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Special Topic: Mean-Field Game and Control of Large Population Systems: From Theory to Practice

Non-zero-sum linear quadratic stochastic differential games with jump diffusion and input delay: an asymmetric information framework

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Abstract We investigate a two-player non-zero-sum linear-quadratic stochastic differential game under asymmetric information, where the state dynamics is governed by a jump diffusion. The asymmetry in information stems from the structure of the players' strategies: while one player makes decisions based on full information, the other operates under delayed information due to a time delay in his/her control input. Using the maximum principles and the orthogonal decomposition and reorganization technique, we transform the problem of finding open-loop Nash equilibria to that of solving an auxiliary system of forward-backward stochastic delayed differential equations (FBSDDEs) with mutually orthogonal strategies. Under the assumption of a unique Nash equilibrium, we derive explicit solutions to the auxiliary FBSDDEs and hence obtain an explicit form of the open-loop Nash equilibrium based on a generalized Riccati equation developed in this work. Numerical examples are provided to validate the theoretical results.

Keywords linear quadratic stochastic differential game, asymmetric information, open-loop Nash equilibrium, forward-backward stochastic delayed differential equations, jump diffusion

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1 Introduction

Linear quadratic (LQ) stochastic differential games (SDGs) are a specialized class of dynamic games formulated within the framework of stochastic differential equations (SDEs) and characterized by linear system dynamics and quadratic cost functionals. For two-player games, the payoff structure categorizes the games into two paradigms: zero-sum and non-zero-sum games. Zero-sum games, where one player's gain is offset by the other's loss, offer simplified yet powerful models for analyzing adversarial interactions. Non-zero-sum games generalize zero-sum games by allowing the possibility of simultaneous gains or losses for the players. Among the core issues in two-player LQ SDGs is the study of open-loop and closed-loop Nash equilibria, which are called saddle points in zero-sum games. The two types of Nash equilibria (saddle points) exhibit different mathematical structures, and their existence is not equivalent (see [1–4] and the references therein).

Asymmetric information in a two-player game denotes scenarios in which players have unequal information when making decisions, leading to a knowledge imbalance. A class of LQ SDGs considers information asymmetry arising when the strategies of different players are adapted to non-identical filtrations. For instance, in leader-follower games, Shi et al. [5] incorporated asymmetric information by restricting the follower's knowledge to a sub-σ-algebra of the leader's information. Non-zero-sum LQ differential games driven by backward SDEs were examined in [6]. With some special symmetric information structures, the feedback Nash equilibria were uniquely given by forward-backward SDEs, the filters of forward-backward SDEs and the associated Riccati equations. Li and Wu [7] extended LQ SDGs by integrating mean-field terms into both the system dynamics and cost functionals, and provided feedback form of open-loop Nash equilibria for certain special cases under asymmetric information. Shi et al. [8] introduced overlapping information into stochastic LQ Stackelberg games, an extension that captures complex information

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structures where players share partial knowledge while maintaining non-nested filtrations. A subsequent study in [9] investigated non-zero-sum LQ SDGs with backward SDEs under overlapping information, deriving coupled Riccati equations and conditional mean-field SDEs to characterize feedback-form Nash equilibria.

The stochastic control problem can be formulated as a one-player stochastic differential game. For a comprehensive survey on the stochastic optimal control theory, see the monograph [10]. Extensive studies have been conducted on stochastic control problems with delayed information, primarily focusing on deriving various versions of maximum principles to identify optimal controllers via solutions to associated forward-backward stochastic delayed differential equations (FBSDDEs). Notable contributions include the maximum principles for Itô stochastic systems with both state and input delays [11], forward-backward stochastic systems with jumps and partial information [12], mean-field stochastic delay systems with jump diffusion [13], forward-backward delay systems involving impulse controls [14], control systems with time-varying delays [15], and stochastic systems with state and control delays under non-convex control domains [16].

In the context of LQ stochastic control problems with time delays, one stream of research focuses on characterizing optimal controllers through time-delay or time-advanced forward-backward SDEs (see [17, 18]). Another important direction centers on deriving necessary and sufficient solvability conditions and constructing explicit optimal controllers via Riccati-type equations. Progress encompasses: for a stochastic discrete system with a single delayed control input, Zhang et al. [19] obtained the explicit optimal controller in terms of state prediction and the solution to a Riccati-ZXL difference equation (a generalized Riccati-type equation established in [19]). Zhang and Xu [20] developed these results to Itô stochastic systems with a single delayed input, and a new modified Riccati differential equation was defined to derive necessary and sufficient solvability conditions and explicit optimal controllers. However, the single-input-delay approach fails for systems with multiple control inputs and delays due to input interdependence across different measurabilities. To resolve this, Wang et al. [21] proposed the orthogonal decomposition and reorganization technique to convert multiple control inputs with different time delays into mutually orthogonal control inputs. This technique led to new Riccati-type equations, with which necessary and sufficient solvability conditions and explicit optimal controllers were provided for stochastic discrete-time systems [21] and Itô stochastic systems [22].

There has been growing interest in non-zero-sum SDGs with delays. For example, An and Øksendal [23] established maximum principles for SDGs with jump diffusions under asymmetric delayed information. Wang et al. [24] extended the results in [23] to the SDGs with singular controls. Chen and Yu [25] derived maximum principles for non-zero-sum SDGs involving delays, where both of the state delay and control delay enter drift and volitality terms. In the LQ SDGs setting, Chen and Wu [26] presented Nash equilibria for non-zero-sum LQ SDGs with delayed states using generalized forward-backward SDEs. Huang and Li [27] studied LQ mean-field games with delays, verifying ϵ-Nash equilibria through anticipated forward-backward SDEs. Xu et al. [28] addressed leader-follower LQ SDGs with time delays in the leader's strategy and provided the open-loop solution based on symmetric Riccati equations.

Jump-diffusion models extend the Itô diffusion framework by incorporating both continuous paths and discrete jumps. This dual structure allows the model to capture not only the gradual evolution of system states over time, but also abrupt changes from rare yet impactful events such as market crashes, policy interventions, or natural disasters. Merton [29] pioneered the use of jump-diffusion processes in financial asset pricing, aiming to better reflect the effects of unexpected shocks on stock returns. Since then, jump-diffusion models have been widely adopted across diverse fields including economics, insurance, and engineering (see the monograph by [30]). Recent research has further extended the application of jump-diffusion models to LQ optimal control and LQ SDGs (see [31–33]).

In this paper, we are concerned with a two-player non-zero-sum LQ SDG under asymmetric information, where the state dynamics is governed by jump diffusions. The information asymmetry arises from the structure of the players' control strategies. To illustrate, let full information filtration $\{\mathcal{F}_s\}_{s\geqslant 0}$ be generated by the driving process in the state. Then, at each time $t \in [0, T]$, Player 1's strategy $u_1(t)$ is measurable to the σ-algebra $\mathcal{F}_t$, whereas Player 2's control input $u_2(t-\delta)$ incorporates a time delay and is restricted to be $\mathcal{F}_{t-\delta}$-measurable. The parameter $\delta \geqslant 0$ denotes the length of the delay, representing the information lag. This framework captures real-world strategic interactions such as high-frequency trading markets [34], where algorithmic traders (Player 1) execute orders using real-time information $\mathcal{F}_t$, while general traders (Player 2) operate with delayed market data $\mathcal{F}_{t-\delta}$ due to technological and infrastructural constraints.

The objective of this paper is to establish explicit representations of open-loop Nash equilibria. Our game framework presents two fundamental challenges. (i) The strategic interdependence between players, compounded by their differing information structures (manifested through distinct σ-algebra measurability constraints), creates inherent analytical complexity in equilibrium characterization. (ii) Unlike single-objective LQ control problems with mutiple time delays [21, 22], the game-theoretic setting introduces multiple competing objectives that generate nonlinear FBSDDEs with multiple backward SDEs. To address these challenges, we introduce a novel methodological framework that integrates maximum principle techniques with an enhanced orthogonal decomposition and reorganization technique. This framework transforms asymmetrically constrained strategies into mutually orthogonal components and restructures the resulting two-backward-SDEs system into a computationally tractable form. The framework enables the establishment of a new generalized Riccati equation, based on which the explicit representations for Nash equilibria can be derived.

Our work makes the following key contributions to the existing literature. We generalize the classical non-zero-sum LQ SDG in [1] to incorporate jump diffusions and asymmetric information structures, where time-delayed strategies induce distinct measurability constraints between players. In addition, we extend the orthogonal decomposition and reorganization technique, originally developed for LQ stochastic control with multiple input delays [21,22], to the context of LQ SDGs with asymmetric information. The players' distinct objectives result in a nonlinear system of FBSDDEs involving two backward SDEs. The derivation of explicit Nash equilibrium solutions is achieved by transforming the players' strategies into mutually orthogonal components and reformulating the corresponding two backward SDEs into a more tractable form. Furthermore, we establish a new class of generalized Riccati equations that accommodate delay-induced couplings and jump-related integrals. Unlike symmetric solutions in LQ control setting [20, 22], our Riccati system exhibits asymmetry, reflecting fundamental differences between LQ control problems and LQ SDGs.

The remainder of this paper is organized as follows. Section 2 presents the mathematical formulation of the two-player non-zero-sum LQ SDG with jump diffusion. The main theoretical results are developed in Section 3. Section 4 provides a numerical example demonstrating the practical implementation of our theoretical framework. Finally, concluding remarks are given in Section 5.

Notations. Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space with the usual Euclidean norm $|\cdot|$ and the usual Euclidean scalar product $\langle\cdot,\cdot\rangle$. $\mathbb{R}_0^n := \mathbb{R}^n - \{0\}$. Let $\mathbb{R}^{n\times m}$ denote the set of all $n\times m$ real matrices. $\mathcal{S}^n$ symbolizes the set of all $n\times n$ symmetric matrices. The superscript T indicates the transpose of a vector or a matrix. In a filtered probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P\right)$, $\mathbb{E}[\cdot]$ is the expectation with respect to the probability measure P, and $\mathbb{E}[\cdot|\,\mathcal{F}_t]$ is the conditional expectation given $\mathcal{F}_t$. The following notations are used in this paper.

- $\mathbb{E}[\cdot|\,\mathcal{F}_s^t] = \mathbb{E}[\cdot|\,\mathcal{F}_t] - \mathbb{E}[\cdot|\,\mathcal{F}_s]$.
- $C^1(\Gamma,M) = \{f(t):\Gamma\to M|\,f(t)\ \text{ is continuously differentiable}\}$.
- $\bar{C}(\Gamma,M) = \{f(t):\Gamma\to M|\,f(t)\ \text{ is continuous with }\ \sup_{t\in\Gamma}|f(t)|<\infty\}$.
- $L^k(\Omega;\mathbb{R}^n) = \left\{X(\omega):\Omega\to\mathbb{R}^n|\,\mathbb{E}\,|X|^k<\infty\right\}$.
- $L^k_{\mathbb{F}}(\Gamma;\mathbb{R}^n) = \left\{X(t,\omega):\Gamma\times\Omega\to\mathbb{R}^n|\,X(t)\ \text{ is }\ \mathcal{F}_t\text{-predictable with }\mathbb{E}\left[\int_\Gamma |X(t)|^k\,\mathrm{d}t\right]<\infty\right\}$.
- $\mathcal{S}^k(\Gamma;\mathbb{R}^n) = \left\{X(t,\omega):\Gamma\times\Omega\to\mathbb{R}^n|\,X(t)\text{ is }\mathcal{F}_t\text{-predictable càdlàg with }\mathbb{E}\left[\sup_{t\in\Gamma}|X(t)|^k\right]<\infty\right\}$.

