SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

November 2025, Vol. 68, Iss. 11, 210208:1–210208:15
https://doi.org/10.1007/s11432-025-4659-x

Special Topic: Mean-Field Game and Control of Large Population Systems: From Theory to Practice

Sufficient stochastic maximum principle for mean-field control problems with regime switching in an infinite horizon

Na LI[1], Siyu LV[2*], Zhen WU[3] & Jie XIONG[4]

[1]School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
[2]School of Mathematics, Southeast University, Nanjing 211189, China
[3]School of Mathematics, Shandong University, Jinan 250100, China
[4]Department of Mathematics and SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen 518055, China

Received 14 February 2025/Revised 12 July 2025/Accepted 28 October 2025/Published online 5 November 2025

Abstract This paper is concerned with an optimal control problem for stochastic system with regime switching and mean-field interactions in an infinite horizon. The discounted framework is adopted to ensure the stability of the state equation and the well-posedness of the cost functional. By choosing an appropriate discount factor, we first, as a preliminary, establish the global solvability for infinite horizon conditional mean-field (forward and backward) stochastic differential equations with Markov chains and the asymptotic property of their solutions when time goes to infinity. Then, we prove a sufficient stochastic maximum principle for the infinite horizon optimal control problem by means of a dual approach under some convexity condition of the associated Hamiltonian function. Finally, the maximum principle is applied to solve a cash flow management problem of an insurance firm, which turns out to be a linear quadratic optimal control problem. An explicit optimal premium policy and the minimum cost are obtained based on two algebraic Riccati equations and an additional linear equation. Numerical experiments are reported to illustrate the theoretical results.

Keywords sufficient maximum principle, infinite horizon problem, Markov chain, mean-field interaction, algebraic Riccati equation

Citation Li N, Lv S Y, Wu Z, et al. Sufficient stochastic maximum principle for mean-field control problems with regime switching in an infinite horizon. Sci China Inf Sci, 2025, 68(11): 210208, https://doi.org/10.1007/s11432-025-4659-x

1 Introduction

It is well known that the maximum principle is one of the fundamental approaches in optimal control theory, which originated from the pioneering work by Pontryagin et al. [1] on deterministic systems. The maximum principle provides a set of necessary conditions in the sense that any optimal control together with the corresponding optimal state trajectory must solve the so-called Hamiltonian system plus a maximum condition of the Hamiltonian function. Bensoussan [2] and Peng [3] first studied maximum principles for stochastic systems, where they utilized the convex variation and spike variation techniques to establish the stochastic maximum principles in local and general forms, respectively. Then, various versions of stochastic maximum principles for different problems, settings, or contexts were obtained; see [4–6]. On the other hand, there has been an increasing interest in finding the sufficient conditions for optimal control problems since the middle of 1990s, which can be used to identify or construct directly an optimal control. Zhou [7] derived the first sufficient stochastic maximum principle with certain convexity conditions on control domain and Hamiltonian function. Following this work, many extensions and generalizations were motivated and conducted; see [8–10]. It is worth mentioning that most (no matter necessary or sufficient) stochastic maximum principles focus on finite time horizon problems, while there are very few works on infinite time horizon problems.

Historically, mean-field stochastic differential equations (SDEs) can be traced back to the so-called McKean-Vlasov SDEs, which were first suggested and studied by Kac [11] and McKean [12]. In the dynamics of a mean-field SDE, one replaces the interactions of all the particles by their average or mean

* Corresponding author (email: lvsiyu@seu.edu.cn)

to reduce the complexity involved. In the last decade, optimal control problems for mean-field SDEs have attracted considerable attention from the control and systems community. Note that the adjoint equation of a controlled mean-field SDE is a mean-field backward stochastic differential equation (BSDE), so it is not until Buchdahn et al. [13,14] introduced the mean-field BSDEs that the stochastic maximum principle for mean-field optimal control problems has become a popular topic; see, for example, [15–20]. Another feature of this paper is the utilization of regime switching model, which exhibits a hybrid characteristic in the sense that continuous and discrete dynamics coexist. Compared with traditional diffusion models, regime switching model has two distinct advantages: one is the capacity to depict the discrete events that have greater impact on long-term system behavior, the other one is the tractability that enables advanced mathematical analysis and feasible numerical schemes to be developed. For these reasons, regime switching model enjoys a wide range of applications; see, among others, [21–26].

It is somewhat unexpectedly that the study of mean-field optimal control problems with regime switching, which blends the above two research areas of recent interests together, is still in an early stage. The main obstacle may be the lack of a proper formulation for such kind of problems. Recently, Nyugen et al. [27] established the theory of mean-field SDEs with regime switching, where, quite differently, the mean-field term is characterized as the conditional expectation of the state process given the history of the underlying Markov chain; in this sense, the Markov chain serves as a common noise. This work paves the way for treating complex dynamic systems with mean-field interactions and regime switching. Based on this finding, Nyugen et al. [28, 29] developed stochastic maximum principles for mean-field optimal control problems with regime switching in local and general forms, respectively. Gutiérrez et al. [30] investigated the well-posedness of mean-field forward-backward stochastic differential equations (FBSDEs) with Markov chains. They also solved a linear quadratic (LQ) nonzero-sum stochastic differential games and obtained an open-loop Nash equilibrium via the FBSDE theory; see also Lv et al. [31] for a closed-loop Nash equilibrium via Riccati equations. Lv et al. [32] studied an LQ leader-follower stochastic differential game and derived an open-loop Stackelberg equilibrium and its state feedback representation. Jian et al. [33] considered the convergence rate of N-player linear quadratic Gaussian game with a Markov chain common noise towards its asymptotic mean-field game. Note that all the above studies are, once again, concerned with the problems on a finite time horizon.

As a further step along this research direction and motivated by many infinite horizon optimization problems in finance, economics, and management (see [34–36]), in this paper we consider the mean-field optimal control problem with regime switching on an infinite horizon and in a discounted framework. An appropriate discounted factor is introduced to treat the issue of system stability, and our objective is to establish a sufficient stochastic maximum principle for optimal controls. To this end, we first, as a preliminary and also of particular interest in their own rights, establish the global solvability of infinite horizon mean-field SDEs and BSDEs with Markov chains (as state equation and adjoint equation, respectively) and asymptotic property of their solutions when time goes to infinity, which are crucial to the well-posedness of the corresponding Hamiltonian system and the proof of our maximum principle. Then, the sufficient stochastic maximum principle, is established based on a dual approach under some certain convexity condition on the associated Hamiltonian function. As an application, we apply the maximum principle to solve a cash flow management problem of an insurance firm. It is essentially an LQ problem, and two algebraic Riccati equations (AREs) and an additional linear equation will arise in this procedure, based on which we are able to construct an explicit feedback optimal control or optimal premium policy. Numerical examples are also conducted to illustrate the theoretical results.

The rest of this paper is organized as follows. Section 2 formulates the optimal control problem. Subsection 3.1 establishes the solvability of infinite horizon mean-field SDEs and BSDEs with Markov chains. Subsection 3.2 proves the sufficient stochastic maximum principle. Section 4 applies the maximum principle to solve a cash flow management problem and provides some numerical tests. Finally, Section 5 concludes the paper with some further remarks.

2 Problem formulation

Let R^n be the n-dimensional Euclidean space with Euclidean norm $|\cdot|$ and Euclidean inner product $\langle\cdot,\cdot\rangle$. Let $R^{n\times k}$ be the space of all $(n\times k)$ matrices. f_x and f_{xx} denote the first and second order derivatives of a function f with respect to x, respectively.

Let $(\Omega,\mathcal{F},P)$ be a fixed probability space on which a 1-dimensional standard Brownian motion W_t,

$t \geqslant 0$, and a Markov chain α_t, $t \geqslant 0$, are defined. The Markov chain α takes values in a finite state space $\mathcal{M} = \{1, \ldots, m\}$. Let $Q = (\lambda_{ij})_{i,j \in \mathcal{M}}$ be the generator of $\alpha(\cdot)$ with $\lambda_{ij} \geqslant 0$ for $i \neq j$ and $\sum_{j \in \mathcal{M}} \lambda_{ij} = 0$ for each $i \in \mathcal{M}$. Assume that W and α are independent. For $t \geqslant 0$, we define $\mathcal{F}_t^W = \sigma\{W(s) : 0 \leqslant s \leqslant t\}$, $\mathcal{F}_t^\alpha = \sigma\{\alpha(s) : 0 \leqslant s \leqslant t\}$, and $\mathcal{F}_t = \sigma\{W(s), \alpha(s) : 0 \leqslant s \leqslant t\}$.

