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Special Topic: Mean-Field Game and Control of Large Population Systems: From Theory to Practice

Minimum number of prejudiced agents needed for consensus of weighted median opinion dynamics

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Abstract How to intervene in a group of agents such that their opinions reach consensus is an important issue in the social sciences. In this paper, we investigate this problem where the opinions of agents evolve according to the weighted median mechanism inspired by the cognitive dissonance theory in psychology. Some agents, referred to as prejudiced agents, are informed of the prejudice, while the other agents, called unprejudiced agents, do not have such information. We provide quantitative results on the minimum number of prejudiced agents needed to make opinions of all agents reach the expected consensus over three classes of proximity-based graphs: k-nearest-neighbor cycle, grid graph, and random geometric graph. In addition to this, we construct the methods to appropriately choose prejudiced agents such that the system reaches consensus on the prejudice over these three graphs. Simulation results are given to verify the effectiveness of theoretical results.

Keywords consensus, opinion dynamics, weighted median, prejudice

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1 Introduction

Opinion dynamics mainly focuses on the interaction mechanisms and evolutionary patterns of individual opinions within a group, and is becoming one of the significant research directions in social sciences. It has attracted considerable research interests in diverse fields including politics, economy and psychology [1–3], and enjoys wide applications in election forecasting [4], cultural dissemination [5], and public sentiment monitoring [6]. Besides the experiments or the computer simulations, theoretical investigations are also conducted to understand the phenomena observed in real society, e.g., opinion formation, diffusion, and polarization.

Consensus, meaning that opinions of all agents reach agreement by interaction between agents, is an important collective behavior in social networks. The DeGroot model [7, 8] is one of the earliest opinion dynamics models and shows that consensus can be reached if the network is aperiodic and irreducible. To better reflect the social phenomenon that people are more influenced by similar opinions, bounded-confidence models were proposed in [9, 10], showing that agents reach consensus only when the interaction radius is large enough [11]. However, the two aforementioned models fail to explain the difference of opinions in connected social networks [12, 13]. To understand this, some new mechanisms are proposed in opinion dynamics to characterize the evolution of opinions. For example, the Altafini model [14] allows negative weights in the networks. They show that the bipartite consensus can be achieved if the network is strongly connected and structurally balanced, and consensus at zero can be achieved over unbalanced networks [15].

We note that the opinion dynamics can exhibit consensus behavior under some conditions, but the consensus value depends on the system parameters and initial states, which may not be the desired one [16, 17]. How to intervene in the collective behavior of a group of agents is an important issue in social sciences and has wide applications in many practical scenarios such as public health [18], political

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elections [19], social movements [20], and crisis management [21]. Introducing a subset of agents with prior information of the expected opinion is a feasible way. These agents are called leaders [22] or prejudiced agents [23], which can affect the dynamical behavior of opinions within the whole group [22, 24]. For example, in [22], leaders that can broadcast at certain moments are introduced, and the minimum number of broadcasts required for consensus is investigated. In [24], a two-timescale opinion dynamics model with prejudiced agents is analyzed, and sufficient conditions for reaching consensus in time-varying networks are established.

As far as we know, most existing investigations on opinion dynamics focus on the weighted average mechanism, which has the property that the greater the distance between agents' opinions, the stronger the attraction [25, 26]. To overcome this limitation, the opinion dynamics with the weighted median mechanism [25] was proposed based on the cognitive dissonance theory in psychology [27]. Different from the weighted average mechanism, agents update their opinions based on the weighted median of their neighbors' opinions, which can avoid the problem that the influence increases with the difference of opinions. Theoretical results are obtained for the behavior of opinion dynamics with the weighted median mechanism in [28]. Furthermore, the weighted median opinion dynamics with prejudiced agents was investigated in [29]. Intuitively, if a group of unprejudiced agents form a cohesive set, their opinions can only be disseminated in a closed environment, potentially resulting in an echo chamber [28]. Therefore, avoiding such cohesive sets is essential for reaching consensus. But how many prejudiced agents are needed for the expected behavior is unsolved. In fact, the experiments in social sciences [30, 31] show that a small fraction of prejudiced agents can effectively guide a large group toward consensus. Related problems have also attracted significant interest in the control community, such as pinning control, and the minimal controllability problem. However, in the study of the pinning control problem, the relationship between agents is typically explicitly coupled and is generally described by the Laplacian matrix (cf., [32, 33]), and the minimum controllability problem mainly focused on the networked linear systems (cf., [34]). These methods and results are not suitable for the study of the minimum number of prejudiced agents needed in the weighted median opinion dynamics.