2 Problem formulation

Let $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P\right)$ be the filtered probability space, on which $W(t)=W(t,\omega):[0,\infty)\times\Omega\to\mathbb{R}$ is a one-dimensional standard Brownian motion and $N(\mathrm{d}t,\mathrm{d}z)$ is a one-dimensional Poisson random measure. The compensated Poisson random measure is defined as $\tilde{N}(\mathrm{d}t,\mathrm{d}z) = N(\mathrm{d}t,\mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$ with Lévy measure $\nu(\cdot)$ satisfying $\int_{\mathbb{R}_0^n}\min\left\{1,|z|^2\right\}\nu(\mathrm{d}z)<\infty$. The filtration $\mathbb{F}=\{\mathcal{F}_t\}_{t\geqslant 0}$, which is generated by both the Brownian motion $W(s)$ and the Poisson random measure $N(\mathrm{d}s,\mathrm{d}z)$ for $s\leqslant t$, describes full information available to the players in the SDG.

We consider a non-zero-sum SDG with two players (Player 1 and Player 2). Suppose that the state process $x(t)=x(t,\omega):[0,T]\times\Omega\to\mathbb{R}^n$ evolves according to a linear controlled jump diffusion as follows:

$$\begin{cases} \mathrm{d}x(t)=[Ax(t)+B_1u_1(t)+B_2u_2(t-\delta)]\,\mathrm{d}t+\left[\bar{A}x(t)+\bar{B}_1u_1(t)+\bar{B}_2u_2(t-\delta)\right]\mathrm{d}W(t)\\ \qquad\qquad +\int_{\mathbb{R}_0^n}\left[\breve{A}(z)x(t)+\breve{B}_1(z)u_1(t)+\breve{B}_2(z)u_2(t-\delta)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}z), \quad t\in[\delta,T],\\ x(t)=\xi(t),\quad u_2(t)=\mu_2(t),\quad t\in[0,\delta), \end{cases} \tag{1}$$

where $u_1(t):=u_1(t,\omega):[\delta,T]\times\Omega\to U_1$ is the strategy of Player 1 and $u_2(t):=u_2(t,\omega):[0,T-\delta]\times\Omega\to U_2$ is the action of Player 2. Here the control domain U_i is a nonempty convex subset of $\mathbb{R}^{m_i}$, $i=1,2$, and $\delta\geqslant 0$ is a given positive constant that represents the time delay in the action $u_2(\cdot)$. The initial state path $\xi(t)\in\mathcal{S}^2([0,\delta);\mathbb{R}^n)$ and the initial control path $\mu(t)\in\bar{C}([0,\delta),\mathbb{R}^{m_2})$ are predetermined. In addition, the coefficients $\breve{A}(z):\mathbb{R}_0^n\to\mathbb{R}^{n\times n}$, $\breve{B}_1(z):\mathbb{R}_0^n\to\mathbb{R}^{n\times m_1}$ and $\breve{B}_2(z):\mathbb{R}_0^n\to\mathbb{R}^{n\times m_2}$ are matrix-valued functions, while $A,B_1,B_2,\bar{A},\bar{B}_1,\bar{B}_2$ are constant matrices with compatible dimensions.

The set of admissible open-loop controls for Player i is defined by $\mathcal{A}^i:=L^2_{\mathbb{F}}([\delta,T];U_i)$ and meanwhile $u_i(\cdot)\in\mathcal{A}^i$ is referred to as an admissible open-loop control for Player i. Suppose that the cost functional of Player i is defined as

$$\mathcal{J}_i(u_1(\cdot),u_2(\cdot))=\frac{1}{2}\mathbb{E}\left[\int_\delta^T\left(x(t)^\top Q_ix(t)+u_1(t)^\top R_i^1u_1(t)+u_2(t-\delta)^\top R_i^2u_2(t-\delta)\right)\mathrm{d}t+x(T)^\top G_ix(T)\right], \tag{2}$$

where Q_i,G_i are positive semi-definite and R_i^1,R_i^2 are positive definite matrices of compatible dimensions. Apparently, if the control pair $(u_1(\cdot),u_2(\cdot))\in\mathcal{A}^1\times\mathcal{A}^2$ and the initial paths $\xi(t)\in\mathcal{S}^2([0,\delta);\mathbb{R}^n)$ and $\mu(\cdot)\in L^2_{\mathbb{F}}([0,\delta);U_2)$, then the system (1) admits a unique strong solution $x^{u_1,u_2}(\cdot)\in\mathcal{S}^2([\delta,T];\mathbb{R}^n)$ and the performance functional $\mathcal{J}_i(u_1,u_2)$ is well-defined (see Lemma 2.1 in [13] for details).

We consider a game-theoretic setting with asymmetric information accessibility: while Player 1 operates with full information access, Player 2 is subject to an inherent information delay of $\delta\geqslant 0$ due to input delay, thus being restricted to delayed information. Mathematically, this asymmetry manifests via distinct measurability: for any time $t\in[\delta,T]$, Player 1's strategy $u_1(t)$ is $\mathcal{F}_{t-}$-measurable with $\mathcal{F}_{t-}:=\sigma\left(\bigcup_{s<t}\mathcal{F}_s\right)$, while Player 2's strategy $u_2(t-\delta)$ remains $\mathcal{F}_{(t-\delta)-}$-measurable. This structural difference in information availability creates a fundamental asymmetry in the players' decision-making capabilities. Within this framework, we study an LQ SDG with asymmetric information as follows.

Problem-LQSDG. Find a pair of admissible strategies $(\hat{u}_1(\cdot),\hat{u}_2(\cdot))\in\mathcal{A}^1\times\mathcal{A}^2$ such that

$$\begin{aligned} \mathcal{J}_1(\hat{u}_1(\cdot),\hat{u}_2(\cdot))&\leqslant\mathcal{J}_1(u_1(\cdot),\hat{u}_2(\cdot)),\quad \text{for all}\quad u_1(\cdot)\in\mathcal{A}^1,\\ \mathcal{J}_2(\hat{u}_1(\cdot),\hat{u}_2(\cdot))&\leqslant\mathcal{J}_2(\hat{u}_1(\cdot),u_2(\cdot)),\quad \text{for all}\quad u_2(\cdot)\in\mathcal{A}^2. \end{aligned}$$

In this case, the pair $(\hat{u}_1(\cdot),\hat{u}_2(\cdot))\in\mathcal{A}^1\times\mathcal{A}^2$ (if it exists) is referred to as an open-loop Nash equilibrium of Problem-LQSDG.

3 Main results

In this section, we aim to present the explicit form of the open-loop Nash equilibrium for Problem-LQSDG under the assumption of unique Nash equilibrium. The major obstacles in solving Problem-LQSDG are the interaction between the two coupled cost functionals and the inconsistent measurability of the strategies u_1 and u_2 at time t. To address these challenges, we apply the maximum principles and the orthogonal decomposition and reorganization technique to transform the problem of determining Nash equilibria into that of solving an auxiliary system of FBSDDEs with mutually orthogonal strategies. Then, under the uniqueness assumption, we provide the explicit solutions to the auxiliary FBSDDEs and thereby obtain the explicit open-loop Nash equilibrium via a new-established generalized Riccati differential equation.

3.1 An auxiliary system of FBSDDEs

To begin with, we employ the maximum principles established in [23] to present sufficient and necessary conditions for a Nash equilibrium $(\hat{u}_1(\cdot),\hat{u}_2(\cdot))$.

Lemma 1. The pair of control strategies $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ is a Nash equilibrium for Problem-LQSDG if and only if $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ satisfies the stationary conditions,

$$R_1^1 \hat{u}_1(t) + B_1^\top \hat{p}_1(t) + \bar{B}_1^\top \hat{q}_1(t) + \int_{\mathbb{R}_0^n} \breve{B}_1^\top(z) \hat{r}_1(t,z) \nu(\mathrm{d}z) = 0, \tag{3}$$

$$R_2^2 \hat{u}_2(t-\delta) + \mathbb{E}\left[B_2^\top \hat{p}_2(t) + \bar{B}_2^\top \hat{q}_2(t) + \int_{\mathbb{R}_0^n} \breve{B}_2^\top(z) \hat{r}_2(t,z) \nu(\mathrm{d}z) \middle| \mathcal{F}_{(t-\delta)^-} \right] = 0, \tag{4}$$

where the triple $(\hat{p}_i(t), \hat{q}_i(t), \hat{r}_i(t,z))$, $i = 1, 2$, of $\mathcal{F}_t$-predictable processes satisfies the adjoint equation given by

$$\begin{cases} \mathrm{d}\hat{p}_i(t) = -\left[Q_i \hat{x}(t) + A^\top \hat{p}_i(t) + \bar{A}^\top \hat{q}_i(t) + \int_{\mathbb{R}_0^n} \breve{A}^\top(z) \hat{r}_i(t,z) \nu(\mathrm{d}z) \right] \mathrm{d}t \\ \qquad\qquad + \hat{q}_i(t) \mathrm{d}W(t) + \int_{\mathbb{R}_0^n} \hat{r}_i(t,z) \tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \in [\delta, T], \\ \hat{p}_i(T) = G_i \hat{x}(T), \end{cases} \tag{5}$$

with $\hat{x}(\cdot) = x^{\hat{u}_1, \hat{u}_2}(\cdot)$ being the state trajectory corresponding to the pair $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$.

Remark 1. With the assumptions that Q_i, G_i are positive semi-definite and R_i^1, R_i^2 are positive definite matrices, Lemma 1 follows from the necessary and sufficient maximum principles established in [23]. More specifically, these matrix definiteness conditions ensure the convexity of the Hamiltonian and the terminal utilities, under which the stationary conditions (3) and (4) are necessary and sufficient for $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ to be a Nash equilibrium. Parallel results in the Brownian-motion setting see Theorem 2.2.1 in [1].

Based on Lemma 1, the problem of finding a Nash equilibrium $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ has been reformulated to the task of solving the following system of FBSDDEs:

$$\begin{cases} \mathrm{d}\hat{x}(t) = [A\hat{x}(t) + B_1 \hat{u}_1(t) + B_2 \hat{u}_2(t-\delta)] \mathrm{d}t + \left[\bar{A}\hat{x}(t) + \bar{B}_1 \hat{u}_1(t) + \bar{B}_2 \hat{u}_2(t-\delta) \right] \mathrm{d}W(t) \\ \qquad\qquad + \int_{\mathbb{R}_0^n} \left[\breve{A}(z)\hat{x}(t) + \breve{B}_1(z) \hat{u}_1(t) + \breve{B}_2(z) \hat{u}_2(t-\delta) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \in [\delta, T], \\ \mathrm{d}\hat{p}_i(t) = -\left[Q_i \hat{x}(t) + A^\top \hat{p}_i(t) + \bar{A}^\top \hat{q}_i(t) + \int_{\mathbb{R}} \breve{A}^\top(z) \hat{r}_i(t,z) \nu(\mathrm{d}z) \right] \mathrm{d}t \\ \qquad\qquad + \hat{q}_i(t) \mathrm{d}W(t) + \int_{\mathbb{R}_0^n} \hat{r}_i(t,z) \tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \in [\delta, T], \quad i = 1, 2, \\ \hat{x}(t) = \xi(t), \quad \hat{u}_i(t) = \mu_i(t), \quad \text{for} \quad t \in [0, \delta); \quad \hat{p}_i(T) = G_i \hat{x}(T), \quad i = 1, 2, \end{cases} \tag{6}$$

with the stationary conditions (3) and (4).

