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Special Topic: Mean-Field Game and Control of Large Population Systems: From Theory to Practice

Optimal tax-subsidy incentive for population games based on mean field approximation

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Cooperation is pivotal for social prosperity and population survival, but it often comes at the expense of private interests, thereby generating an inherent conflict between collective and private interests. Several exogenous incentive mechanisms have been identified to mitigate such conflicts [1, 2]. Extending this line of research, recent studies have further focused on optimal incentive mechanisms to minimize implementation costs incurred by the institution [3–5]. However, the cost-based incentive mechanisms may expose the institution to financial problems from a long-term perspective.

In response to these financial challenges, this study proposes an adaptive tax-subsidy incentive mechanism to maintain budget balance. Specifically, taxes collected from all players form an incentive budget and a portion of this budget is allocated as subsidies to reward cooperative behaviors. To verify the effectiveness of such a mechanism in driving cooperation, the prisoner's dilemma is employed as a metaphor to characterize the aforementioned inherent conflict. Meanwhile, the analysis is confined to unstructured populations to eliminate the interference of network reciprocity.

Building on the above setup, this study formulates the mean-field equation that approximates the aggregate behavior of the population. For this equation, the conditions for the tax-subsidy incentive to achieve a target cooperative state are first derived. Notably, the value of the target state is contingent upon the objective, typically involving either improving cooperation or enhancing social welfare, which is assumed by default to promote cooperation in this study. When these conditions are satisfied, the expression of the optimal incentive policy, which minimizes implementation costs during allocation, is further derived. Moreover, the conditions to prevent the conflict between cooperation improvement and social welfare enhancement under optimal policy are also identified. Finally, the validity of theoretical results is corroborated through Monte Carlo simulations.

Problem formulation. Consider a well-mixed population that consists of large but finite anonymous players, denoted as $N = \{1, \cdots, n\}$. Within this population, each player repeatedly interacts with others. Departing from the best response strategy, players revise their strategies based on a revision protocol, as exemplified in [4]. In the interaction, every player $i \in N$ selects a strategy $s_i \in \{C, D\}$ to engage in the prisoner's dilemma with others, where C, D respectively represent cooperation and defection. A C-player (cooperator) incurs a cost c to provide a benefit b $(b > c > 0)$ to its interaction opponent. In contrast, a D-player (defector) saves the cost and provides nothing to the opponent. Then the instantaneous payoff for s-strategists is

$$\pi_s(x^n) = \begin{cases} \frac{x^n n-1}{n-1} b - c, & s = C, \\ \frac{x^n n}{n-1} b, & s = D, \end{cases}$$

where x^n, referred to as the population state, denotes the fraction of cooperators in the n-player population.

The revision protocol follows the pairwise comparison rule, which aligns with the rule employed in [4]. To formalize this evolutionary process explicitly, at each step $\tau \in \mathbb{N}$, a focal player i is randomly selected for strategy revision. Subsequently, another player j is uniformly chosen as an exemplar. The probability that player i imitates the strategy of player j under the population state x^n is determined by the Fermi function

$$W = \left(1 + e^{-\omega\left(\pi_{s_j}(x^n) - \pi_{s_i}(x^n)\right)}\right)^{-1},$$

where $\omega \geqslant 0$ denotes the intensity of selection, and s_i denotes the strategy of player i.

To facilitate cooperation, a controller referred to as the institution is introduced. The institution implements a tax-subsidy incentive mechanism. More concretely, this institution levies a capitation tax, which is a uniform tax imposed on every player. A fraction α of the collected taxes is retained to cover implementation costs, with $\alpha \in [0, 1]$ representing the inefficiency ratio. After deducting the implementation costs, the remaining taxes are distributed equally among cooperators, providing each with a subsidy of $(1 - \alpha)u$. Here, $u \in [0, b]$ specifies the subsidy amount in the absence of inefficiency, and it serves as the incentive parameter.

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The stochastic process $\{x^n(\tau)\}_{\tau\in\mathbb{N}}$ with incentive policy $\{u^n(\tau)\}_{\tau\in\mathbb{N}}$ for the large population $n \gg 1$ with weak selection $\omega \ll 1$ could be approximated by the solution of a mean field equation

$$\dot{x} = \frac{1}{2}x(1-x)\big[(1-\alpha)u - c\big], \tag{1}$$

after a re-scaled time $t = \tau\omega/n$, where $u(t) = u^n(\lfloor tn/\omega\rfloor), t \geqslant 0$ denotes the re-scaled incentive policy. Appendix A shows the detailed derivation of (1).

The magnitude of the implementation costs for the incentive u can be indirectly reflected as $\frac{1}{2}(ux)^2$. Accordingly, the optimal tax-subsidy incentive problem for the target state x_1 can be equivalently formulated as

$$\begin{aligned} &\min_{u\in U_{[t_0,T]}} \quad J(u) = \int_{t_0}^{T} \frac{1}{2}(ux)^2 dt \\ &\text{s.t.} \quad \begin{cases} \dot{x} = \frac{1}{2}x(1-x)\big[(1-\alpha)u - c\big], \\ x(t_0) = x_0, \\ x(T) = x_1, \end{cases} \end{aligned} \tag{2}$$

where $T = \inf\{t \geqslant 0 |\ x(t) = x_1\}$ represents the stopping time for hitting the target state x_1, $U_{[t_0,T]}$ denotes the set of feasible incentives on $[t_0, T]$ with values in $[0, b]$, and $x_0 \in (0, 1)$ stands for the initial state.

