst-case upper bound for losses like the minimax criterion. In minimax, the loss remains below the upper bound regardless of the uncertainty A. In contrast, the losses of GRS can be large for some unfavorable assumptions.

Then, we introduce some specific criteria for the GRS.

GRS minimax (GRM). Incorporating the minimax idea along with the information from the optimal cost function $C(\cdot)$, the GRM criterion is here defined by

$$\|C_{A^*}(\cdot) - C(\cdot)\|_c = \max\left\{C_{A^*}(A) - C(A) | A \in \mathcal{A}\right\}, \quad (2)$$

i.e., A^G of the GRM solution is given by

$$A^G = \min_{A^* \in \mathcal{A}} \max\left\{C_{A^*}(A) - C(A) | A \in \mathcal{A}\right\}. \quad (3)$$

According to (2) and (3), it is easy to see that the regret estimation in [4] and the set-membership estimation in [5] are the two special cases of the above GRM solutions.

Minimum average deviation (MAD). If the uncertain assumption A is a parameter with no prior knowledge of its distribution on $\mathcal{A}$, one typically assumes A is uniformly distributed and prefers to minimize the average deviation (MAD) of $C_{A^*}(A) - C(A)$ over $\mathcal{A}$. Thus, the MAD criterion in (1) can be defined as

$$\|C_{A^*}(\cdot) - C(\cdot)\|_c = \int_{\mathcal{A}} (C_{A^*}(A) - C(A)) \mathrm{d}A. \quad (4)$$

Since $\int_{\mathcal{A}} C(A)\mathrm{d}A$ is a constant that does not affect the choice of the optimal value and can be ignored, the optimization problem can be simplified to finding

$$A^G = \arg\min_{A^* \in \mathcal{A}} \int_{\mathcal{A}} C_{A^*}(A)\mathrm{d}A. \quad (5)$$

Minimum expectation deviation (MED). If the uncertain assumption A is a parameter with known statistical information, such as its probability density $q(A)$ on $\mathcal{A}$. This prior knowledge is valuable and should be utilized to enhance the solution's performance. Therefore, the MED in (1) can be defined as

$$\|C_{A^*}(\cdot) - C(\cdot)\|_c = \int_{\mathcal{A}} q(A)(C_{A^*}(A) - C(A)) \mathrm{d}A. \quad (6)$$

Similarly, since $\int_{\mathcal{A}} q(A)C(A)\mathrm{d}A$ is a constant that can be ignored, we only need to consider solving

$$A^G = \arg\min_{A^* \in \mathcal{A}} \int_{\mathcal{A}} q(A) C_{A^*}(A)\mathrm{d}A. \quad (7)$$

Notably, when the exact distribution of the parameters is unknown, we can focus on coarse information, such as identifying a robust solution by determining where the parameters are concentrated at a particular point.

Remark 4. The MAD can be seen as a form of MED when the uncertainty follows a uniform distribution, as their objectives differ by a constant factor. Additionally, the GRM (minimax) criterion, even within the GRS scheme, cannot fully utilize global knowledge, like the probability density $q(A)$. This shows the GRM has some degree of locality, but it still provides the advantage of guaranteeing a minimal upper bound on deviations between the GRM and the optimal cost across all cases.

Remark 5. When the uncertainty follows a distribution with single point or very small area around the least favorable case, the MED criterion takes the same optimization equation as the traditional minimax criterion. However, they both come respectively from two completely different optimization frameworks.

Conclusion. We have established the new robust scheme—the GRS for uncertain problems in a more global sense. The new scheme can utilize various optimization criteria to sufficiently take advantage of different prior problem knowledge and satisfy practical requirements. It may have more extensive applications than the traditional minimax solution. Since uncertain problems exist everywhere, and the traditional minimax is extensively applied in many areas, such as communication, signal detection, medical diagnosis, automation, control, and image processing, the GRS method can also be extensively applied to those areas.

Supporting information Appendixes A and B. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

Acknowledgements The work was supported by Sichuan Science and Technology Program (Grant No. 2025ZNSFSC0073).

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