Lemma 2. Problem-LQSDG admits a unique Nash equilibrium $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ if and only if there exists a unique solution $(\hat{x}(\cdot), \hat{p}_1(\cdot), \hat{q}_1(\cdot), \hat{r}_1(\cdot,\cdot), \hat{p}_2(\cdot), \hat{q}_2(\cdot), \hat{r}_2(\cdot,\cdot), \hat{u}_1(\cdot), \hat{u}_2(\cdot))$ satisfying FBSDDEs (6) with the stationary conditions (3) and (4).

Clearly, Lemma 2 is a consequence of Lemma 1.

In FBSDDEs (6), one forward SDE and two backward SDEs are coupled through the stationary conditions (3) and (4). The conditional expectation in the stationary condition (4) introduces nonlinearity and meanwhile the two inputs $u_1(t)$ and $u_2(t-\delta)$ at time t are correlated and asynchronous, with $u_2(t-\delta)$ exhibiting a time delay of δ. These characteristics give rise to considerable difficulty in seeking an explicit solution to FBSDDEs (6). To overcome the difficulty, we extend the orthogonal decomposition and reorganization technique, proposed in [21, 22], to derive an equivalent auxiliary FBSDDEs with orthogonal strategies.

Orthogonal decomposition. Let the strategy $u_1(t)$ of Player 1 at time t be decomposed into

$$u_1(t) = u_1^\delta(t) + u_1^\perp(t), \tag{7}$$

where

$$u_1^\delta(t) = \mathbb{E}\left[u_1(t) \middle| \mathcal{F}_{(t-\delta)-} \right]$$

is $\mathcal{F}_{(t-\delta)-}$-measurable for $t \in [\delta, T]$, and

$$u_1^\perp(t) = u_1(t) - \mathbb{E}\left[u_1(t) \middle| \mathcal{F}_{(t-\delta)-} \right]$$

with $\mathbb{E}\left[u_1^\perp(t) \middle| \mathcal{F}_{(t-\delta)-} \right] = 0$. Then the two components $u_1^\delta(t)$ and $u_1^\perp(t)$ are orthogonal in $L^2(\Omega; \mathbb{R}^n)$ due to the fact that $\mathbb{E}\left[\left\langle u_1^\delta(t), u_1^\perp(t) \right\rangle \right] = 0$ for $t \in [\delta, T]$. Moreover, at time $t \in [\delta, T]$, the control strategy $u_1^\perp(t)$ is orthogonal to any $\mathcal{F}_{t-\delta}$-predictable $\upsilon(t)$.

With the orthogonal decomposition (7), the cost functional $\mathcal{J}_i(u_1(\cdot),u_2(\cdot))$ for Player i is rewritten as

$$\begin{aligned}\mathcal{J}_i(u_1^\delta(\cdot),u_1^\perp(\cdot),u_2(\cdot)) =&\frac{1}{2}\mathbb{E}\Bigg[\int_0^T \big(x(t)^\top Q_i x(t)+u_1^\delta(t)^\top R_i^1 u_1^\delta(t)+u_1^\perp(t)^\top R_i^1 u_1^\perp(t)\\&+u_2(t-\delta)^\top R_i^2 u_2(t-\delta)\big)\,\mathrm{d}t+x(T)^\top G_i x(T)\Bigg],\quad i=1,2.\end{aligned}\tag{8}$$

Furthermore, Problem-LQSDG is equivalent to the LQ SDEs as follows.

Problem-LQSDG-Eq. Find a triple of admissible controls $(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),\hat{u}_2(\cdot))\in\mathcal{A}^1\times\mathcal{A}^1\times\mathcal{A}^2$ such that

$$\begin{aligned}&\mathcal{J}_1\left(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),\hat{u}_2(\cdot)\right)\leqslant\mathcal{J}_1\left(u_1^\delta(\cdot),u_1^\perp(\cdot),\hat{u}_2(\cdot)\right),\quad\text{for all}\quad (u_1^\delta(\cdot),u_1^\perp(\cdot))\in\mathcal{A}^1\times\mathcal{A}^1,\\&\mathcal{J}_2\left(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),\hat{u}_2(\cdot)\right)\leqslant\mathcal{J}_2\left(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),u_2(\cdot)\right),\quad\text{for all}\quad u_2(\cdot)\in\mathcal{A}^2.\end{aligned}$$

The triple $(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),\hat{u}_2(\cdot))\in\mathcal{A}^1\times\mathcal{A}^1\times\mathcal{A}^2$ (if it exists) is referred to as an open-loop Nash equilibrium of Problem-LQSDG-Eq.

With the aid of Lemmas 1 and 2, we have the sufficient and necessary conditions for Nash equilibria of Problem-LQSDG-Eq.

Lemma 3. Problem-LQSDG-Eq admits a unique open-loop Nash equilibrium $\left(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),\hat{u}_2(\cdot)\right)$ if and only if the triple $(\hat{u}_1^\delta(\cdot),\hat{u}_1^\perp(\cdot),\hat{u}_2(\cdot))$ satisfies the following system of FBSDDEs:

$$\begin{cases}\mathrm{d}\hat{x}(t)=\left[A\hat{x}(t)+B_1\hat{u}_1^\delta(t)+B_1\hat{u}_1^\perp(t)+B_2\hat{u}_2(t-\delta)\right]\mathrm{d}t\\\qquad\quad+\left[\bar{A}\hat{x}(t)+\bar{B}_1\hat{u}_1^\delta(t)+\bar{B}_1\hat{u}_1^\perp(t)+\bar{B}_2\hat{u}_2(t-\delta)\right]\mathrm{d}W(t)\\\qquad\quad+\int_{\mathbb{R}_0^n}\left[\breve{A}(z)\hat{x}(t)+\breve{B}_1(z)\hat{u}_1^\delta(t)+\breve{B}_1(z)\hat{u}_1^\perp(t)+\breve{B}_2(z)\hat{u}_2(t-\delta)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\\mathrm{d}\hat{p}_i(t)=-\left[Q_i\hat{x}(t)+A^\top\hat{p}_i(t)+\bar{A}^\top\hat{q}_i(t)+\int_{\mathbb{R}_0^n}\breve{A}^\top(z)\hat{r}_i(t,z)\nu(\mathrm{d}z)\right]\mathrm{d}t\\\qquad\quad+\hat{q}_i(t)\mathrm{d}W(t)+\int_{\mathbb{R}_0^n}\hat{r}_i(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\quad i=1,2,\\\hat{x}(t)=\xi(t),\quad\hat{u}_2(t)=\mu(t),\quad\text{for}\quad t\in[0,\delta];\quad\hat{p}_i(T)=G_i\hat{x}(T),\quad i=1,2\end{cases}\tag{9}$$

with the following stationary conditions:

$$R_1^1\hat{u}_1^\delta(t)+\mathbb{E}\left[B_1^\top\hat{p}_1(t)+\bar{B}_1^\top\hat{q}_1(t)+\int_{\mathbb{R}_0^n}\breve{B}_1^\top(z)\hat{r}_1(t,z)\nu(\mathrm{d}z)\middle|\mathcal{F}_{(t-\delta)-}\right]=0,\tag{10}$$

$$R_1^1\hat{u}_1^\perp(t)+\mathbb{E}\left[B_1^\top\hat{p}_1(t)+\bar{B}_1^\top\hat{q}_1(t)+\int_{\mathbb{R}_0^n}\breve{B}_1^\top(z)\hat{r}_1(t,z)\nu(\mathrm{d}z)\middle|\mathcal{F}^t_{(t-\delta)-}\right]=0,\tag{11}$$

$$R_2^2\hat{u}_2(t-\delta)+\mathbb{E}\left[B_2^\top\hat{p}_2(t)+\bar{B}_2^\top\hat{q}_2(t)+\int_{\mathbb{R}_0^n}\breve{B}_2^\top(z)\hat{r}_2(t,z)\nu(\mathrm{d}z)\middle|\mathcal{F}_{(t-\delta)-}\right]=0.\tag{12}$$

Here $\hat{x}(t),\hat{p}_i(t),\hat{q}_i(t),\hat{r}_i(t,z)$ are the state process and the adjoint processes corresponding to the triple $\left(\hat{u}_1^\delta(t),\hat{u}_1^\perp(t),\hat{u}_2(t-\delta)\right)$, respectively.

Reorganization. Define

$$v^\delta(t):=\left(u_1^\delta(t)\ \ u_2(t-\delta)\right)^\top.\tag{13}$$

Then $v^\delta(t)$ is $\mathcal{F}_{(t-\delta)-}$-measurable and $v^\delta(t)$ is orthogonal to $u_1^\perp(t)$. Moreover, let I_m denote the identity matrix of size m and define the block matrices as follows:

$$\begin{aligned}&\mathbb{I}_m=\left(I_m\ \ I_m\right)^\top,\quad B=\left(B_1\ \ B_2\right),\quad\bar{B}=\left(\bar{B}_1\ \ \bar{B}_2\right),\quad\breve{B}(z)=\left(\breve{B}_1(z)\ \ \breve{B}_2(z)\right),\\&\mathbf{B}=\begin{pmatrix}B&0\\0&B\end{pmatrix},\quad\bar{\mathbf{B}}=\begin{pmatrix}\bar{B}&0\\0&\bar{B}\end{pmatrix},\quad\breve{\mathbf{B}}(z)=\begin{pmatrix}\breve{B}(z)&0\\0&\breve{B}(z)\end{pmatrix},\quad\mathbf{A}=\begin{pmatrix}A&0\\0&A\end{pmatrix},\quad\bar{\mathbf{A}}=\begin{pmatrix}\bar{A}&0\\0&\bar{A}\end{pmatrix},\\&\breve{\mathbf{A}}(z)=\begin{pmatrix}\breve{A}(z)&0\\0&\breve{A}(z)\end{pmatrix},\quad\mathbf{Q}=\begin{pmatrix}Q_1&0\\0&Q_2\end{pmatrix},\quad\mathbf{G}=\begin{pmatrix}G_1&0\\0&G_2\end{pmatrix},\quad\mathbf{R}=\begin{pmatrix}R_1&0\\0&R_2\end{pmatrix},\\&R_i=\begin{pmatrix}R_i^1&0\\0&R_i^2\end{pmatrix},\quad\mathbf{p}(t)=\begin{pmatrix}p_1(t)\\p_2(t)\end{pmatrix},\quad\mathbf{q}(t)=\begin{pmatrix}q_1(t)\\q_2(t)\end{pmatrix},\quad\mathbf{r}(t,z)=\begin{pmatrix}r_1(t,z)\\r_2(t,z)\end{pmatrix}.\end{aligned}$$

Apparently, $\mathbf{Q}$, $\mathbf{G}$ are positive semi-definite and $\mathbf{R}$ is positive definite.