For the Markov chain $\alpha(\cdot)$, associated with each pair $(i, j) \in \mathcal{M} \times \mathcal{M}$ with $i \neq j$, define

$$[M_{ij}](t) = \sum_{0 \leqslant s \leqslant t} 1_{\{\alpha(s-)=i\}} 1_{\{\alpha(s)=j\}}, \quad \langle M_{ij} \rangle(t) = \int_0^t \lambda_{ij} 1_{\{\alpha(s-)=i\}} ds.$$

It follows that the process

$$M_{ij}(t) \doteq [M_{ij}](t) - \langle M_{ij} \rangle(t)$$

is null at origin and is a purely discontinuous and square-integrable martingale with respect to $\mathcal{F}_t^\alpha$ (see [28, 29]). In fact, the processes $[M_{ij}](t)$ and $\langle M_{ij} \rangle(t)$ are the optional and predictable quadratic variations of $M_{ij}(t)$, respectively. For convenience, we define $M_{ii}(t) = [M_{ii}](t) = \langle M_{ii} \rangle(t) = 0$ for each $i \in \mathcal{M}$. From the definition of optional quadratic covariations (see Lipster and Shiryayev [37]), we have the following orthogonality relations:

$$[M_{ij}, W] = 0, \quad [M_{ij}, M_{pq}] = 0$$

for $(i, j) \neq (p, q)$, where $[M_{ij}, W]$ and $[M_{ij}, M_{pq}]$ are the optional quadratic covariations of the pairs (M_{ij}, W) and (M_{ij}, M_{pq}), respectively.

Let $r > 0$ be the discounted factor, which will be determined later. Let $\mathcal{L}_{\mathcal{F}}^{2,r}(0, \infty; R^n)$ be the set of all R^n-valued $\mathcal{F}_t$-adapted processes X on $[0, \infty)$ such that

$$E\left[\int_0^\infty \mathrm{e}^{-rt} |X_t|^2 dt\right] < \infty.$$

Let $\mathcal{K}_{\mathcal{F}}^{2,r}(0, \infty; R^n)$ be the set of all collections of R^n-valued $\mathcal{F}_t$-adapted processes $(K_{ij}(\cdot))_{i,j \in \mathcal{M}}$ on $[0, \infty)$ such that $K_{ii}(t) = 0$ for each $i \in \mathcal{M}$ and

$$\sum_{i,j \in \mathcal{M}} E\left[\int_0^\infty \mathrm{e}^{-rt} |K_{ij}(t)|^2 d[M_{ij}](t)\right] < \infty.$$

For convenience, we also define $K(\cdot) = (K_{ij}(\cdot))_{i,j \in \mathcal{M}}$ and

$$\int_0^t K(s) \bullet dM(s) \doteq \sum_{i,j \in \mathcal{M}} \int_0^t K_{ij}(s) dM_{ij}(s), \quad K(s) \bullet dM(s) \doteq \sum_{i,j \in \mathcal{M}} K_{ij}(s) dM_{ij}(s).$$

Moreover, we define $\mathcal{H}_{\mathcal{F}}^{2,r}(0, \infty; R^n) = (\mathcal{L}_{\mathcal{F}}^{2,r}(0, \infty; R^n))^2 \times \mathcal{K}_{\mathcal{F}}^{2,r}(0, \infty; R^n)$.

In this paper, we consider the following controlled system, which is an infinite horizon conditional mean-field SDE with regime switching:

$$\begin{cases} dX_t = b(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)dt + \sigma(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)dW_t, & t \geqslant 0, \\ X_0 = x \in R^n, & \alpha_0 = i \in \mathcal{M}, \end{cases} \tag{1}$$

where X is the state process with values in R^n, u is the control process with values in a convex subset $U \subset R^k$, and $b, \sigma : R^n \times R^n \times \mathcal{M} \times U \mapsto R^n$ are two given functions. The cost functional is defined as

$$J(x, i; u) = E\int_0^\infty \mathrm{e}^{-rt} f(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)dt, \tag{2}$$

where $f : R^n \times R^n \times \mathcal{M} \times U \mapsto R$ is a given function. We aim to find an optimal control $u^* \in \mathcal{U}_{ad}$ to minimize (2):

$$J(x, i; u^*) = \inf_{u \in \mathcal{U}_{ad}} J(x, i; u),$$

subject to (1), where the admissible control set $\mathcal{U}_{ad}$ is defined to be $\mathcal{L}_{\mathcal{F}}^{2,r}(0, \infty; U)$.

Remark 1. In fact, SDE (1) is obtained as the mean-square limit as $N \to \infty$ of a system of interacting particles in the following form:

$$\begin{cases} dX_t^{l,N} = b\left(X_t^{l,N}, \dfrac{1}{N}\displaystyle\sum_{l=1}^{N} X_t^{l,N}, \alpha_t, u_t^l\right) dt + \sigma\left(X_t^{l,N}, \dfrac{1}{N}\displaystyle\sum_{l=1}^{N} X_t^{l,N}, \alpha_t, u_t^l\right) dW_t^l, \quad t \geqslant 0, \\ X_0^{l,N} = x \in R^n, \quad 1 \leqslant l \leqslant N, \quad \alpha_0 = i \in \mathcal{M}, \end{cases}$$

where $\{W^l\}_{l=1}^N$ is a collection of independent standard Brownian motions and the Markov chain α serves as a common noise for all particles, which leads to the conditional expectation rather than expectation in (1).

Intuitively, since all the particles depend on the history of α, their average and thereby its limit as $N \to \infty$ should also depend on this process. This intuition has been justified by the law of large numbers established by Nguyen et al. [27, Theorem 2.1], which shows that the joint process $(\frac{1}{N}\sum_{l=1}^N X^{l,N}(\cdot), \alpha(\cdot))$ converges weakly to $(\mu_\alpha(\cdot), \alpha(\cdot))$, where $(\mu_\alpha(t), \alpha(t)) \doteq (E[X(t)|\mathcal{F}_t^\alpha], \alpha(t))$ and $X(\cdot)$ is exactly the solution to SDE (1).

On the other hand, there is also another framework to treat mean-field games with many particles and regime switching, which requires to have an idiosyncratic Markov chain for each particle (see Wang and Zhang [38]). Thus, there are as many Markov chains as that of particles and all of these Markov chains are independent. Within this framework, Zhang et al. [39] studied a mean-field control problem including regime switching and Poisson jump. In both [38, 39], it is the expectation $E[X_t]$, rather than the conditional expectation $E[X_t|\mathcal{F}_t^\alpha]$, to represent the mean-field limit of finite population of weakly interacting particles.

In this paper, we impose the following assumptions on the coefficients b, σ, f.

(A1) The functions $b(x, y, i, u)$ and $\sigma(x, y, i, u)$ are continuously differentiable and have linear growth w.r.t. (x, y, u), and the partial derivatives of b and σ w.r.t. x, y are bounded. More precisely, for $\varphi = b, \sigma$, there exists a constant $\kappa_\varphi > 0$ such that

$$\begin{aligned} &|\varphi_x(x, y, i, u)| \leqslant \kappa_\varphi, \quad |\varphi_y(x, y, i, u)| \leqslant \kappa_\varphi, \\ &|\varphi(x, y, i, u)| \leqslant \kappa_\varphi(1 + |x| + |y| + |u|). \end{aligned}$$

(A2) The function $f(x, y, i, u)$ is continuously differentiable w.r.t. (x, y, u), and the partial derivatives of f w.r.t. (x, y) have linear growth in (x, y, u). More precisely, there exists a constant $\kappa_f > 0$ such that

$$|f_x(x, y, i, u)| \leqslant \kappa_f(1 + |x| + |y| + |u|), \quad |f_y(x, y, i, u)| \leqslant \kappa_f(1 + |x| + |y| + |u|).$$

Remark 2. Other than the mean-field formulation adopted in this paper, where the mean-field term is represented by the conditional expectation $E[X_t|\mathcal{F}_t^\alpha]$ of the state process, there is another more general mean-field formulation in the literature, i.e., the mean-field term is represented by the conditional law $P_{X_t}^\alpha$ of X_t given $\mathcal{F}_t^\alpha$; in this connection, we refer to [15, 28, 29, 39] for the former formulation, and to [40–43] for the latter one.