In this paper, we consider the weighted median opinion dynamics with prejudice, and quantitatively investigate the minimum number of prejudiced agents needed to guarantee all agents reach the expected consensus over three types of proximity-based graphs: k-nearest-neighbor cycle, grid graph, and random geometric graph. The main contributions can be summarized as follows.

- For the k-nearest-neighbor cycle, the exact value for the minimum number of prejudiced agents is obtained by identifying the cohesive sets of the k-nearest-neighbor cycle.
- For the grid graph, a necessary condition (lower bound) and a sufficient condition (upper bound) for the minimum number of prejudiced agents are established by characterizing the circle structure in the grid graph. The orders of magnitude of the upper and lower bounds are essentially the same.
- For the random geometric graph, a sufficient condition for the minimum number of prejudiced agents is given by explicitly estimating the number of neighbors of agents.
- We construct methods to select prejudiced agents to ensure that the system reaches consensus on the prejudice over these three types of graphs.

The paper is organized as follows. In Section 2, we introduce the consensus problem of the weighted median opinion dynamics with prejudice. In Sections 3–5, we separately provide the theoretical results on the minimum number of the prejudice agents over k-nearest-neighbor cycle, grid graph, and random geometric graph. In Section 6, we present some simulation results to verify the effectiveness of the theoretical results. Concluding remarks are made in Section 7.

2 Problem formulation

In this paper, we consider the consensus problem of the weighted median opinion dynamics. The weighted median opinion dynamics was proposed in [25], which is inspired by the cognitive dissonance theory in psychology [27]. The weighted median mechanism can avoid the problem of the weighted average mechanism that the greater the distance between agents' opinions, the stronger the attraction [25, 26]. Moreover, this mechanism has the ability to demonstrate the diversity of opinion evolution, such as consensus, polarization and divergence [25, 26].

We first introduce the concept of weighted median. Let $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)^\top$ be a weight vector satisfying $0 \leqslant \vartheta_i \leqslant 1$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n \vartheta_i = 1$. For a vector $\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, we say that

$x^* \in \cup_{i=1}^n \{x_i\}$ is a weighted median of $\boldsymbol{x}$ associated with $\boldsymbol{\vartheta}$ if

$$\sum_{i:x_i<x^*} \vartheta_i \leqslant \frac{1}{2} \text{ and } \sum_{i:x_i>x^*} \vartheta_i \leqslant \frac{1}{2}.$$

It is clear that the weighted median exists and may not be unique.

This paper considers a group of n agents. The evolution of each agent's opinion is affected by its neighbors. The influence between agents is represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $W = (w_{ij})$ denotes the weighted adjacency matrix. For any two agents $i, j \in \mathcal{V}$, the weights satisfy $w_{ij} \geqslant 0$, and $w_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. Moreover, we assume that the matrix W is stochastic, i.e., $\sum_{j\in\mathcal{V}} w_{ij} = 1$ holds for all $i \in \mathcal{V}$. Let $W_i = (w_{i1}, \ldots, w_{in})^\top$ denote the weight vector of the agent i. Let $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ be the neighbor set of the vertex i, and the cardinality of $\mathcal{N}_i$ is denoted as N_i. For convenience of analysis, we assume that for $(i, j) \in \mathcal{E}$ the weights are taken as $w_{ij} = \frac{1}{N_i}$. This is a classical assumption for agent interactions and is widely adopted in multi-agent systems, such as the well-known Vicsek model [35].

Now, we introduce the weighted median opinion dynamics. For each agent $i \in \mathcal{V}$, its opinion $x_i(t) \in \mathbb{R}$ at the time instant $t \in \mathbb{N}$ obeys a discrete-time dynamical model. Let $\boldsymbol{x}(t) := (x_1(t), \ldots, x_n(t))^\top$ be the vector formed by the opinions of all agents. Then the weighted median opinion dynamics are described by

$$x_i(t+1) = \mathrm{Med}(\boldsymbol{x}(t); W_i), \tag{1}$$

where $\mathrm{Med}(\boldsymbol{x}(t); W_i)$ denotes the weighted median of $\boldsymbol{x}(t)$ associated with the weight W_i. If the weighted median is not unique, $\mathrm{Med}(\boldsymbol{x}(t); W_i)$ is taken as the weighted median closest to $x_i(t)$.