Through the extended orthogonal decomposition and reorganization technique, we further convert FBSDDEs (6) into an equivalent auxiliary FBSDDEs with the orthogonal strategies $\hat{v}^{\delta}(\cdot)$ and $\hat{v}^{\perp}(\cdot)$.

Lemma 4. FBSDDEs (6) with the stationary conditions (3) and (4) admit a unique solution if and only if there exists a unique $\big(\hat{x}(\cdot), \hat{\mathbf{p}}(\cdot), \hat{\mathbf{q}}(\cdot), \hat{\mathbf{r}}(\cdot,\cdot), \hat{v}^{\delta}(\cdot), \hat{u}_1^{\perp}(\cdot)\big)$ satisfying the following FBSDDEs:

$$\begin{cases} \mathrm{d}\hat{x}(t) = \big[A\hat{x}(t) + B\hat{v}^{\delta}(t) + B_1\hat{u}_1^{\perp}(t)\big]\,\mathrm{d}t + \big[\bar{A}\hat{x}(t) + \bar{B}\hat{v}^{\delta}(t) + \bar{B}_1\hat{u}_1^{\perp}(t)\big]\,\mathrm{d}W(t) \\ \qquad\qquad + \int_{\mathbb{R}_0^n}\Big[\breve{A}(z)\hat{x}(t) + \breve{B}(z)\hat{v}^{\delta}(t) + \breve{B}_1(z)\hat{u}_1^{\perp}(t)\Big]\tilde{N}(\mathrm{d}t,\mathrm{d}z), \qquad t\in[\delta,T], \\ \mathrm{d}\hat{\mathbf{p}}(t) = -\Big[\mathbf{Q}\mathbb{I}_n\hat{x}(t) + \mathbf{A}^{\top}\hat{\mathbf{p}}(t) + \bar{\mathbf{A}}^{\top}\hat{\mathbf{q}}(t) + \int_{\mathbb{R}_0^n}\breve{\mathbf{A}}^{\top}(z)\hat{\mathbf{r}}(t,z)\nu(\mathrm{d}z)\Big]\,\mathrm{d}t \\ \qquad\qquad + \hat{\mathbf{q}}(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0^n}\hat{\mathbf{r}}(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \quad t\in[\delta,T], \\ \hat{x}(t) = \xi(t), \quad \hat{u}_2(t) = \mu_2(t) \quad \text{for} \quad t\in[0,\delta); \quad \hat{\mathbf{p}}(T) = \mathbf{G}\mathbb{I}_n\hat{x}(T) \end{cases} \tag{14}$$

with the stationary conditions

$$\mathbf{M}^{\top}\mathbf{R}\mathbf{M}\hat{v}^{\delta}(t) + \mathbb{E}\left[\mathbf{M}^{\top}\mathbf{B}^{\top}\hat{\mathbf{p}}(t) + \mathbf{M}^{\top}\bar{\mathbf{B}}^{\top}\hat{\mathbf{q}}(t) + \int_{\mathbb{R}_0^n}\mathbf{M}^{\top}\breve{\mathbf{B}}^{\top}(z)\hat{\mathbf{r}}(t,z)\nu(\mathrm{d}z)\,\middle|\,\mathcal{F}_{(t-\delta)-}\right] = 0, \tag{15}$$

$$\mathbf{L}^{\top}\mathbf{R}\mathbf{L}\hat{u}_1^{\perp}(t) + \mathbb{E}\left[\mathbf{L}^{\top}\mathbf{B}^{\top}\hat{\mathbf{p}}(t) + \mathbf{L}^{\top}\bar{\mathbf{B}}^{\top}\hat{\mathbf{q}}(t) + \int_{\mathbb{R}_0^n}\mathbf{L}^{\top}\breve{\mathbf{B}}^{\top}(z)\hat{\mathbf{r}}(t,z)\nu(\mathrm{d}z)\,\middle|\,\mathcal{F}^t_{(t-\delta)-}\right] = 0, \tag{16}$$

where

$$\mathbf{M} := \begin{pmatrix} I_{m_1} & 0_{m_2\times m_1} & 0_{m_1\times m_1} & 0_{m_2\times m_1} \\ 0_{m_1\times m_2} & 0_{m_2\times m_2} & 0_{m_1\times m_2} & I_{m_2} \end{pmatrix}^{\top} \quad \text{and} \quad \mathbf{L} := \begin{pmatrix} I_{m_1} & 0_{m_2\times m_1} & 0_{m_1\times m_1} & 0_{m_2\times m_1} \end{pmatrix}^{\top}.$$

3.2 Solutions to Problem-LQSDG

In order to identify the unique solution to FBSDDEs (14) under the uniqueness assumption, we define a generalized Riccati differential equation for $\mathbb{P}(\cdot)\in C^1([\delta,T],\mathbb{R}^{2n\times n})$ as follows:

$$\begin{cases} -\dot{\mathbb{P}}(t) = \mathbf{Q}\mathbb{I}_n + \mathbb{P}(t)A + \mathbf{A}^{\top}\mathbb{P}(t) + \bar{\mathbf{A}}^{\top}\mathbb{P}(t)\bar{A} + \int_{\mathbb{R}_0^n}\breve{\mathbf{A}}^{\top}(z)\mathbb{P}(t)\breve{A}(z)\nu(\mathrm{d}z) - \Pi(t,t+\delta) - \Pi_1(t,t), \\ \mathbb{P}(T) = \mathbf{G}\mathbb{I}_n, \end{cases} \tag{17}$$

where

$$\Pi_1(t,t) = \phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^{\top}\lambda(t), \tag{18}$$

and $\Pi(t,s)$ satisfies the following ordinary differential equation (ODE):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Pi(t,s) = &-\Pi(t,s)\left\{A - B_1\Psi_1^{-1}(t)\mathbf{L}^{\top}\left(\lambda(t) - \mathbf{B}^{\top}\int_s^{t+\delta}\Pi(t,u)\mathrm{d}u\right)\right\} \\ &- \left\{\mathbf{A}^{\top} - \left(\phi_1(t) - \left(\int_s^{t+\delta}\Pi(t,u)\mathrm{d}u\right)B_1\right)\Psi_1^{-1}(t)\mathbf{L}^{\top}\mathbf{B}^{\top}\right\}\Pi(t,s), \quad s\in[t,t+\delta] \end{aligned} \tag{19}$$

with $\Pi(T,s)=0$ for $s\in[T,T+\delta]$ and

$$\begin{aligned} \Pi(t,t) = &\left(\phi(t) - \left(\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)B\right)\Psi^{-1}(t)\mathbf{M}^{\top}\left(\lambda(t) - \mathbf{B}^{\top}\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right) \\ &- \left(\phi_1(t) - \left(\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)B_1\right)\Psi_1^{-1}(t)\mathbf{L}^{\top}\left(\lambda(t) - \mathbf{B}^{\top}\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right). \end{aligned} \tag{20}$$

Here

$$\Psi(t) = \mathbf{M}^{\top}\left(\mathbf{R}\mathbf{M} + \bar{\mathbf{B}}^{\top}\mathbb{P}(t)\bar{B} + \int_{\mathbb{R}_0^n}\breve{\mathbf{B}}^{\top}(z)\mathbb{P}(t)\breve{B}(z)\nu(\mathrm{d}z)\right), \tag{21}$$

$$\Psi_1(t) = \mathbf{L}^\top \left(\mathbf{R}\mathbf{L} + \bar{\mathbf{B}}^\top \mathbb{P}(t)\bar{B}_1 + \int_{\mathbb{R}_0^n} \breve{\mathbf{B}}^\top(z)\mathbb{P}(t)\breve{B}_1(z)\nu(\mathrm{d}z)\right), \tag{22}$$

$$\lambda(t) = \mathbf{B}^\top \mathbb{P}(t) + \bar{\mathbf{B}}^\top \mathbb{P}(t)\bar{A} + \int_{\mathbb{R}_0^n} \breve{\mathbf{B}}^\top(z)\mathbb{P}(t)\breve{A}(z)\nu(\mathrm{d}z), \tag{23}$$

$$\phi(t) = \bar{\mathbf{A}}^\top \mathbb{P}(t)\bar{B} + \int_{\mathbb{R}_0^n} \breve{\mathbf{A}}^\top(z)\mathbb{P}(t)\breve{B}(z)\nu(\mathrm{d}z) + \mathbb{P}(t)B, \tag{24}$$

$$\phi_1(t) = \bar{\mathbf{A}}^\top \mathbb{P}(t)\bar{B}_1 + \int_{\mathbb{R}_0^n} \breve{\mathbf{A}}^\top(z)\mathbb{P}(t)\breve{B}_1(z)\nu(\mathrm{d}z) + \mathbb{P}(t)B_1. \tag{25}$$

Remark 2. The generalized Riccati differential equation (17)–(25) extends the Riccati-type equations typically encountered in LQ stochastic control problems with time delays (see [20, 22]). Unlike the Riccati-type equations in [20, 22], whose solutions are symmetric matrices in $\mathcal{S}^n$, the solution $\mathbb{P}(t)$ of our generalized system takes values in $\mathbb{R}^{2n\times n}$ and loses symmetry due to the coupling among multiple objective functionals and asymmetric information structures. Specifically, let

$$\mathbb{P}(t) = \begin{bmatrix}\mathbb{P}_1(t) & \mathbb{P}_2(t)\end{bmatrix}^\top \quad \text{with} \quad \mathbb{P}_i(t) : [\delta, T] \to \mathbb{R}^{n\times n}, \quad i = 1, 2.$$

Then $\mathbb{P}_1(t)$ and $\mathbb{P}_2(t)$ are inherently coupled through the term $\Pi(t, t+\delta)$, which comes from Player 2's delayed control input $u_2(t-\delta)$. This coupling prevents the system (17) from being decomposed into two independent Riccati-type equations for $\mathbb{P}_1(t)$ and $\mathbb{P}_2(t)$. Furthermore, the presence of $\Pi(t, s)$, governed by ODE (19), introduces a non-local dependence on future values of $\mathbb{P}(t)$, reflecting the influence of time delays on the Nash equilibrium. Moreover, the jump components in the system dynamics introduce integral terms with respect to the Lévy measure $\nu(\mathrm{d}z)$, which contribute additional structure to the generalized Riccati equation and illustrate how discontinuous uncertainties shape Nash equilibria.