Moreover, if one adopts the former formulation, then $E[X_t|\mathcal{F}_t^\alpha]$ takes values in R^n and the Hamiltonian function $H(x, y, i, u, p, q)$, where the position y stands for the mean-field term, maps $(x, y, i, u, p, q) \in R^n \times R^n \times \mathcal{M} \times U \times R^n \times R^n$ to R. In this case, the derivative of H with respect to y, denoted by H_y, is defined in the usual sense as a function from R^n to R.

On the other hand, if one adopts the latter formulation, then $P_{X_t}^\alpha$ belongs to $\mathcal{P}_2(R^n)$, i.e., the space of all probability measures μ on $(R^n, \mathcal{B}(R^n))$ with finite second moment $\int_{R^n} |x|^2 \mu(dx) < \infty$. In this case, the differentiability of H with respect to y will be defined in the sense of the so-called L-derivative based on the technique of lifting of functions and the notion of Fréchet differentiability; see Carmona and Delarue [41, Chapter 5.2] for details.

The key aim of this paper is to develop the conditional mean-field framework (in the presence of a Markov chain as common noise) recently established by Nguyen et al. [27–29] to the infinite horizon context. In order to display our main idea more clearly, we adopt the former (relatively concise, compared to the latter one) mean-field formulation in this paper.

3 Stochastic maximum principle

3.1 Conditional mean-field SDEs and BSDEs

In this subsection, we prove the global solvability of infinite horizon mean-field SDEs and BSDEs with Markov chains, and the asymptotic behavior of their solutions in the infinity. Note that the mean-field terms in these two equations become conditional expectations due to the inclusion of the Markov chain (as a common noise), and the infinite horizon feature needs to be carefully treated within a discounted framework. For convenience, we define $\widehat{X}_t = E[X_t|\mathcal{F}_t^\alpha]$ for a stochastic process X in this paper. In the sequel, we shall define by C a generic constant, which may vary from line to line.

Theorem 1. Let (A1) hold and $r > 4\kappa_b + 4\kappa_\sigma^2$. For any $u \in \mathcal{U}_{ad}$, SDE (1) admits a unique strong solution $X \in \mathcal{L}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$. Moreover, for any $\varepsilon \in (0,(r-4\kappa_\sigma^2-4\kappa_b)/3)$, we have

$$(r-4\kappa_\sigma^2-4\kappa_b-3\varepsilon)E\bigg[\int_0^\infty \mathrm{e}^{-rs}|X_s|^2 ds\bigg] \leqslant |x|^2 + \frac{C}{\varepsilon}E\bigg[\int_0^\infty \mathrm{e}^{-rs}(1+|u_s|^2)ds\bigg], \tag{3}$$

and

$$\lim_{T\to\infty} E[\mathrm{e}^{-rT}|X_T|^2] = 0. \tag{4}$$

Proof. By classical SDE theory, under (A1), SDE (1) admits a unique solution X on $[0,\infty)$; see Wei and Yu [44]. In what follows, let us show that such a solution X belongs to the space $\mathcal{L}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$ (i.e., the estimate (3)) and its asymptotic behavior at infinity (i.e., the limit (4)).

For any $T>0$, applying Itô's formula to $\mathrm{e}^{-rs}|X_s|^2$ over $[t,T]$, we have

$$\begin{aligned}\mathrm{e}^{-rT}|X_T|^2 =& \mathrm{e}^{-rt}|X_t|^2 + \int_t^T \mathrm{e}^{-rs}[-r|X_s|^2 + 2\langle X_s, b(X_s,\widehat{X}_s,\alpha_s,u_s)\rangle + |\sigma(X_s,\widehat{X}_s,\alpha_s,u_s)|^2]ds \\ &+ 2\int_t^T \mathrm{e}^{-rs}\langle X_s, \sigma(X_s,\widehat{X}_s,\alpha_s,u_s)\rangle dW_s.\end{aligned} \tag{5}$$

Taking expectation on both sides of (5) and using Jensen's inequality $E[|\widehat{X}_s|^2] = E[|E[X_s|\mathcal{F}_s^\alpha]|^2] \leqslant E[E[|X_s|^2|\mathcal{F}_s^\alpha]] = E[|X_s|^2]$, together with the elementary inequality $2ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ for any $\varepsilon > 0$, it follows that

$$\begin{aligned}
&E[\mathrm{e}^{-rT}|X_T|^2]\\
=&E\left[\mathrm{e}^{-rt}|X_t|^2 + \int_t^T \mathrm{e}^{-rs}[-r|X_s|^2 + |\sigma(X_s,\widehat{X}_s,\alpha_s,u_s) - \sigma(0,0,\alpha_s,u_s) + \sigma(0,0,\alpha_s,u_s)|^2]ds\right]\\
&+E\left[\int_t^T \mathrm{e}^{-rs}[2\langle X_s, b(X_s,\widehat{X}_s,\alpha_s,u_s) - b(0,0,\alpha_s,u_s)\rangle + 2\langle X_s, b(0,0,\alpha_s,u_s)\rangle]ds\right]\\
\leqslant&E\left[\mathrm{e}^{-rt}|X_t|^2 + \int_t^T \mathrm{e}^{-rs}\bigg(-r|X_s|^2 + (2\kappa_\sigma^2+\varepsilon)(|X_s|^2+|\widehat{X}_s|^2) + \bigg(1+\frac{2\kappa_\sigma^2}{\varepsilon}\bigg)|\sigma(0,0,\alpha_s,u_s)|^2\bigg)ds\right]\\
&+E\left[\int_t^T \mathrm{e}^{-rs}\bigg(2\kappa_b|X_s|^2 + 2\kappa_b|X_s||\widehat{X}_s| + \varepsilon|X_s|^2 + \frac{1}{\varepsilon}|b(0,0,\alpha_s,u_s)|^2\bigg)ds\right]\\
\leqslant&E\left[\mathrm{e}^{-rt}|X_t|^2 + (4\kappa_\sigma^2+4\kappa_b+3\varepsilon-r)\int_t^T \mathrm{e}^{-rs}|X_s|^2 ds\right] + \frac{C}{\varepsilon}E\left[\int_t^T \mathrm{e}^{-rs}(1+|u_s|^2)ds\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}&E[\mathrm{e}^{-rT}|X_T|^2] + (r-4\kappa_b-4\kappa_\sigma^2-3\varepsilon)E\left[\int_t^T \mathrm{e}^{-rs}|X_s|^2 ds\right]\\ &\leqslant E[\mathrm{e}^{-rt}|X_t|^2] + \frac{C}{\varepsilon}E\left[\int_t^T \mathrm{e}^{-rs}(1+|u_s|^2)ds\right].\end{aligned} \tag{6}$$

Choosing $r > 4\kappa_b + 4\kappa_\sigma^2$ and $0 < \varepsilon < (r-4\kappa_b-4\kappa_\sigma^2)/3$, taking $t=0$ and sending $T\to\infty$, and then applying the monotonic convergence theorem, it follows that the estimate (3) holds and the solution X to (1) belongs to the space $\mathcal{L}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$.