After formulating the optimization problem, we state the following results for the feasible conditions and the expressions for optimal policies.

Theorem 1. For the optimization problem (2), no feasible policy exists if $x_1 > x_0$ and $\alpha \in [1 - \frac{c}{b}, 1]$. Otherwise, the optimal tax-subsidy policy is given by

$$u^* = \begin{cases} 0, & x_1 \leqslant x_0, \\ \min\big\{\frac{2c}{1-\alpha}, b\big\}, & x_1 > x_0,\ \alpha \in [0, 1-\frac{c}{b}), \end{cases} \tag{3}$$

and its corresponding terminal time is $T^* = \kappa^{-1}\ln\big(\zeta\cdot\eta^{-1}\big) + t_0$, where the auxiliary parameters are defined as $\kappa = \frac{1}{2}\left[(1-\alpha)u^* - c\right]$, $\zeta = \frac{1-x_0}{x_0}$, and $\eta = \frac{1-x_1}{x_1}$.

Moreover, the optimal state dynamics is governed by $x^*(t) = \big(1 + \zeta e^{-\kappa(t-t_0)}\big)^{-1}$, and the optimal value of the cost function J^* is

$$\begin{aligned} J^* = \frac{1}{2}{u^*}^2\kappa^{-1}\big[&\ln(1+\eta^{-1}) + (1+\eta^{-1})^{-1} \\ &- \ln(1+\zeta^{-1}) - (1+\zeta^{-1})^{-1}\big]. \end{aligned}$$

Remark 1. For $x_1 \leqslant x_0$, the system (1) without incentives tends to exhibit a decrease in cooperation, which implies that it can spontaneously reach the target state x_1. It naturally follows that the optimal policy is $u^* = 0$, which signifies no incentives to be implemented.

Since the objective is to promote cooperation, we assume $x_1 > x_0$ by default. The qualitative presentations for optimal values u^*, T^*, and J^* are shown in Figure 1. For the cases where $b/c > 2$ and $\alpha \in [0, 1 - 2c/b)$, the optimal policy $u^* = \frac{2c}{1-\alpha}$ ensures that the payoffs for cooperators are greater than those of defectors by c, which ultimately aligns with the principle of equivalent exchange.

However, the improvement in cooperation does not necessarily lead to an increase in social welfare under institutional incentives, where the definition of social welfare is presented in Appendix C. This assertion stems from the fact that institutional inefficiency reduces social welfare. The institution can ensure that improved cooperation translates into increased welfare by adopting the following suitable inefficiency ratio α.

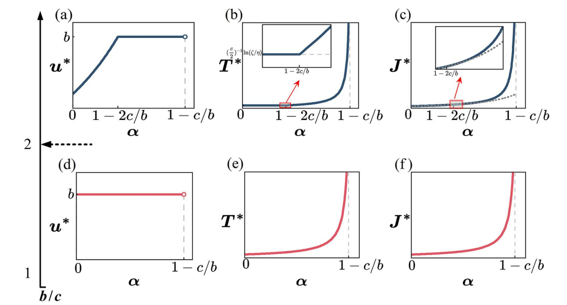

Figure 1 (Color online) The optimal values u^*, T^*, and J^* for various inefficiency ratio α and benefit-to-cost ratio b/c. (a)–(c) present results for $b/c > 2$, while (d)–(f) display those for $1 < b/c \leqslant 2$. To emphasize the differences between u^* and $\widetilde{u}$, the associated results for $\widetilde{u}$ (gray dashed line) are added in panels (b) and (c). Parameters are fixed to x_0, x_1 $(> x_0)$, b, c, t_0 $(= 0)$.

Corollary 1. An enhancement of the level of cooperation under the optimal policy (3) does not impair the social welfare if the inefficiency ratio α satisfies

$$\begin{cases} \alpha \in \big[0, \frac{1}{2} - \frac{c}{2b}\big], & b/c \in (1, 3], \\ \alpha \in \big[0, \frac{b-c}{b+3c}\big], & b/c > 3. \end{cases}$$

Simulation. To verify the applicability of the optimal policy (3) in large but finite population, we compare the discrepancies between the following pairs of values.

- The aggregate behavior of the n-player population $\mathbb{E}\big(x^n(\tau\omega/n)\big)$, $\tau \in \mathbb{N}\cap[0, T^*n/\omega]$ versus the theoretical one $x^*(\tau\omega/n)$.
- The cumulative cost of the n-player population $\mathbb{E}(J^n)$ versus the theoretical one J^*.

Monte Carlo simulations are executed to obtain $x^n(\tau\omega/n)$, J^n for the n-player population. The simulations are divided into a series of Monte Carlo steps, where each step, $\Delta\tau = 1$, consists of two elementary operations: interaction and strategy updating. The above operations are iterated T^*n/ω times before termination. To mitigate stochastic fluctuations, 200 independent Monte Carlo trials are conducted. The comparison analysis between theoretical values and simulation ones is provided in Appendix D.

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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