Lemma 5. Let $\mathbb{P}(\cdot) \in C^1([\delta, T], \mathbb{R}^{2n\times n})$ be a solution to the generalized Riccati differential equation (17)–(25) such that $\Psi(t)$ and $\Psi_1(t)$ are invertible. Suppose that the triple $(\hat{\mathbf{p}}(t), \hat{\mathbf{q}}(t), \hat{\mathbf{r}}(t,z))$ satisfies the backward SDE in FBSDDEs (14). Define the processes $\mathbb{Y}(t)$, $\mathbb{Z}_1(t)$ and $\mathbb{Z}_2(t)$ by

$$\mathbb{Y}(t) = \hat{\mathbf{p}}(t) - \mathbb{P}(t)\hat{x}(t), \tag{26}$$

$$\mathbb{Z}_1(t) = \hat{\mathbf{q}}(t) - \mathbb{P}(t)\bar{A}\hat{x}(t) - \mathbb{P}(t)\bar{B}\hat{v}^\delta(t) - \mathbb{P}(t)\bar{B}_1\hat{u}_1^\perp(t), \tag{27}$$

$$\mathbb{Z}_2(t,z) = \hat{\mathbf{r}}(t,z) - \mathbb{P}(t)\breve{A}(z)\hat{x}(t) - \mathbb{P}(t)\breve{B}(z)\hat{v}^\delta(t) - \mathbb{P}(t)\breve{B}_1(z)\hat{u}_1^\perp(t), \tag{28}$$

respectively. Then the tuple $(\hat{x}(\cdot), \mathbb{Y}(\cdot), \mathbb{Z}_1(\cdot), \mathbb{Z}_2(\cdot,\cdot))$ satisfies the following FBSDDEs:

$$\begin{cases}
\mathrm{d}\hat{x}(t) = \Big\{A\hat{x}(t) - B\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}_{(t-\delta)-}\right] - B_1\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}^t_{(t-\delta)-}\right] \\
\qquad -B\Psi^{-1}(t)\mathbf{M}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}_{(t-\delta)-}\right] - B_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}t \\
\qquad + \Big\{\bar{A}\hat{x}(t) - \bar{B}\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}_{(t-\delta)-}\right] - \bar{B}_1\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}^t_{(t-\delta)-}\right] \\
\qquad -\bar{B}\Psi^{-1}(t)\mathbf{M}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}_{(t-\delta)-}\right] - \bar{B}_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}W(t) \\
\qquad + \int_{\mathbb{R}_0^n}\Big\{\breve{A}(z)\hat{x}(t) - \breve{B}(z)\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}_{(t-\delta)-}\right] \\
\qquad -\breve{B}_1(z)\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}^t_{(t-\delta)-}\right] - \breve{B}(z)\Psi^{-1}(t)\mathbf{M}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}_{(t-\delta)-}\right] \\
\qquad -\breve{B}_1(z)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\
\mathrm{d}\mathbb{Y}(t) = -\Big\{\mathbf{A}^\top\mathbb{Y}(t) + \bar{\mathbf{A}}^\top\mathbb{Z}_1(t) + \int_{\mathbb{R}_0^n}\breve{\mathbf{A}}^\top(z)\mathbb{Z}_2(t,z)\nu(\mathrm{d}z) + \Pi(t,t+\delta)\hat{x}(t) + \Pi_1(t,t)\hat{x}(t) \\
\qquad -\phi(t)\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}_{(t-\delta)-}\right] - \phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)|\,\mathcal{F}^t_{(t-\delta)-}\right] \\
\qquad -\phi(t)\Psi^{-1}(t)\mathbf{M}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}_{(t-\delta)-}\right] - \phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbb{E}\left[\Xi(t)|\,\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}t \\
\qquad +\mathbb{Z}_1(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0^n}\mathbb{Z}_2(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\
\mathbb{Y}(T) = 0; \quad \hat{x}(t) = \xi(t), \qquad \hat{u}(t) = \mu(t), \quad \text{for} \quad t\in[0,\delta),
\end{cases} \tag{29}$$

with

$$\Xi(t) := \mathbf{B}^\top\mathbb{Y}(t) + \bar{\mathbf{B}}^\top\mathbb{Z}_1(t) + \int_{\mathbb{R}_0^n}\breve{\mathbf{B}}^\top(z)\mathbb{Z}_2(t,z)\nu(\mathrm{d}z).$$

Moreover, there exists a unique solution to FBSDDEs (14) with the stationary conditions (15) and (16) if and only if FBSDDEs (29) admits a unique solution.

Proof. The proof of Lemma 5 is detailed in Appendix A.

Before presenting explicit solutions for FBSDDEs (29), we need the following preliminary result.

Lemma 6. For $t \in [\delta, T]$, let the stochastic process $\eta(t)$ be defined by

$$\eta(t) := \int_t^{t+\delta} \Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta, \tag{30}$$

where $\Pi(t,\theta)$ is given by ODE (20) for $\theta \in [t, t+\delta]$ and $\hat{x}(t)$ is the solution of the forward SDE in FBSDDEs (29). Then we have

$$\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(t-\delta)-}\right] = \left(\int_t^{t+\delta} \Pi(t,\theta)\mathrm{d}\theta\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right], \tag{31}$$

and meanwhile we get, for $\theta \in (t, t+\delta)$,

$$\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right] = \int_t^{\theta} \Pi(t,s)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(s-\delta)-}\right]\mathrm{d}s + \left(\int_\theta^{t+\delta} \Pi(t,s)\mathrm{d}s\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]. \tag{32}$$

Proof. The proof of Lemma 6 can be found in Appendix B.

We now turn our attention to the explicit solution for FBSDDEs (29) under the assumption of unique Nash equilibrium.

Theorem 1. Assume that Problem-LQSDG-Eq admits a unique Nash equilibrium $(\hat{u}_1^\delta(\cdot), \hat{u}_1^\perp(\cdot), \hat{u}_2(\cdot))$. Let $\mathbb{P}(\cdot) \in C^1([\delta, T], \mathbb{R}^{2n\times n})$ be a solution to the generalized Riccati differential equation (17)–(25) such that $\Psi(t)$ and $\Psi_1(t)$ are invertible. Let $\Pi(t,s)$ be the solution to ODE (19) for $s \in [t, t+h]$. Then, for $t \in [\delta, T]$, the solution $(\hat{x}(t), \mathbb{Y}(t), \mathbb{Z}_1(t), \mathbb{Z}_2(t,z))$ to FBSDDEs (29) is uniquely given as follows:

$$\mathbb{Y}(t) = -\eta(t) = -\int_t^{t+\delta} \Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta, \tag{33}$$

$$\mathbb{Z}_1(t) = \mathbb{Z}_2(t,z) = 0, \quad P\text{-a.s.}, \tag{34}$$

and the state process $\hat{x}(t)$ evolves according to the following SDE:

$$\begin{aligned}
\mathrm{d}\hat{x}(t) = &\Big\{A\hat{x}(t) - B\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - B_1\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\
&- B\Psi^{-1}(t)\mathbf{M}^\top\mathbf{B}^\top\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - B_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\eta(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}t\\
&+ \Big\{\bar{A}\hat{x}(t) - \bar{B}\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \bar{B}_1\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\
&- \bar{B}\Psi^{-1}(t)\mathbf{M}^\top\mathbf{B}^\top\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \bar{B}_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\eta(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}W(t)\\
&+ \int_{\mathbb{R}_0^n}\Big\{\breve{A}(z)\hat{x}(t) - \breve{B}(z)\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \breve{B}_1(z)\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\
&- \breve{B}(z)\Psi^{-1}(t)\mathbf{M}^\top\mathbf{B}^\top\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \breve{B}_1(z)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\eta(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\tilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned} \tag{35}$$

Proof. Refer to Appendix C for the proof of Theorem 1.

Combining the results of Lemma 5 and Theorem 1, we obtain the explicit solution of FBSDDEs (14), which is summarized as follows.

Corollary 1. Assume that Problem-LQSDG-Eq is uniquely solvable with the Nash equilibrium point denoted by $(\hat{u}_1^\delta(\cdot), \hat{u}_1^\perp(\cdot), \hat{u}_2(\cdot))$. Let $\mathbb{P}(\cdot) \in C^1([\delta, T], \mathbb{R}^{2n\times n})$ be the solution to the generalized Riccati equation (17)–(25) such that $\Psi(t)$ and $\Psi_1(t)$ are invertible. Let $\Pi(t,s)$ be the solution to ODE (19) for $s \in [t, t+h]$. Then, for $t \in [\delta, T]$, the solution $(\hat{x}(t), \hat{\mathbf{p}}(t), \hat{\mathbf{q}}(t), \hat{\mathbf{r}}(t,\cdot))$ to FBSDDEs (14) is uniquely given by

$$\begin{aligned}
\hat{\mathbf{p}}(t) &= \mathbb{P}(t)\hat{x}(t) - \int_t^{t+\delta} \Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta,\\
\hat{\mathbf{q}}(t) &= \mathbb{P}(t)\bar{A}\hat{x}(t) + \mathbb{P}(t)\bar{B}\hat{v}^\delta(t) + \mathbb{P}(t)\bar{B}_1\hat{u}_1^\perp(t),\\
\hat{\mathbf{r}}(t,z) &= \mathbb{P}(t)\breve{A}(z)\hat{x}(t) + \mathbb{P}(t)\breve{B}(z)\hat{v}^\delta(t) + \mathbb{P}(t)\breve{B}_1(z)\hat{u}_1^\perp(t).
\end{aligned} \tag{36}$$

Additionally, the state process $\hat{x}(t)$ evolves according to the SDE (35).

Now we are in the position of the Nash equilibrium for Problem-LQSDG-Eq.

Theorem 2. Assume that Problem-LQSDG-Eq is uniquely solvable with the Nash equilibrium denoted by $(\hat{u}_1^\delta(t), \hat{u}_1^\perp(t), \hat{u}_2(t-\delta))$ for $t \in [\delta, T]$. Let $\mathbb{P}(\cdot) \in C^1([\delta, T], \mathbb{R}^{2n\times n})$ be the solution to the generalized Riccati equation (17)–(25) such that $\Psi(t)$ and $\Psi_1(t)$ are invertible. Let $\Pi(t,s)$ be the solution to ODE (19) for $s \in [t, t+h]$. Then the Nash equilibrium $(\hat{u}_1^\delta(t), \hat{u}_1^\perp(t), \hat{u}_2(t-\delta))$ is given by

$$\hat{u}_1^\delta(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{v}^\delta(t), \tag{37}$$

$$\hat{u}_2(t-\delta) = \begin{pmatrix} 0 & 1 \end{pmatrix} \hat{v}^\delta(t), \tag{38}$$

and

$$\begin{aligned} \hat{u}_1^\perp(t) = &- \Psi_1^{-1}(t)\mathbf{L}^\top \left(\lambda(t)\hat{x}(t) - \mathbf{B}^\top \int_t^{t+\delta} \Pi(t,\theta)\mathbb{E}\left[\hat{x}(t) \middle| \mathcal{F}_{(\theta-\delta)-} \right] \mathrm{d}\theta \right) \\ &+ \Psi_1^{-1}(t)\mathbf{L}^\top \left(\lambda(t) - \mathbf{B}^\top \int_t^{t+\delta} \Pi(t,\theta)\mathrm{d}\theta \right) \mathbb{E}\left[\hat{x}(t) \middle| \mathcal{F}_{(t-\delta)-} \right]. \end{aligned} \tag{39}$$

Here

$$\hat{v}^\delta(t) = -\Psi^{-1}(t)\mathbf{M}^\top \left(\lambda(t) - \mathbf{B}^\top \int_t^{t+\delta} \Pi(t,\theta)\mathrm{d}\theta \right) \mathbb{E}\left[\hat{x}(t) \middle| \mathcal{F}_{(t-\delta)-} \right], \tag{40}$$

$$\mathbb{E}\left[\hat{x}(t) \middle| \mathcal{F}_{(\theta-\delta)-} \right] = \mathrm{e}^{A(t-\theta+\delta)}\hat{x}(\theta-\delta) + \int_{\theta-\delta}^t \mathrm{e}^{A(t-s)} \left(B\hat{v}^\delta(s) + B_1\mathbb{E}\left[\hat{u}_1^\perp(s) \middle| \mathcal{F}_{(\theta-\delta)-} \right] \right) \mathrm{d}s \tag{41}$$

for $\theta \in [t, t+h]$, and

$$\mathbb{E}\left[\hat{x}(t) \middle| \mathcal{F}_{(t-\delta)-} \right] = \mathrm{e}^{A\delta}\hat{x}(t-\delta) + \int_{t-\delta}^t \mathrm{e}^{A(t-s)} B\hat{v}^\delta(s)\mathrm{d}s. \tag{42}$$

Proof. The proof is presented in Appendix D.