Next, substituting T with T_2 and t with T_1 in (6), we obtain

$$\begin{aligned}&|E[\mathrm{e}^{-rT_1}|X_{T_1}|^2] - E[\mathrm{e}^{-rT_2}|X_{T_2}|^2]|\\&\leqslant \frac{C}{\varepsilon}E\left[\int_{T_1}^{T_2}\mathrm{e}^{-rs}(1+|u_s|^2)ds\right] + (r-4\kappa_b-4\kappa_\sigma^2-3\varepsilon)E\left[\int_{T_1}^{T_2}\mathrm{e}^{-rs}|X_s|^2ds\right].\end{aligned}$$

In view of $X \in \mathcal{L}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$ and $u \in \mathcal{L}_{\mathcal{F}}^{2,r}(0,\infty;R^k)$, it follows that the map $T \mapsto E[\mathrm{e}^{-rT}|X_T|^2]$ is uniformly continuous, and then $E[\int_0^\infty \mathrm{e}^{-rs}|X_s|^2ds] < \infty$ implies that $\lim_{T\to\infty} E[\mathrm{e}^{-rT}|X_T|^2] = 0$.

The Hamiltonian of the discounted optimal control problem is given by

$$H(x,y,i,u,p,q) = \langle b(x,y,i,u),p\rangle + \langle \sigma(x,y,i,u),q\rangle + f(x,y,i,u) - r\langle x,p\rangle, \tag{7}$$

and the adjoint equation reads

$$dP_t = -(H_x(t) + E[H_y(t)|\mathcal{F}_t^\alpha])dt + Q_tdW_t + K_t \bullet dM_t, \quad t \geqslant 0,$$

i.e.,

$$\begin{aligned}dP_t =& -[b_x(t)P_t + \sigma_x(t)Q_t + f_x(t) - rP_t]dt - E[b_y(t)P_t + \sigma_y(t)Q_t + f_y(t)|\mathcal{F}_t^\alpha]dt\\&+ Q_tdW_t + K_t \bullet dM_t, \quad t \geqslant 0,\end{aligned} \tag{8}$$

where we define

$$\zeta_z(t) = \zeta_z(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)$$

for $\zeta = b, \sigma, f$ and $z = x, y$, and

$$\begin{aligned}H_x(t) =& H_x(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t, P_t, Q_t)\\ \equiv& b_x(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)P_t + \sigma_x(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)Q_t\\&+ f_x(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - rP_t,\end{aligned}$$

and

$$\begin{aligned}H_y(t) =& H_y(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t, P_t, Q_t)\\ \equiv& b_y(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)P_t + \sigma_y(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t)Q_t\\&+ f_y(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t),\end{aligned}$$

where the triple $(P_t, Q_t, K_t) \in \mathcal{H}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$ denotes the solution to the adjoint equation (8). Recall that the martingale term $K_t \bullet dM_t$ is defined as $\sum_{i,j\in\mathcal{M}} K_{ij}(t)dM_{ij}(t)$, which appears due to the Markov chain involved. Also note that there is no terminal condition given in advance in (8).

To proceed, for notational simplicity, we also define

$$\begin{aligned}&F(t,P,Q,\alpha_t) = b_x(t)P + \sigma_x(t)Q + E[b_y(t)P + \sigma_y(t)Q|\mathcal{F}_t^\alpha],\\&\varphi(t,\alpha_t) = f_x(t) + E[f_y(t)|\mathcal{F}_t^\alpha].\end{aligned}$$

Then, the adjoint equation (8) can be rewritten as the following form:

$$-dP_t = [F(t,P_t,Q_t,\alpha_t) - rP_t + \varphi(t,\alpha_t)]dt - Q_tdW_t - K_t \bullet dM_t, \quad t \geqslant 0. \tag{9}$$

Noting that the partial derivatives of b and σ in (x,y) are bounded (see Assumption (A1)), we have

$$\begin{aligned}&|F(t,P_1,Q_1,\alpha_t) - F(t,P_2,Q_2,\alpha_t)|\\&\leqslant \kappa_b|P_1-P_2| + \kappa_\sigma|Q_1-Q_2| + \kappa_bE[|P_1-P_2||\mathcal{F}_t^\alpha] + \kappa_\sigma E[|Q_1-Q_2||\mathcal{F}_t^\alpha].\end{aligned} \tag{10}$$

Moreover, it is easy to verify that $\varphi \in \mathcal{L}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$ for $r > 4\kappa_b + 4\kappa_\sigma^2$; in fact, from the Assumption (A2) and the well-posedness of X in Theorem 1, one has

$$E\left[\int_0^\infty \mathrm{e}^{-rs}|\varphi(s,\alpha_s)|^2ds\right] \leqslant 16\kappa_f^2\left(1 + E\left[\int_0^\infty \mathrm{e}^{-rs}(|X_s|^2+|u_s|^2)ds\right]\right) < \infty.$$

The following lemma provides a priori estimate for the solution (P,Q,K) to (9) in $\mathcal{H}_{\mathcal{F}}^{2,r}(0,\infty;R^n)$.

Lemma 1. Let (A1) and (A2) hold and $r > 4\kappa_b + 4\kappa_\sigma^2$. Let $(P, Q, K) \in \mathcal{H}_{\mathcal{F}}^{2,r}(0, \infty; R^n)$ be a solution to BSDE (9), then

$$\lim_{T\to\infty} E[\mathrm{e}^{-rT}|P_T|^2] = 0. \tag{11}$$

In addition, for any $\varepsilon \in (0, (r - 4\kappa_b - 4\kappa_\sigma^2)/3)$, we have

$$\begin{aligned}&E\left[|P_0|^2 + \int_0^\infty \mathrm{e}^{-rs}\left[(r - 4\kappa_b - 4\kappa_\sigma^2 - 3\varepsilon)|P_s|^2 + \frac{\varepsilon}{2\kappa_\sigma^2 + \varepsilon}|Q_s|^2\right]ds + \int_0^T \mathrm{e}^{-rs}|K_s|^2 \bullet d[M]_s\right]\\ &\leqslant \frac{1}{\varepsilon}E\left[\int_0^\infty \mathrm{e}^{-rs}|\varphi(s, \alpha_s)|^2 ds\right].\end{aligned} \tag{12}$$

Proof. In the form of forward SDE, the BSDE (9) becomes

$$P_t = P_0 - \int_0^t [F(s, P_s, Q_s, \alpha_s) - rP_s + \varphi(s, \alpha_s)]ds + \int_0^t Q_s dW_s + \int_0^t K_s \bullet dM_s, \quad t \geqslant 0.$$

Similar as the proof of Theorem 1, for any $T_2 > T_1$, there exists some constant $C > 0$ such that

$$\begin{aligned}&|E[\mathrm{e}^{-rT_2}|P_{T_2}|^2] - E[\mathrm{e}^{-rT_1}|P_{T_1}|^2]|\\ &\leqslant C\left[\int_{T_1}^{T_2} \mathrm{e}^{-rs}(|P_s|^2 + |Q_s|^2 + |\varphi(s, \alpha_s)|^2)ds + \int_{T_1}^{T_2} \mathrm{e}^{-rs}|K_s|^2 \bullet d[M]_s\right].\end{aligned}$$

It follows from $P, Q, \varphi \in \mathcal{L}_{\mathcal{F}}^{2,r}(0, \infty; R^n)$ and $K \in \mathcal{M}_{\mathcal{F}}^{2,r}(0, \infty; R^n)$ as well as the uniform continuity of the map $T \mapsto E[\mathrm{e}^{-rT}|P_T|^2]$ that Eq. (11) holds.