In order to intervene in the weighted median opinion dynamics such that all agents reach the desired opinion (referred to as prejudice u), a subset of agents is chosen to be informed about this prejudice. These selected agents are called prejudiced agents. The opinions of prejudiced agents evolve as the balance between the weighted median of $\boldsymbol{x}(t)$ associated with the corresponding weight vector and the prejudice u. Thus, the dynamics of these agents are described by

$$x_i(t+1) = \lambda_i u + (1-\lambda_i)\mathrm{Med}(\boldsymbol{x}(t); W_i),$$

where $\lambda_i \in (0, 1]$ is a constant.

Denote the set of prejudiced agents as $\mathcal{V}_1$, and unprejudiced agents as $\mathcal{V}_2 = \mathcal{V} \setminus \mathcal{V}_1$. Then, the weighted median opinion dynamics with prejudice is described by

$$x_i(t+1) = \begin{cases} \lambda_i u + (1-\lambda_i)\mathrm{Med}(\boldsymbol{x}(t); W_i), & i \in \mathcal{V}_1, \\ \mathrm{Med}(\boldsymbol{x}(t); W_i), & i \in \mathcal{V}_2, \end{cases} \tag{2}$$

where $\lambda_i \in (0, 1]$ for all $i \in \mathcal{V}_1$.

In this paper, we will investigate how many prejudiced agents are required such that the opinion dynamics (2) reaches consensus on the common prejudice u, i.e.,

$$\lim_{t\to\infty} x_i(t) = u, i \in \mathcal{V}$$

holds for any initial opinion $\boldsymbol{x}(0) \in \mathbb{R}$.

By analyzing the dynamics (2), we see that if all agents are prejudiced, then the system can reach consensus on u. However, experimental results (cf., [30,31]) show that only a small minority of prejudiced agents are needed to guide a large group to reach consensus. We are interested in how many prejudiced agents are needed and how we select these prejudiced agents such that the whole group reaches the consensus on the prejudiced opinion. In order to investigate this problem, we introduce the following definition of the cohesive set [28].

Definition 1 (Cohesive set). If a non-empty subset $\mathcal{M} \subset \mathcal{V}$ satisfies $\sum_{j\in\mathcal{M}} w_{ij} \geqslant \frac{1}{2}$ for any $i \in \mathcal{M}$, then we say that $\mathcal{M}$ is a cohesive set of the graph $\mathcal{G}$.

If a cohesive set $\mathcal{M}$ consists of unprejudiced agents only, i.e., $\mathcal{M} \subset \mathcal{V}_2$, we call it an unprejudiced cohesive set. It is clear that if there exists an unprejudiced cohesive set $\mathcal{M}$, then all agents in $\mathcal{M}$ do not adopt the opinion beyond the set $\mathcal{M}$ during the evolution of the opinion dynamics. In this sense, an unprejudiced cohesive set can be seen as an echo chamber, in which opinions are disseminated and reinforced inside a closed system (cf., [28]). So, to reach consensus, the system should not contain unprejudiced cohesive sets to avoid the generation of echo chambers. In our previous paper [29], we provided a necessary and sufficient condition for consensus on the prejudiced opinion of the system (2).

Lemma 1 ([29])**.** The system (2) reaches consensus on the common prejudice u if and only if the graph $\mathcal{G}$ does not contain unprejudiced cohesive sets.

Remark 1. By Lemma 1, we see that the parameters λ_i will not influence the convergence property of the weighted median opinion dynamics (2). But according to Lemmas 4.3 and 4.6 in [29], the system (2) converges with a certain exponential convergence rate which is affected by the parameters λ_i.

According to Lemma 1, the issues of how many prejudiced agents are required and how to select them to achieve consensus have not been solved yet. In this paper, we study the minimum number v^* of prejudiced agents required for the system (2) to reach consensus. The meaning of the minimal number v^* is twofold. One is that there exists at least a set of prejudiced agents whose cardinality is exactly v^* such that the system (2) reaches consensus on u. The other is that if the number of prejudiced agents is less than v^*, then the system (2) cannot reach consensus no matter how we choose prejudiced agents.

We note that when all cohesive sets $\mathcal{M}_1, \ldots, \mathcal{M}_k$ are identified, finding the minimum number v^* can be reformulated as solving the following binary linear programming problem.

$$\min_{\xi} \sum_{j=1}^{n} \xi_j,\ \xi = (\xi_1, \ldots, \xi_n)^\top, \xi_j \in \{0,1\}, \tag{3a}$$

$$\text{s.t.} \ \sum_{j=1}^{n} p_{ij}\xi_j \geqslant 1,\ \forall i = 1, 2, \ldots, k, \tag{3b}$$

$$p_{ij} = \begin{cases} 1, & j \in \mathcal{M}_i, \\ 0, & j \notin \mathcal{M}_i. \end{cases} \tag{3c}$$

For general graphs, it is difficult to identify the cohesive sets and solve the above binary linear programming problem. Thus, finding the accurate minimum number v^* of prejudiced agents for general graphs is very hard. In this paper we will focus on quantitatively analyzing the minimum number v^* over three types of classical graphs: k-nearest-neighbor cycle $\mathcal{G}_{n,k}$, grid graph $\mathcal{G}_{m\times n}$ and random geometric graphs $\mathcal{G}(n, r_n)$.