We conclude this section by presenting the Nash equilibrium for Problem-LQSDG under the uniqueness assumption.

Theorem 3. Assume that Problem-LQSDG is uniquely solvable with the Nash equilibrium denoted by $(\hat{u}_1(t), \hat{u}_2(t-\delta))$ for $t \in [\delta, T]$. Let $\mathbb{P}(\cdot) \in C^1([\delta, T], \mathbb{R}^{2n\times n})$ be the solution to the generalized Riccati equation (17)–(25) such that $\Psi(t)$ and $\Psi_1(t)$ are invertible. Let $\Pi(t,s)$ be the solution to ODE (19) for $s \in [t, t+h]$. Then the Nash equilibrium $(\hat{u}_1(t), \hat{u}_2(t-\delta))$ is determined by

$$\hat{u}_1(t) = \hat{u}_1^\delta(t) + \hat{u}_1^\perp(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \hat{v}^\delta(t) + \hat{u}_1^\perp(t), \tag{43}$$

and

$$\hat{u}_2(t-\delta) = \begin{pmatrix} 0 & 1 \end{pmatrix} \hat{v}^\delta(t), \tag{44}$$

where $\hat{v}^\delta(t)$ and $\hat{u}_1^\perp(t)$ are defined by (40) and (39), respectively.

Theorem 3 is a consequence of Theorem 2, combined with the orthogonal decomposition and reorganization (7) and (13).

Remark 3. Regarding LQ stochastic control problems with input delays, considered in [20, 22], the established Riccati-like equations not only yield the explicit optimal controllers but also allow us derive the sufficient and necessary solvability conditions via the completion of squares method. However, in our LQ SDG framework, we can only derive an explicit open-loop Nash equilibrium based on the generalized Riccati equation (17)–(25). The existence and uniqueness of the solutions to the generalized Riccati equation do not imply the existence and uniqueness of the Nash equilibrium. This limitation is caused by the fact that $\mathbb{P}_1(t)$ and $\mathbb{P}_2(t)$ are inherently coupled through the term $\Pi(t, t+\delta)$ so that the completion of squares method is invalid. Moreover, the limitation implies that LQ SDGs are not trivial extensions of LQ stochastic control problems.

4 Numerical example

To compute the Nash equilibrium for Problem-LQSDG, we implement the following steps.

(1) Solve the generalized Riccati equation (17)–(25) for $\mathbb{P}(t)$, $\Pi(t,t)$, and $\Pi(t,s)$.

(2) Compute $\hat{u}_1^{\perp}(t)$, $\hat{v}^{\delta}(t)$, and the optimal state $\hat{x}(t)$ via (39), (40), and the forward SDE in (14).

(3) Construct the Nash equilibrium $(\hat{u}_1(t), \hat{u}_2(t-\delta))$ using (43) and (44).

Let $N(\cdot,\cdot)$ be a Poisson random measure generated by a compound Poisson process $\sum_{k=1}^{N(t)} Z_i$, where $N(t)$ is a Poisson process with intensity θ and $\{Z_i\}$ are independent and identically distributed random variables with $Z_i \sim N(0,\sigma^2)$. Then the corresponding Lévy measure is given by $\nu(\mathrm{d}z) = \theta\phi(z)\mathrm{d}z$ with $\phi(z) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{z^2}{2\sigma^2}\right\}$. In this situation, the system (1) is written as

$$
\begin{aligned}
x(t) =& \xi(\delta) + \int_{\delta}^{T} [Ax(s) + B_1u_1(s) + B_2u_2(s-\delta)]\,\mathrm{d}s + \int_{\delta}^{T}\left[\bar{A}x(s) + \bar{B}_1u_1(s) + \bar{B}_2u_2(s-\delta)\right]\mathrm{d}W(s)\\
&+ \sum_{i=1}^{N(t)-N(\delta)} \left[\breve{A}(Z_i)x(s_i^-) + \breve{B}_1(Z_i)u_1(s_i^-) + \breve{B}_2(Z_i)u_2((s_i-\delta)^-)\right],
\end{aligned}
$$

where $\{s_i\}_{i=1}^{N(t)}$ are the arrival times at which jumps happen.

Consider the system (1) and the cost functional (2) with the following parameters:

$$
\begin{aligned}
&A = 0.5, \quad B_1 = 0.15, \quad B_2 = 0.17, \quad \bar{A} = 0.5, \quad \bar{B}_1 = 0.21, \quad \bar{B}_2 = 0.22, \quad \xi(t) = 1, \quad \mu(t) = 0,\\
&\breve{A}(Z_i) = \upsilon_0 Z_i, \quad \breve{B}_1(Z_i) = \upsilon_1 Z_i, \quad \breve{B}_2(Z_i) = \upsilon_2 Z_i, \quad \upsilon_0 = \upsilon_1 = \upsilon_2 = 0.2, \quad \theta = 1, \quad \sigma = 2,\\
&Q_1 = Q_2 = 0.1, \quad R_1^1 = R_1^2 = R_2^1 = R_2^2 = 0.1, \quad G_1 = 1, \quad G_2 = 1.2, \quad \delta = 0.1, \quad T = 5.
\end{aligned}
$$

Let $\Upsilon_{12} = \begin{pmatrix}\upsilon_1 & \upsilon_2\end{pmatrix}$, $\Upsilon_{24} = \begin{pmatrix}\upsilon_1 & \upsilon_2 & 0 & 0\\ 0 & 0 & \upsilon_1 & \upsilon_2\end{pmatrix}$, $\Upsilon_{22} = \begin{pmatrix}\upsilon_0 & 0\\ 0 & \upsilon_0\end{pmatrix}$. Then we have

$$
\int_{\mathbb{R}_0} \breve{\mathbf{B}}^{\top}(z)\mathbb{P}(t)\breve{B}(z)\nu(\mathrm{d}z) = \Upsilon_{24}^{\top}\mathbb{P}(t)\Upsilon_{12}\int_{\mathbb{R}_0} z^2\nu(\mathrm{d}z) = \Upsilon_{24}^{\top}\mathbb{P}(t)\Upsilon_{12}\theta\sigma^2,
$$

which gives rise to

$$
\Psi(t) = \mathbf{M}^{\top}\left(\mathbf{RM} + \bar{\mathbf{B}}^{\top}\mathbb{P}(t)\bar{B} + \Upsilon_{24}^{\top}\mathbb{P}(t)\Upsilon_{12}\theta\sigma^2\right).
$$

Similarly, we obtain

$$
\begin{aligned}
\Psi_1(t) &= \mathbf{L}^{\top}\left(\mathbf{RL} + \bar{\mathbf{B}}^{\top}\mathbb{P}(t)\bar{B}_1 + \Upsilon_{24}^{\top}\mathbb{P}(t)\upsilon_1\theta\sigma^2\right), \quad \lambda(t) = \mathbf{B}^{\top}\mathbb{P}(t) + \bar{\mathbf{B}}^{\top}\mathbb{P}(t)\bar{A} + \Upsilon_{24}^{\top}\mathbb{P}(t)\upsilon_0\theta\sigma^2,\\
\phi(t) &= \bar{\mathbf{A}}^{\top}\mathbb{P}(t)\bar{B} + \Upsilon_{22}^{\top}\mathbb{P}(t)\Upsilon_{12}\theta\sigma^2 + \mathbb{P}(t)B, \quad \phi_1(t) = \bar{\mathbf{A}}^{\top}\mathbb{P}(t)\bar{B}_1 + \Upsilon_{22}^{\top}\mathbb{P}(t)\upsilon_1\theta\sigma^2 + \mathbb{P}(t)B_1.
\end{aligned}
$$

Applying Euler-Maruyama method with $\Delta t = 0.1$, we provide the numerical solutions of $\mathbb{P}(t) = \begin{pmatrix}\mathbb{P}_1(t) & \mathbb{P}_2(t)\end{pmatrix}^{\top}$, the Nash equilibrium $(\hat{u}_1(t), \hat{u}_2(t-\delta))$ and the optimal state $\hat{x}(t)$, shown in Figures 1–3.

5 Conclusion

We have examined a two-player non-zero-sum LQ SDG with jump-diffusions under asymmetric information. The information asymmetry is triggered by the assumption that, at time t, Player 1's strategy $u_1(t)$ is measurable to $\mathcal{F}_t$ while Player 2's action $u_2(t-\delta)$ incorporates a time delay and is restricted to be $\mathcal{F}_{t-\delta}$-measurable. By applying the maximum principles, we reduced the problem of finding the open-loop Nash equilibria to solving a system of FBSDDEs (6). The different measurabilities of the strategies from the players pose a considerable challenge in finding explicit Nash equilibrium solutions. We addressed this difficulty through extending the orthogonal decomposition and reorganization technique, which allows us to construct equivalent auxiliary FBSDDEs (14) with mutually orthogonal strategies. Under the assumption of a unique Nash equilibrium, we derived explicit solutions to the auxiliary FBSDDEs using the generalized Riccati equation (17)–(25), ultimately obtaining an explicit form of the open-loop Nash equilibrium. It is worth noting that several issues remain open: (i) rigorous solvability conditions for the

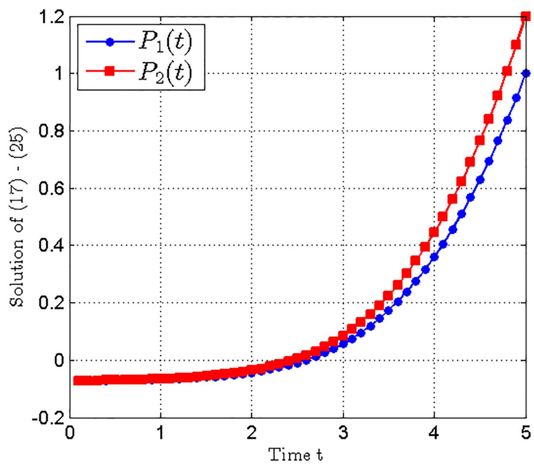

Figure 1 (Color online) Numerical solution of $\mathbb{P}(t)$.

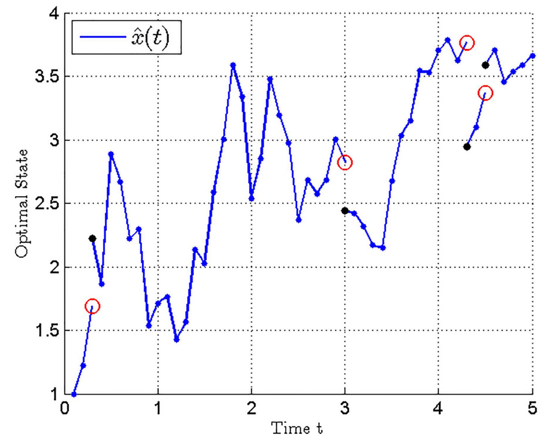

Figure 2 (Color online) One sample path of optimal state $\hat{x}(t)$.