Next, we establish the priori estimate (12). Applying Itô's formula to $\mathrm{e}^{-rt}|P_t|^2$ yields

$$\begin{aligned}&\mathrm{e}^{-rT}|P_T|^2\\ &=|P_0|^2 + \int_0^T \mathrm{e}^{-rs}[-r|P_s|^2 - 2\langle P_s, F(s, P_s, Q_s, \alpha_s) - rP_s + \varphi(s, \alpha_s)\rangle + |Q_s|^2]ds\\ &\quad + \int_0^T \mathrm{e}^{-rs}|K_s|^2 \bullet d[M]_s + 2\int_0^T \mathrm{e}^{-rs}\langle P_s, Q_s\rangle dW_s + 2\int_0^T \mathrm{e}^{-rs}\langle P_s, K_s \bullet dM_s\rangle.\end{aligned} \tag{13}$$

Taking expectation on both sides of (13) leads to

$$\begin{aligned}&E\left[|P_0|^2 + \int_0^T \mathrm{e}^{-rs}[r|P_s|^2 + |Q_s|^2]ds + \int_0^T \mathrm{e}^{-rs}|K_s|^2 \bullet d[M]_s\right]\\ &=E\left[\mathrm{e}^{-rT}|P_T|^2 + 2\int_0^T \mathrm{e}^{-rs}[\langle P_s, F(s, P_s, Q_s, \alpha_s) - F(s, 0, Q_s, \alpha_s)\rangle]ds\right]\\ &\quad + 2E\left[\int_0^T \mathrm{e}^{-rs}\langle P_s, F(s, 0, Q_s, \alpha_s) - F(s, 0, 0, \alpha_s)\rangle ds\right] + 2E\left[\int_0^T \mathrm{e}^{-rs}\langle P_s, \varphi(s, \alpha_s)\rangle ds\right].\end{aligned} \tag{14}$$

From the Lipschitz condition (10) on F, for any $\varepsilon > 0$, one has

$$\begin{cases}2\langle P_s, F(s, P_s, Q_s, \alpha_s) - F(s, 0, Q_s, \alpha_s)\rangle \leqslant 2\kappa_b(|P_s|^2 + |P_s|E[|P_s||\mathcal{F}_s^\alpha]),\\ 2\langle P_s, F(s, 0, Q_s, \alpha_s) - F(s, 0, 0, \alpha_s)\rangle \leqslant 2(2\kappa_\sigma^2 + \varepsilon)|P_s|^2 + \dfrac{\kappa_\sigma^2}{2\kappa_\sigma^2 + \varepsilon}(|Q_s|^2 + |E[|Q_s||\mathcal{F}_s^\alpha]|^2),\\ 2\langle P_s, \varphi(s, \alpha_s)\rangle \leqslant \varepsilon|P_s|^2 + \frac{1}{\varepsilon}|\varphi(s, \alpha_s)|^2.\end{cases} \tag{15}$$

Substituting (15) into (14) and using the fact that

$$E[|P_s|E[|P_s||\mathcal{F}_s^\alpha]] \leqslant E[|P_s|^2], \quad |E[|Q_s||\mathcal{F}_s^\alpha]|^2 \leqslant E[|Q_s|^2],$$

it follows that

$$\begin{aligned}&E\left[|P_0|^2+\int_0^T \mathrm{e}^{-rs}\left[(r-4\kappa_b-4\kappa_\sigma^2-3\varepsilon)|P_s|^2+\frac{\varepsilon}{2\kappa_\sigma^2+\varepsilon}|Q_s|^2\right]ds+\int_0^T \mathrm{e}^{-rs}|K_s|^2\bullet d[M]_s\right]\\&\leqslant E\left[\mathrm{e}^{-rT}|P_T|^2+\frac{1}{\varepsilon}\int_0^T \mathrm{e}^{-rs}|\varphi(s,\alpha_s)|^2ds\right].\end{aligned}$$

By letting $T\to\infty$ and noting that $\lim_{T\to\infty}E[\mathrm{e}^{-rT}|P_T|^2]=0$, we obtain the desired result (12).

Based on Lemma 1 and similar to Peng and Shi [45] (see also Yu [46] and Wei and Yu [44]), we have the following existence and uniqueness results for BSDE (9).

Theorem 2. Let (A1) and (A2) hold and $r>4\kappa_b+4\kappa_\sigma^2$. Then, the adjoint equation (9) admits a unique solution $(P,Q,K)\in\mathcal{H}^{2,r}_{\mathcal{F}}(0,\infty;R^n)$.

Proof. The uniqueness can be obtained immediately from the priori estimate (12). We now prove the existence of a solution to (9) via three steps.

Step 1: truncation and finite horizon solution. For each natural number N, let the truncated version of φ be defined as $\varphi_N(t,\alpha_t)=\varphi(t,\alpha_t)1_{[0,N]}(t)$. Consider the following finite horizon mean-field BSDE with regime switching:

$$\begin{cases}dP_t^{(N)}=-[F(t,P_t^{(N)},Q_t^{(N)},\alpha_t)-rP_t+\varphi_N(t,\alpha_t)]dt+P_t^{(N)}dW_t+K_t^{(N)}\bullet dM_t, \quad t\in[0,N],\\ P_N^{(N)}=0.\end{cases}\tag{16}$$

From Nguyen et al. [29, Theorem 2.6], the BSDE (16) admits a unique solution $(P^{(N)},Q^{(N)},K^{(N)})$ for each N.

Step 2: extension to infinite horizon solution. For $\Phi=P,Q,K$, we extend the finite horizon solution to an infinite horizon solution in the following way:

$$\widetilde{\Phi}_t^{(N)}=\Phi_t^{(N)}1_{[0,N]}(t),$$

and it is straightforward to verify that the triple $(\widetilde{P}^{(N)},\widetilde{Q}^{(N)},\widetilde{K}^{(N)})$ satisfies the following infinite horizon mean-field BSDE with regime switching:

$$dP_t=-[F(t,P_t,Q_t,\alpha_t)-rP_t+\varphi_N(t,\alpha_t)]dt+Q_tdW_t+K_t\bullet dM_t.$$

In the next, we establish the convergence in norm of the sequence $\{(\widetilde{P}^{(N)},\widetilde{Q}^{(N)},\widetilde{K}^{(N)})\}_{N=1}^\infty$.

Step 3: convergence. From Lemma 1, there exists a constant $C>0$ such that, for any positive integers N and L large enough,

$$\begin{aligned}&E\left[\int_0^\infty \mathrm{e}^{-rt}\left[\left|\widetilde{P}_t^{(N)}-\widetilde{P}_t^{(L)}\right|^2+\left|\widetilde{Q}_t^{(N)}-\widetilde{Q}_t^{(L)}\right|^2\right]dt+\int_0^\infty \mathrm{e}^{-rt}\left|\widetilde{K}_t^{(N)}-\widetilde{K}_t^{(L)}\right|^2\bullet d[M]_t\right]\\&\leqslant CE\left[\int_0^\infty \mathrm{e}^{-rt}|\varphi_N(t,\alpha_t)-\varphi_L(t,\alpha_t)|^2dt\right]\\&=CE\left[\int_{N\wedge L}^{N\vee L}\mathrm{e}^{-rt}|\varphi(t,\alpha_t)|^2dt\right],\end{aligned}$$

which yields that $\{(\widetilde{P}^{(N)},\widetilde{Q}^{(N)},\widetilde{K}^{(N)})\}_{N=1}^\infty$ is a Cauchy sequence and hence admits a limit (P,Q,K) in $\mathcal{H}^{2,r}_{\mathcal{F}}(0,\infty;R^n)$, in other words,

$$\lim_{N\to\infty}E\left\{\int_0^\infty \mathrm{e}^{-rs}\left[\left|\widetilde{P}_t^{(N)}-P_t\right|^2+\left|\widetilde{Q}_t^{(N)}-Q_t\right|^2\right]dt+\int_0^\infty \mathrm{e}^{-rt}\left|\widetilde{K}_t^{(N)}-K_t\right|^2\bullet d[M]_t\right\}=0.$$

Finally, for any $T>0$ and $N>T$, by letting $N\to\infty$ in the following equation:

$$\begin{aligned}\widetilde{P}_t^{(N)}=&\widetilde{P}_T^{(N)}+\int_t^T\left[F\left(s,\widetilde{P}_s^{(N)},\widetilde{Q}_s^{(N)},\alpha_s\right)-r\widetilde{P}_s^{(N)}+\varphi_N(s,\alpha_s)\right]ds\\&-\int_t^T\widetilde{Q}_s^{(N)}dW_s-\int_t^T\widetilde{K}_s^{(N)}\bullet dM_s,\quad t\in[0,T],\end{aligned}$$

it follows that the limiting triple (P,Q,K) solves the adjoint equation (9).

3.2 Sufficient stochastic maximum principle

In this subsection, we establish the sufficient stochastic maximum principle (SMP) for our infinite horizon discounted optimal control problem. The proof is based on a duality method together with the asymptotic property of solutions to the state equation and adjoint equation.

Theorem 3. Let $u^* \in \mathcal{U}_{ad}$ and (X^*, P, Q, K) be the corresponding solution to the state equation (1) and adjoint equation (8). Suppose that

(i) $H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*, P_t, Q_t) = \min_{u\in U} H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u, P_t, Q_t)$, a.e. $t \in [0,\infty)$, P-a.s.;

(ii) $(x, y, u) \mapsto H(x, y, \alpha_t, u, P_t, Q_t)$ is a convex function, a.e. $t \in [0,\infty)$, P-a.s.

Then we have u^* is an optimal control.

Proof. Let u be an arbitrary element in $\mathcal{U}_{ad}$ and X be the corresponding solution to (1). On the one hand, in view of the forms of the cost functional (2) and the Hamiltonian (7), we have

$$
\begin{aligned}
&J(x,i;u) - J(x,i;u^*)\\
=&E\int_0^\infty \mathrm{e}^{-rt}\big[f(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - f(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*)\big]dt\\
=&E\int_0^\infty \mathrm{e}^{-rt}\big[H(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t, P_t, Q_t) - H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*, P_t, Q_t)\\
&-\big\langle b(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - b(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*), P_t\big\rangle\\
&-\big\langle \sigma(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - \sigma(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*), Q_t\big\rangle + r\big\langle X_t - X_t^*, P_t\big\rangle\big]dt.
\end{aligned}
\tag{17}
$$

On the other hand, for any $T > 0$, applying Itô's formula to $\mathrm{e}^{-rt}\langle X_t - X_t^*, P_t\rangle$ on $[0, T]$, we obtain

$$
\begin{aligned}
E\big[\mathrm{e}^{-rT}\big\langle X_T - X_T^*, P_T\big\rangle\big] =&E\Bigg[\int_0^T \mathrm{e}^{-rt}\big[-r\big\langle X_t - X_t^*, P_t\big\rangle\\
&+\big\langle b(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - b(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*), P_t\big\rangle\\
&+\big\langle X_t - X_t^*, -H_x(t) - E[H_y(t)|\mathcal{F}_t^\alpha]\big\rangle\\
&+\big\langle \sigma(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - \sigma(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*), Q_t\big\rangle\big]dt\Bigg].
\end{aligned}
\tag{18}
$$

From (4) and (11), it follows that

$$
\big|E\big[\mathrm{e}^{-rT}\big\langle X_T - X_T^*, P_T\big\rangle\big]\big| \leqslant \frac{1}{2}E[\mathrm{e}^{-rT}|X_T - X_T^*|^2] + \frac{1}{2}E[\mathrm{e}^{-rT}|P_T|^2] \to 0, \quad T \to \infty.
$$

Letting $T \to \infty$ in (18), we get

$$
\begin{aligned}
0 =& \lim_{T\to\infty} E\big[\mathrm{e}^{-rT}\big\langle X_T - X_T^*, P_T\big\rangle\big]\\
=&E\Bigg[\int_0^\infty \mathrm{e}^{-rt}\big[-r\big\langle X_t - X_t^*, P_t\big\rangle\\
&+\big\langle b(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - b(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*), P_t\big\rangle\\
&+\big\langle X_t - X_t^*, -H_x(t) - E[H_y(t)|\mathcal{F}_t^\alpha]\big\rangle\\
&+\big\langle \sigma(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t) - \sigma(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*), Q_t\big\rangle\big]dt\Bigg].
\end{aligned}
\tag{19}
$$

Combining (17) and (19), we have

$$
\begin{aligned}
&J(x,i;u) - J(x,i;u^*)\\
=&E\int_0^\infty \mathrm{e}^{-rt}\big[H(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t, P_t, Q_t) - H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*, P_t, Q_t)\\
&+\big\langle X_t - X_t^*, -H_x(t) - E[H_y(t)|\mathcal{F}_t^\alpha]\big\rangle\big]dt.
\end{aligned}
\tag{20}
$$

The convexity condition (ii) implies that

$$\begin{aligned} & H(X_t, E[X_t|\mathcal{F}_t^\alpha], \alpha_t, u_t, P_t, Q_t) - H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*, P_t, Q_t) \\ & \geqslant \big\langle X_t - X_t^*, H_x(t)\big\rangle + \big\langle E[X_t - X_t^*|\mathcal{F}_t^\alpha], H_y(t)\big\rangle + \big\langle u_t - u_t^*, H_u(t)\big\rangle, \end{aligned} \tag{21}$$

and the minimization condition (i) yields

$$\begin{aligned} & \big\langle u_t - u_t^*, H_u(t)\big\rangle \\ & = \lim_{\theta\to 0}\frac{1}{\theta}\big[H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^\theta, P_t, Q_t) - H(X_t^*, E[X_t^*|\mathcal{F}_t^\alpha], \alpha_t, u_t^*, P_t, Q_t)\big] \\ & \geqslant 0, \end{aligned} \tag{22}$$

where $u_t^\theta \doteq \theta u_t + (1-\theta)u_t^* \equiv u_t^* + \theta(u_t - u_t^*) \in \mathcal{U}_{ad}$ for any $\theta \in [0,1]$.

Moreover, we note that

$$E\big[\big\langle E[X_t - X_t^*|\mathcal{F}_t^\alpha], H_y(t)\big\rangle\big] = E\big[\big\langle X_t - X_t^*, E[H_y(t)|\mathcal{F}_t^\alpha]\big\rangle\big]. \tag{23}$$

From (20)–(23), we have

$$J(x,i;u) - J(x,i;u^*) \geqslant 0, \quad \forall u \in \mathcal{U}_{ad},$$

which means that u^* is an optimal control.

Remark 3. Indeed, we can still obtain the expression (22) and the conclusion of Theorem 3 without the differentiability assumption of the coefficients with respect to u. The alternative theoretical tool is the so-called Clarke's generalized gradient (see Lemma 2.3 of Section 3 in Yong and Zhou [47] for its precise definition) when the functions are non-smooth in u. Then, we can prove Theorem 3 by following exactly the proof of Theorem 5.2 of Section 3 in Yong and Zhou [47]. Here, we adopt the differentiability assumption of the coefficients with respect to u just for simplicity of presentation.

4 Application to an optimal premium problem

4.1 Model and solution

In this subsection, we will formulate and solve an optimal premium problem of an insurance firm in a regime switching market (it is an LQ problem in nature), by means of the sufficient SMP established in the previous section. An optimal premium policy in a feedback form is obtained with the help of two algebraic Riccati equations and an additional linear equation.

Let $X \in R$ be the cash flow process of an insurance firm and $u \in U = R$ be the premium policy of the firm. The cash flow process X is described by

$$\begin{cases} dX_t = [A(\alpha_t)X_t + \widehat{A}(\alpha_t)E[X_t|\mathcal{F}_t^\alpha] + B(\alpha_t)u_t]dt + \sigma(\alpha_t)dW_t, & t \geqslant 0, \\ X_0 = x \in R, & \alpha_0 = i \in \mathcal{M}, \end{cases} \tag{24}$$

where $A(i)$, $\widehat{A}(i)$, $B(i)$, $\sigma(i)$, $i \in \mathcal{M}$, are given constants. The insurance firm aims to minimize the following cost functional:

$$J(x,i;u) = \frac{1}{2}E\int_0^\infty \mathrm{e}^{-rt}[R(\alpha_t)(X_t - c(\alpha_t))^2 + N(\alpha_t)u_t^2]dt, \tag{25}$$

where the positive constants $c(i)$, $i \in \mathcal{M}$, are some preset cash levels (as a benchmark), and $R(i) > 0$ and $N(i) > 0$, $i \in \mathcal{M}$, are given weighting coefficients. Note that in this problem the insurance firm has two objectives: one is to minimize the deviation between the cash flow process X and the dynamic benchmark, and the other one is to minimize the cost of premium policy u over the infinite time horizon.

It is also mentioned that the above formulation of cash flow model and optimal premium problem is motivated by Huang et al. [48] and Wang and Wu [19]. It is modified in this paper by incorporating the conditional mean into consideration. In the formulation, the Markov chain α, as a common noise, is usually used to represent the market trend (for example, bull market and bear market), and the conditional mean provides a more realistic model to allow the cash flow process to depend on its average;

such a model has been commonly adopted in finance and economics to describe the so-called systematic risk of inter-bank borrowing and lending (see Carmona et al. [49]).