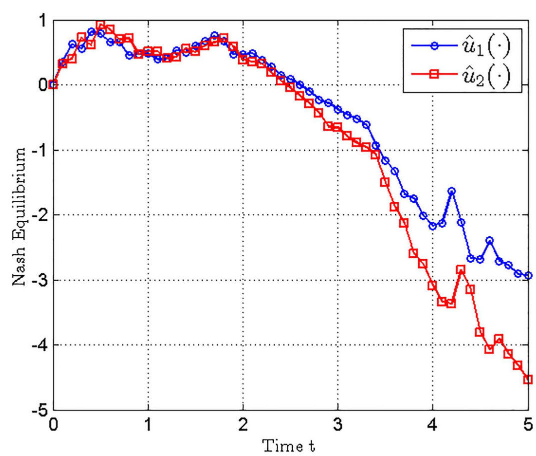

Figure 3 (Color online) One sample path of Nash equilibrium $(\hat{u}_1(t), \hat{u}_2(t-\delta))$.

coupled FBSDDEs deserve dedicated analysis, particularly given the jump-diffusion setting and asymmetric information structure; (ii) the solvability of the generalized Riccati equation (17)–(25) requires further theoretical investigation; (iii) the restriction to the time interval $[\delta, T]$ rather than $[0, T]$ simplifies the analysis, as Player 2's strategy is predetermined on $[0, \delta)$. As shown in [22] for delayed LQ control problems, the solutions exhibit fundamentally different forms on these intervals. These issues will be investigated in our subsequent work.

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References

1 Sun J, Yong J. Stochastic Linear-Quadratic Optimal Control Theory: Differential Games and Mean-field Problems. Cham: Springer, 2020
2 Sun J, Yong J. Linear quadratic stochastic differential games: open-loop and closed-loop saddle points. SIAM J Control Optim, 2014, 52: 4082–4121
3 Sun J. Two-person zero-sum stochastic linear-quadratic differential games. SIAM J Control Optim, 2021, 59: 1804–1829
4 Li X, Shi J, Yong J. Mean-field linear-quadratic stochastic differential games in an infinite horizon. ESAIM-COCV, 2021, 27: 81
5 Shi J, Wang G, Xiong J. Leader-follower stochastic differential game with asymmetric information and applications. Automatica, 2016, 63: 60–73
6 Wang G, Xiao H, Xiong J. A kind of LQ non-zero sum differential game of backward stochastic differential equation with asymmetric information. Automatica, 2018, 97: 346–352
7 Li M, Wu Z. Linear-quadratic non-zero sum differential game for mean-field stochastic systems with asymmetric information. J Math Anal Appl, 2021, 504: 125315
8 Shi J, Wang G, Xiong J. Stochastic linear quadratic stackelberg differential game with overlapping information. ESAIM–COCV, 2020, 26: 83
9 Wu S. Linear-quadratic non-zero sum backward stochastic differential game with overlapping information. IEEE Trans Automat Contr, 2023, 68: 1800–1806
10 Yong J, Zhou X Y. Stochastic Controls: Hamiltonian Systems and HJB Equations. Berlin: Springer, 1999

11 Chen L, Wu Z. Maximum principle for the stochastic optimal control problem with delay and application. Automatica, 2010, 46: 1074–1080
12 Øksendal B, Sulem A. Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. SIAM J Control Optim, 2009, 48: 2945–2976
13 Shen Y, Meng Q, Shi P. Maximum principle for mean-field jump-diffusion stochastic delay differential equations and its application to finance. Automatica, 2014, 50: 1565–1579
14 Wang S, Wu Z. Stochastic maximum principle for optimal control problems of forward-backward delay systems involving impulse controls. J Syst Sci Complex, 2017, 30: 280–306
15 Han Y, Li Y. Stochastic maximum principle for control systems with time-varying delay. Syst Control Lett, 2024, 191: 105864
16 Meng W, Shi J, Wang T, et al. A general maximum principle for optimal control of stochastic differential delay systems. SIAM J Control Optim, 2024, 63: 175–205
17 Wu S, Wang G. Optimal control problem of backward stochastic differential delay equation under partial information. Syst Control Lett, 2015, 82: 71–78
18 Li N, Wang G, Wu Z. Linear-quadratic optimal control for time-delay stochastic system with recursive utility under full and partial information. Automatica, 2020, 121: 109169
19 Zhang H, Li L, Xu J, et al. Linear quadratic regulation and stabilization of discrete-time systems with delay and multiplicative noise. IEEE Trans Automat Contr, 2015, 60: 2599–2613
20 Zhang H, Xu J. Control for Itô stochastic systems with input delay. IEEE Trans Automat Contr, 2017, 62: 350–365
21 Wang H, Zhang H, Li L, et al. LQR and stabilization for discrete-time systems with multiplicative noises and input delays. IEEE Trans Automat Contr, 2024, 69: 3515–3530
22 Wang H, Zhang H, Xie L. Optimal control and stabilization for Itô systems with input delay. J Syst Sci Complex, 2021, 34: 1895–1926
23 An T T K, Øksendal B. Maximum principle for stochastic differential games with partial information. J Optim Theor Appl, 2008, 139: 463–483
24 Wang Y, Wang L, Teo K L. Necessary and sufficient optimality conditions for regular-singular stochastic differential games with asymmetric information. J Optim Theor Appl, 2018, 179: 501–532
25 Chen L, Yu Z. Maximum principle for nonzero-sum stochastic differential game with delays. IEEE Trans Automat Contr, 2015, 60: 1422–1426
26 Chen L, Wu Z. A type of general forward-backward stochastic differential equations and applications. Chin Ann Math Ser B, 2011, 32: 279–292
27 Huang J, Li N. Linear-quadratic mean-field game for stochastic delayed systems. IEEE Trans Automat Contr, 2018, 63: 2722–2729
28 Xu J J, Shi J T, Zhang H S. A leader-follower stochastic linear quadratic differential game with time delay. Sci China Inf Sci, 2018, 61: 112202
29 Merton R C. Option pricing when underlying stock returns are discontinuous. J Financial Economics, 1976, 3: 125–144
30 Øksendal B, Sulèm A. Applied Stochastic Control of Jump Diffusions. New York: Springer-Verlag, 2007
31 Li N, Xiong J, Yu Z. Linear-quadratic generalized Stackelberg games with jump-diffusion processes and related forward-backward stochastic differential equations. Sci China Math, 2021, 64: 2091–2116
32 Yu Z, Zhang B, Zhang F. One kind of linear-quadratic zero-sum stochastic differential game with jumps. Int J Control, 2022, 95: 1470–1481
33 Wang G, Wang W. Indefinite linear-quadratic optimal control of mean-field stochastic differential equation with jump diffusion: an equivalent cost functional method. IEEE Trans Automat Contr, 2024, 69: 7449–7462
34 Saito T, Takahashi A. Stochastic differential game in high frequency market. Automatica, 2019, 104: 111–125

Appendix A Proof of Lemma 5

Applying Itô formula to $\mathbb{P}(t)\hat{x}(t)$, we have

$$\begin{aligned}\mathrm{d}\left(\mathbb{P}(t)\hat{x}(t)\right) = &\left[\dot{\mathbb{P}}(t)\hat{x}(t) + \mathbb{P}(t)A\hat{x}(t) + \mathbb{P}(t)B\hat{v}^{\delta}(t) + \mathbb{P}(t)B_1\hat{u}_1^{\perp}(t)\right]\mathrm{d}t + \Big[\mathbb{P}(t)\bar{A}\hat{x}(t) + \mathbb{P}(t)\bar{B}\hat{v}^{\delta}(t)\\ &+\mathbb{P}(t)\bar{B}_1\hat{u}_1^{\perp}(t)\Big]\mathrm{d}W(t) + \int_{\mathbb{R}_0^n}\left[\mathbb{P}(t)\breve{A}(z)\hat{x}(t) + \mathbb{P}(t)\breve{B}(z)\hat{v}^{\delta}(t) + \mathbb{P}(t)\breve{B}_1(z)\hat{u}_1^{\perp}(t)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned} \tag{A1}$$

In view of (14) and (A1), Eq. (26) implies

$$\mathrm{d}\mathbb{Y}(t) = \mathrm{d}\hat{\mathbf{p}}(t) - \mathrm{d}\left(\mathbb{P}(t)\hat{x}(t)\right) = -\alpha(t)\mathrm{d}t + \mathbb{Z}_1(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0^n}\mathbb{Z}_2(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \tag{A2}$$

where

$$\alpha(t) := \mathbf{Q}\mathbb{I}_n\hat{x}(t) + \mathbf{A}^{\mathsf{T}}\hat{\mathbf{p}}(t) + \bar{\mathbf{A}}^{\mathsf{T}}\hat{\mathbf{q}}(t) + \int_{\mathbb{R}_0^n}\breve{\mathbf{A}}^{\mathsf{T}}(z)\hat{\mathbf{r}}(t,z)\nu(\mathrm{d}z) + \dot{\mathbb{P}}(t)\hat{x}(t) + \mathbb{P}(t)A\hat{x}(t) + \mathbb{P}(t)B\hat{v}^{\delta}(t) + \mathbb{P}(t)B_1\hat{u}_1^{\perp}(t), \tag{A3}$$

and $\mathbb{Z}_1(t)$, $\mathbb{Z}_2(t,z)$ are given by (27) and (28), respectively. Substituting (17), (26), (27) and (28) into (A3), we have

$$\alpha(t) = \mathbf{A}^{\mathsf{T}}\mathbb{Y}(t) + \bar{\mathbf{A}}^{\mathsf{T}}\mathbb{Z}_1(t) + \int_{\mathbb{R}_0^n}\breve{\mathbf{A}}^{\mathsf{T}}(z)\mathbb{Z}_2(t,z)\nu(\mathrm{d}z) + \Pi(t,t+\delta)\hat{x}(t) + \Pi_1(t,t)\hat{x}(t) + \phi(t)\hat{v}^{\delta}(t) + \phi_1(t)\hat{u}_1^{\perp}(t), \tag{A4}$$

and hence it follows from (A2) that

$$\begin{aligned}\mathrm{d}\mathbb{Y}(t) = -&\Bigg[\mathbf{A}^{\mathsf{T}}\mathbb{Y}(t) + \bar{\mathbf{A}}^{\mathsf{T}}\mathbb{Z}_1(t) + \int_{\mathbb{R}_0^n}\breve{\mathbf{A}}^{\mathsf{T}}(z)\mathbb{Z}_2(t,z)\nu(\mathrm{d}z) + \Pi(t,t+\delta)\hat{x}(t) + \Pi_1(t,t)\hat{x}(t)\\ &+ \phi(t)\hat{v}^{\delta}(t) + \phi_1(t)\hat{u}_1^{\perp}(t)\Bigg]\mathrm{d}t + \mathbb{Z}_1(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0^n}\mathbb{Z}_2(t,z)\tilde{N}\left(\mathrm{d}t,\mathrm{d}z\right).\end{aligned} \tag{A5}$$

On the other hand, substituting (17), (26), (27) and (28) into the stationary condition (15), we obtain

$$\hat{v}^\delta(t) = -\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \Psi^{-1}(t)\mathbf{M}^\top\mathbb{E}\left[\Xi(t)\middle|\mathcal{F}_{(t-\delta)-}\right], \tag{A6}$$

due to the fact that $\Psi(t)$ is invertible and $\mathbb{E}\left[\hat{u}_1^\perp(t)\middle|\mathcal{F}_{(t-\delta)-}\right] = 0$. Similarly, combining (17), (26), (27) and (28), the stationary condition (16) gives rise to

$$\hat{u}_1^\perp(t) = -\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right] - \Psi_1^{-1}(t)\mathbf{L}^\top\mathbb{E}\left[\Xi(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right], \tag{A7}$$

since $\Psi_1(t)$ is invertible and $\mathbb{E}\left[\hat{v}^\delta(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right] = 0$. Consequently, combining (14) with (A6), (A7) and (A5), we have (29).