In the following, we will utilize our sufficient SMP to get a closed-form solution for this problem. At first, note that in this example,

$$\kappa_b = \max_{i=1,2}\{|A(i)|, |\widehat{A}(i)|\}, \quad \kappa_\sigma = 0.$$

From Theorems 1 and 2, the discount factor is set to be

$$r > 4\kappa_b = 4\max_{i=1,2}\{|A(i)|, |\widehat{A}(i)|\}$$

to guarantee that the state equation (24) and adjoint equation (26) admit unique solutions, respectively.

The associated Hamiltonian is given by

$$H(x, y, i, u, p, q) = [A(i)x + \widehat{A}(i)y + B(i)u]p + \sigma(i)q + \frac{1}{2}[R(i)(x - c(i))^2 + N(i)u^2] - rxp.$$

It follows that

$$H_x = A(i)p + R(i)(x - c(i)) - rp, \quad H_y = \widehat{A}(i)p, \quad H_u = B(i)p + N(i)u.$$

Then, the corresponding adjoint equation reads

$$\begin{aligned} dP_t = &- [A(\alpha_t)P_t + R(\alpha_t)(X_t - c(\alpha_t)) - rP_t]dt - \widehat{A}(\alpha_t)E[P_t|\mathcal{F}_t^\alpha]dt \\ &+ Q_t dW_t + K_t \bullet dM_t, \quad t \geqslant 0, \end{aligned} \tag{26}$$

and the (open-loop) optimal control turns out to be

$$u_t^* = -N^{-1}(\alpha_t)B(\alpha_t)P_t. \tag{27}$$

By the four-step scheme developed by Ma et al. [50] to decouple an FBSDE, it is natural to guess the adjoint process P and the state process X has the following linear relationship:

$$P_t = \varphi(\alpha_t)X_t + \phi(\alpha_t)E[X_t|\mathcal{F}_t^\alpha] + \psi(\alpha_t) \tag{28}$$

for some deterministic functions $\varphi(i)$, $\phi(i)$, $\psi(i)$, $i \in \mathcal{M}$, which will be determined later.

On the one hand, taking conditional expectation on both sides of (24), we have (recall that we define $\widehat{X}_t = E[X_t|\mathcal{F}_t^\alpha]$)

$$d\widehat{X}_t = [(A(\alpha_t) + \widehat{A}(\alpha_t))\widehat{X}_t + B(\alpha_t)\widehat{u}_t]dt.$$

Then, applying Itô's formula to P_t defined by (28), we obtain (in what follows, the subscript t, the argument α_t, and the martingale terms of dW and dM are sometimes dropped for simplicity of presentation)

$$\begin{aligned} dP = &\Bigg[\bigg(\varphi(AX + \widehat{A}\widehat{X} + Bu) + \sum_{j\in\mathcal{M}} \lambda_{\alpha_t,j}\varphi(j)X\bigg) \\ &+ \bigg(\phi[(A + \widehat{A})\widehat{X} + B\widehat{u}] + \sum_{j\in\mathcal{M}} \lambda_{\alpha_t,j}\phi(j)\widehat{X}\bigg) + \sum_{j\in\mathcal{M}} \lambda_{\alpha_t,j}\psi(j)\Bigg]dt. \end{aligned} \tag{29}$$

From (27) and (28), the optimal control has the following state feedback form:

$$u^* = -N^{-1}B(\varphi X + \phi\widehat{X} + \psi). \tag{30}$$

Inserting (30) into (29), we get

$$\begin{aligned} dP = &\Bigg[\bigg(A\varphi - N^{-1}B^2\varphi^2 + \sum_{j\in\mathcal{M}} \lambda_{\alpha_t,j}\varphi(j)\bigg)X \\ &+ \bigg((A + \widehat{A})\phi + \widehat{A}\varphi - N^{-1}B^2(\phi^2 + 2\varphi\phi) + \sum_{j\in\mathcal{M}} \lambda_{\alpha_t,j}\phi(j)\bigg)\widehat{X} \\ &+ \bigg(- N^{-1}B^2(\varphi + \phi)\psi + \sum_{j\in\mathcal{M}} \lambda_{\alpha_t,j}\psi(j)\bigg)\Bigg]dt. \end{aligned} \tag{31}$$

On the other hand, inserting (28) into (26), it follows that

$$\begin{aligned}dP =& -\big[(-r+A)(\varphi X+\phi\widehat{X}+\psi)+R(X-c)+\widehat{A}[(\varphi+\phi)\widehat{X}+\psi]\big]dt\\ =& -\big[\big((-r+A)\varphi+R\big)X+\big((-r+A)\phi+\widehat{A}(\varphi+\phi)\big)\widehat{X}\\ &+\big((-r+A+\widehat{A})\psi-Rc\big)\big]dt.\end{aligned} \tag{32}$$

Comparing the coefficients of X and $\widehat{X}$ and the nonhomogeneous terms in (31) and (32), we obtain the following two algebraic Riccati equations (AREs):

$$-r\varphi+2A\varphi-N^{-1}B^2\varphi^2+\sum_{j\in\mathcal{M}}\lambda_{ij}\varphi(j)+R=0, \quad i\in\mathcal{M}, \tag{33}$$

and

$$-r\phi+2(A+\widehat{A})\phi+2\widehat{A}\varphi-N^{-1}B^2(\phi^2+2\varphi\phi)+\sum_{j\in\mathcal{M}}\lambda_{ij}\phi(j)=0, \quad i\in\mathcal{M}, \tag{34}$$

and the following linear equation:

$$(-r+A+\widehat{A})\psi-N^{-1}B^2(\varphi+\phi)\psi+\sum_{j\in\mathcal{M}}\lambda_{ij}\psi(j)-Rc=0, \quad i\in\mathcal{M}. \tag{35}$$

In particular, if we define $\widetilde{A}=A+\widehat{A}$ and $\widetilde{\varphi}=\varphi+\phi$, then $\widetilde{\varphi}$ satisfies

$$-r\widetilde{\varphi}+2\widetilde{A}\widetilde{\varphi}-N^{-1}B^2\widetilde{\varphi}^2+\sum_{j\in\mathcal{M}}\lambda_{ij}\widetilde{\varphi}(j)+R=0, \quad i\in\mathcal{M}, \tag{36}$$

which takes a similar form as (33) with the only difference that the coefficient A becoming $\widetilde{A}$; so we can treat (33) and (36) instead of (33) and (34). Then, it follows from the so-called elimination method introduced in Ding et al. [51, Appendix B] that the ARE (33) (respectively, (36)) admits a unique non-negative solution φ (respectively, $\widetilde{\varphi}$) under the condition $r>2A$ (respectively, $r>2\widetilde{A}$), which holds naturally in view of that the discount factor in this example is assumed to be $r>4\kappa_b=4\max_{i=1,2}\{|A(i)|,|\widehat{A}(i)|\}$.

Finally, we compute the minimum cost of the optimal premium problem. Note that

$$d\bigg(\frac{1}{2}\mathrm{e}^{-rt}\varphi(\alpha_t)X_t^2\bigg)=\frac{1}{2}\mathrm{e}^{-rt}\bigg[-r\varphi X^2+2\varphi X(AX+\widehat{A}\widehat{X}+Bu^*)+\sum_{j\in\mathcal{M}}\lambda_{\alpha_t,j}\varphi(j)X^2+\varphi\sigma^2\bigg]dt. \tag{37}$$

Similarly,

$$d\bigg(\frac{1}{2}\mathrm{e}^{-rt}\phi(\alpha_t)\widehat{X}_t^2\bigg)=\frac{1}{2}\mathrm{e}^{-rt}\bigg[-r\phi\widehat{X}^2+2\phi\widehat{X}[(A+\widehat{A})\widehat{X}+B\widehat{u}^*]+\sum_{j\in\mathcal{M}}\lambda_{\alpha_t,j}\phi(j)\widehat{X}^2\bigg]dt. \tag{38}$$

Moreover,

$$d(\mathrm{e}^{-rt}\psi(\alpha_t)\widehat{X}_t)=\mathrm{e}^{-rt}\bigg[-r\psi\widehat{X}+\psi[(A+\widehat{A})\widehat{X}+B\widehat{u}^*]+\sum_{j\in\mathcal{M}}\lambda_{\alpha_t,j}\psi(j)\widehat{X}\bigg]dt. \tag{39}$$