Conversely, by sequentially applying (A6), (A7), (26), (27) and (28), we can rewrite (29) as (14). As a result, FBSDDEs (14) admit a unique solution if and only if FBSDDEs (29) admit a unique solution.

Appendix B Proof of Lemma 6

By the Fubini theorem and the property of conditional expectation, we get

$$\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(t-\delta)-}\right] = \int_t^{t+\delta}\Pi(t,\theta)\mathbb{E}\left[\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\middle|\mathcal{F}_{(t-\delta)-}\right]\mathrm{d}\theta,$$

which results in (31). Next, for $\theta \in (t, t+\delta)$, we consider $\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]$ and obtain

$$\begin{aligned}
\mathbb{E}\left[\eta(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right] =& \mathbb{E}\left[\int_{t-\delta}^t \Pi(t,s+\delta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_s\right]\mathrm{d}s\middle|\mathcal{F}_{(\theta-\delta)-}\right]\\
=& \int_{t-\delta}^{\theta-\delta}\Pi(t,s+\delta)\mathbb{E}\left[\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{s-}\right]\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}s + \int_{\theta-\delta}^t\Pi(t,s+\delta)\mathbb{E}\left[\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{s-}\right]\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}s\\
=& \int_{t-\delta}^{\theta-\delta}\Pi(t,s+\delta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{s-}\right]\mathrm{d}s + \int_{\theta-\delta}^t\Pi(t,s+\delta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}s,
\end{aligned}$$

so that Eq. (32) follows.

Appendix C Proof of Theorem 1

Let $\mathbb{Y}(t)$, $\mathbb{Z}_1(t)$ and $\mathbb{Z}_2(t,z)$ be defined by (33) and (34), respectively. Then Eq. (35) can be directly written as the forward SDE in FBSDDEs (29). Therefore, it suffices to verify that the tuple $(\hat{x}(t), \mathbb{Y}(t), \mathbb{Z}_1(t), \mathbb{Z}_2(t,z))$ satisfies the backward SDE in (29). For this purpose, we start with the forward SDE in (29). With the help of (34), it is simplified to

$$\begin{aligned}
\mathrm{d}\hat{x}(t) =& \Big\{A\hat{x}(t) - B\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - B_1\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\
&-B\Psi^{-1}(t)\mathbf{M}^\top\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - B_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}t\\
&+\Big\{\bar{A}\hat{x}(t) - \bar{B}\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \bar{B}_1\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\
&-\bar{B}\Psi^{-1}(t)\mathbf{M}^\top\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \bar{B}_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\mathrm{d}W(t)\\
&+\int_{\mathbb{R}_0^n}\Big\{\breve{A}(z)\hat{x}(t) - \breve{B}(z)\Psi^{-1}(t)\mathbf{M}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \breve{B}_1(z)\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\
&-\breve{B}(z)\Psi^{-1}(t)\mathbf{M}^\top\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] - \breve{B}_1(z)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\Big\}\tilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}$$

For $\theta \in [t, t+\delta]$, taking the conditional expectation of $\hat{x}(t)$ with respect to $\mathcal{F}_{(\theta-\delta)-}$ and then taking the derivative of $\mathbb{E}\left[x(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]$ with respect to t yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right] =& A\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right] - B\Psi^{-1}(t)\mathbf{M}^\top\left(\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] + \mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\right)\\
&- B_1\Psi_1^{-1}(t)\mathbf{L}^\top\left(\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^{(\theta-\delta)-}_{(t-\delta)-}\right] + \mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}^{(\theta-\delta)-}_{(t-\delta)-}\right]\right).
\end{aligned} \tag{C1}$$

Now we consider $\mathbb{Y}(t)$ given by (33). For $t \in [\delta, T]$, Eq. (31) ensures that the conditional expectation of $\mathbb{Y}(t)$ with respect to $\mathcal{F}_{(t-\delta)-}$ is represented by

$$\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}_{(t-\delta)-}\right] = -\left(\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]. \tag{C2}$$

Then computing the derivative of $\mathbb{Y}(t)$ provides

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}(t) =& -\Pi(t,t+\delta)\hat{x}(t) + \Pi(t,t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\\
&-\int_t^{t+\delta}\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta - \int_t^{t+\delta}\Pi(t,\theta)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta.
\end{aligned} \tag{C3}$$

Plugging (C1), (C2) and (19) into (C3), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}(t) =& -\Pi(t,t+\delta)\hat{x}(t)+\Pi(t,t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]+\int_t^{t+\delta}\left\{\mathbf{A}^\top-\phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\right\}\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta\\ &+\left(\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)B\Psi^{-1}(t)\mathbf{M}^\top\left(\lambda(t)-\mathbf{B}^\top\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\\ &-\left(\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)B_1\Psi_1^{-1}(t)\mathbf{L}^\top\left(\lambda(t)-\mathbf{B}^\top\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right],\end{aligned}\tag{C4}$$

since

$$\begin{aligned}&\int_t^{t+\delta}\Pi(t,\theta)B_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\int_t^{\theta}\Pi(t,s)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(s-\delta)-}\right]\mathrm{d}s\mathrm{d}\theta\\ &\quad=\int_t^{t+\delta}\left(\int_\theta^{t+\delta}\Pi(t,s)\mathrm{d}s\right)B_1\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta.\end{aligned}$$

In conjunction with (20), Eq. (C4) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}(t) =& -\Pi(t,t+\delta)\hat{x}(t)+\phi(t)\Psi^{-1}(t)\mathbf{M}^\top\left(\lambda(t)-\mathbf{B}^\top\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\\ &-\phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]+\phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\\ &+\left\{\mathbf{A}^\top-\phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\right\}\int_t^{t+\delta}\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta.\end{aligned}\tag{C5}$$

Therefore, by (33) and (31), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{Y}(t) =& -\mathbf{A}^\top\mathbb{Y}(t)-\Pi(t,t+\delta)\hat{x}(t)-\Pi_1(t,t)\hat{x}(t)+\phi(t)\Psi^{-1}(t)\mathbf{M}^\top\left(\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]+\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\right)\\ &+\phi_1(t)\Psi_1^{-1}(t)\mathbf{L}^\top\left(\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]+\mathbf{B}^\top\mathbb{E}\left[\mathbb{Y}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]\right).\end{aligned}\tag{C6}$$

Since $\mathbb{Z}_1(t)=\mathbb{Z}_2(t,z)=0$ almost surely, Eq. (C6) can be further expressed as the backward SDE in (29). Consequently, we have shown that $(\hat{x}(t),\mathbb{Y}(t),\mathbb{Z}_1(t),\mathbb{Z}_2(t,z))$ is a solution of FBSDDEs (29).

The uniqueness of $(\hat{x}(t),\mathbb{Y}(t),\mathbb{Z}_1(t),\mathbb{Z}_2(t,z))$ follows from the fact that the uniquely solvable nature of Problem-LQSDG-Eq is equivalent to the existence and uniqueness of the solution to FBSDDEs (29), as established in Lemmas 2 and 5.

Appendix D Proof of Theorem 2

Lemma 3, combined with Lemma 4, implies the unique solvability of Problem-LQSDG-Eq, which is equivalent to the existence of a unique solution for FBSDDEs (14) with the stationary conditions (15) and (16). Suppose $(\hat{x}(\cdot),\hat{\mathbf{p}}(\cdot),\hat{\mathbf{q}}(\cdot),\hat{\mathbf{r}}(\cdot,\cdot)$, $\hat{v}^\delta(\cdot)$, $\hat{u}_1^\perp(\cdot))$ satisfy FBSDDEs (14). Then, plugging (36) into the stationary condition (15), we get

$$\hat{v}^\delta(t)=-\Psi^{-1}(t)\mathbf{M}^\top\left(\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]-\mathbf{B}^\top\mathbb{E}\left[\int_t^{t+\delta}\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta\middle|\mathcal{F}_{(t-\delta)-}\right]\right),\tag{D1}$$

since the matrix $\Psi(t)$ is invertible for $t\in[\delta,T]$ and $\mathbb{E}\left[\hat{u}_1^\perp(t)\middle|\mathcal{F}_{(t-\delta)-}\right]=0$. Then Eq. (D1) together with (31) gives (40). Therefore, Eqs. (37) and (38) follow from the definition of $\hat{v}^\delta(t)$ given by (13).

Similarly, the stationary condition (16), combined with (36), shows that

$$\begin{aligned}\hat{u}_1^\perp(t)=&-\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]+\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\mathbb{E}\left[\int_t^{t+\delta}\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta\middle|\mathcal{F}^t_{(t-\delta)-}\right]\\ =&-\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\hat{x}(t)+\Psi_1^{-1}(t)\mathbf{L}^\top\lambda(t)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\\ &+\Psi_1^{-1}(t)\mathbf{L}^\top\mathbf{B}^\top\left(\int_t^{t+\delta}\Pi(t,\theta)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(\theta-\delta)-}\right]\mathrm{d}\theta-\left(\int_t^{t+\delta}\Pi(t,\theta)\mathrm{d}\theta\right)\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]\right),\end{aligned}$$

which implies (39). Here we used the assumption $\Psi_1(t)$ is invertible for $t\in[\delta,T]$ and the fact $\mathbb{E}\left[\hat{v}^\delta(t)\middle|\mathcal{F}^t_{(t-\delta)-}\right]=0$ as well.

Moreover, let $\hat{x}(t)$ be described by the forward SDE in (14). Then we have

$$\mathrm{d}\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]=\left\{A\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{(t-\delta)-}\right]+B\hat{v}^\delta(t)\right\}\mathrm{d}t,\tag{D2}$$

with $\mathbb{E}\left[\hat{x}(t-\delta)\middle|\mathcal{F}_{(t-\delta)-}\right]=\hat{x}(t-\delta)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{\theta-\delta}\right]=A\mathbb{E}\left[\hat{x}(t)\middle|\mathcal{F}_{\theta-\delta}\right]+B\hat{v}^\delta(t)+B_1\mathbb{E}\left[\hat{u}_1^\perp(s)\middle|\mathcal{F}_{\theta-\delta}\right],\tag{D3}$$

with $\mathbb{E}\left[\hat{x}(\theta-\delta)\middle|\mathcal{F}_{\theta-\delta}\right]=\hat{x}(\theta-\delta)$. Thus Eqs. (42) and (41) follow from (D2) and (D3), respectively.