Combing (37)–(39) and recalling the definition of the cost functional (25), we have

$$\begin{aligned}&J(x,i;u^*)-\frac{1}{2}\varphi x^2-\frac{1}{2}\phi x^2-\psi x\\ =&J(x,i;u^*)+E\int_0^\infty d\bigg(\frac{1}{2}\mathrm{e}^{-rt}\varphi(\alpha_t)X_t^2+\frac{1}{2}\mathrm{e}^{-rt}\phi(\alpha_t)\widehat{X}_t^2+\mathrm{e}^{-rt}\psi(\alpha_t)\widehat{X}_t\bigg)\\ =&\frac{1}{2}E\int_0^\infty \mathrm{e}^{-rt}[\varphi\sigma^2+Rc^2-N^{-1}B^2\psi^2]dt,\end{aligned}$$

i.e., the minimum cost is given by

$$J(x,i;u^*)=\frac{1}{2}\widetilde{\varphi}x^2+\psi x+\frac{1}{2}E\int_0^\infty \mathrm{e}^{-rt}[\varphi\sigma^2+Rc^2-N^{-1}B^2\psi^2]dt.$$

Table 1 Values of model parameters.

Parameter	$i=1$	$i=2$
$A(i)$	0.1	0.4
$\widehat{A}(i)$	-0.4	-0.1
$B(i)$	1	2
$\sigma(i)$	0.6	0.4
$R(i)$	0.5	0.5
$c(i)$	3	5
$N(i)$	0.4	0.4

Table 2 Solutions to Riccati equations and linear equation.

	$\varphi(1)$	$\varphi(2)$	$\|\Delta\varphi\|$	$\phi(1)$	$\phi(2)$	$\|\Delta\phi\|$	$\psi(1)$	$\psi(2)$	$\|\Delta\psi\|$
$\varepsilon=1$	0.205	0.177	0.028	-0.039	-0.013	0.026	-0.596	-0.712	0.116
$\varepsilon=0.1$	0.191	0.185	0.006	-0.026	-0.020	0.006	-0.646	-0.671	0.025
$\varepsilon=0.01$	0.188	0.187	0.001	-0.023	-0.022	0.001	-0.658	-0.661	0.003

4.2 Numerical experiment

In this subsection, we provide a numerical example to illustrate the optimal premium policy as well as the optimal (conditional) cash flow process. In this example, we consider a two-state Markov chain α, which means that the market trend switches between two regimes 1 ("bad" or "bear") and 2 ("good" or "bull"). Let the generator of the Markov chain be given by

$$Q=\frac{1}{\varepsilon}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix},$$

where the small parameter $\varepsilon>0$ is introduced specially to display the fast and slow time scales; the smaller the ε, the faster the transition of α (or the transition of the market trend). In this example, we will consider and compare three cases: $\varepsilon=1,0.1,0.01$, and explore not only the effect of the switching of the market trend but also the influence of different time scales on the solution (i.e., optimal premium policy) of the problem.

In this example, the discount factor is taken to be $r=2$, and the values of the other model parameters for $i=1,2$ are listed in Table 1. Note that the appreciation rate of the firm $A(i)$ and the benchmark level $c(i)$ in a bull market ($i=2$) are supposed naturally to be bigger than those in a bear market ($i=1$).

First, we check that

$$\kappa_b=\max_{i=1,2}\{|A(i)|,|\widehat{A}(i)|\}=0.4,\quad \kappa_\sigma=0,\quad r=2>4\kappa_b+4\kappa_\sigma^2=1.6.$$

So the stability condition is satisfied. The solutions to Riccati equations (33) and (34) and the additional linear equation (35), as well as their differences between $i=1$ and $i=2$ (denoted as $\Delta\varphi$, $\Delta\phi$, $\Delta\psi$) for $\varepsilon=1,0.1,0.01$ are computed and presented in Table 2. Then, the simulation results of X^*, $\widehat{X}^*$, u^*, α for $\varepsilon=1,0.1,0.01$ are plotted in Figure 1. It turns out as follows.

(i) The cash flow process X^* will be brought to and eventually stabilized at the benchmark level under the optimal policy u^*. Moreover, since the benchmark level in a bull market $c(2)=5$ is bigger than that in a bear market $c(1)=3$, the optimal policy u^* in a bull market should also be bigger than that in a bear market in order to bring X^* to $c(i)$ more quickly; it can be seen from each sub-figure in Figure 1 that u^* displays a sharp change immediately the Markov chain jumps from state 1 to state 2, and vice versa. On the other hand, when X^* approaches $c(i)$, u^* will then decrease towards 0 to save cost.

(ii) Table 2 shows that the differences $|\Delta\varphi|$, $|\Delta\phi|$, $|\Delta\psi|$ between $\varphi(1)$ and $\varphi(2)$, $\phi(1)$ and $\phi(2)$, $\psi(1)$ and $\psi(2)$, respectively, converge to zero as ε tends to 0. This phenomenon reflects that as the two states switch more and more rapidly, they are becoming more like a single state; actually, it is the so-called two-time-scale approximation (see Yin and Zhang [23]).

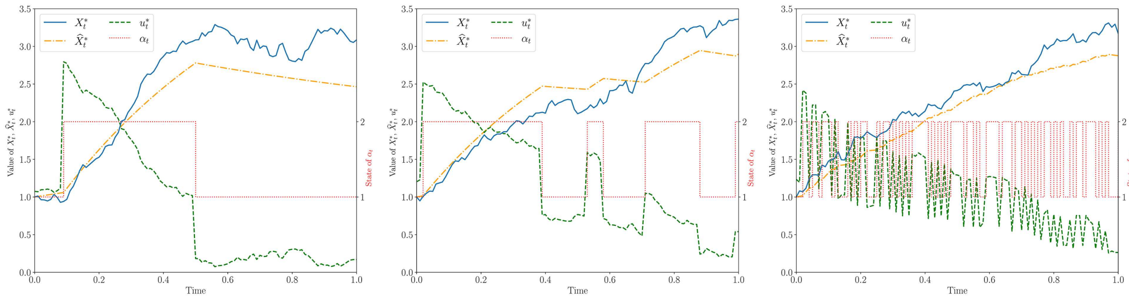

Figure 1 (Color online) Simulations of X^*, $\widehat{X}^*$, u^*, α for $\varepsilon = 1, 0.1, 0.01$.

5 Concluding remarks

There are several interesting questions that deserve further investigation. Firstly, it is natural to proceed to establish the necessary SMP for the infinite horizon optimal control problem considered in this paper, which needs delicate variational techniques, moment estimations, and duality analysis. Secondly, as an application of our sufficient SMP, we provide a cash flow management problem in Section 4, which is a special LQ problem in nature. In the next step, we can continue to study such an LQ problem in the general setting. The key difficulty should be the solvability of the corresponding AREs. Thirdly, this paper considers the limit problem of an original N-agent game problem in the sense of Remark 1. In the future, we will study the convergence of N-agent game towards its asymptotic mean-field game (as well as related numerical experiments) within the framework of, e.g., [52–54], which not only has important mathematical significance, but also enjoys many potential applications in practice.

Acknowledgements This work of Na LI was supported by National Natural Science Foundation of China (Grant Nos. 12571475, 12171279). This work of Siyu LV was supported by National Natural Science Foundation of China (Grant No. 12471414), Natural Science Foundation of Jiangsu Province (Grant No. BK20242023), Jiangsu Province Scientific Research Center of Applied Mathematics (Grant No. BK20233002), and Fundamental Research Funds for the Central Universities (Grant No. 2242025RCB0010). This work of Zhen WU was supported by National Key R&D Program of China (Grant No. 2023YFA1009200), National Natural Science Foundation of China (Grant Nos. 12521001, 62561160159), and Taishan Scholars Climbing Program of Shandong (Grant No. TSPD20210302). This work of Jie XIONG was supported by National Key R&D Program of China (Grant No. 2022YFA1006102) and National Natural Science Foundation of China (Grant No. 12471418).

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