3 Minimum number of prejudiced agents over k-nearest-neighbor cycle

The k-nearest-neighbor cycle is a fundamental concept in graph theory, and in social sciences, it can be used to model the closest social connections of each agent in a social network, helping to understand the structure and spread of information within a social group.

We first introduce some concepts on k-nearest-neighbor cycle. Let $n, k \in \mathbb{N}$ satisfying $n \geqslant 2k+1$, a k-nearest-neighbor cycle $\mathcal{G}_{n,k} = (\mathcal{V}, \mathcal{E}, W)$ (see Figure 1) is a graph with vertex set $\mathcal{V} = \{1, \ldots, n\}$ and edge set $\mathcal{E} = \{(i,j) : |i-j| \leqslant k \text{ or } n - |i-j| \leqslant k\}$. Then $N_i = 2k+1$ for all $i \in \mathcal{V}$. In this section, we consider the system (2) with n agents interacting over a k-nearest-neighbor cycle $\mathcal{G}_{n,k}$ ($n \geqslant 2k+1$). Since $N_i = 2k+1$ for all $i \in \mathcal{V}$, the elements of the weight matrix W of $\mathcal{G}_{n,k}$ satisfy

$$w_{ij} = \begin{cases} \dfrac{1}{2k+1}, & \text{if } (i,j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

In order to analyze the weighted median opinion dynamics with prejudice, we first introduce the following two lemmas concerning the cohesive sets.

Lemma 2. For a subset $\mathcal{M} \in \mathcal{V}$ of the graph $\mathcal{G}_{n,k}$, if each agent i ($i \in \mathcal{M}$) has at least $k+1$ neighbors belonging to $\mathcal{M}$, then $\mathcal{M}$ is a cohesive set.

Proof. For any $i \in \mathcal{M}$, since i has at least $k+1$ neighbors belonging to $\mathcal{M}$, by (4) we have $\sum_{j\in\mathcal{M}} w_{ij} \geqslant (k+1) \cdot \frac{1}{2k+1} > \frac{1}{2}$. By Definition 1, $\mathcal{M}$ is a cohesive set.

Lemma 3. If the agent $i (i \in \mathcal{V})$ has at least $k+1$ prejudiced neighbors, then i does not belong to any unprejudiced cohesive set.

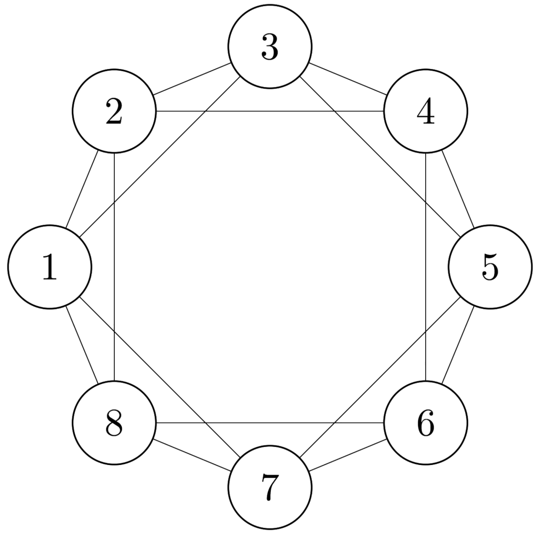

Figure 1 A 2-nearest-neighbor cycle with 8 vertices.

Proof. If the agent i has at least $k+1$ prejudiced neighbors, then for any unprejudiced cohesive set $\mathcal{M}$, by (4) we have

$$\sum_{j\in\mathcal{M}} w_{ij} = 1 - \sum_{j\notin\mathcal{M}} w_{ij} \leqslant 1-(k+1)\cdot\frac{1}{2k+1} < \frac{1}{2},$$

which means that $i \notin \mathcal{M}$.

Let $a \mid b$ denote that a divides b exactly. In the following, we establish a theorem which gives the exact value of the minimum number v^* for consensus of weighted median opinion dynamics with prejudice over k-nearest-neighbor cycle $\mathcal{G}_{n,k}$.

Theorem 1. For the system (2) over k-nearest-neighbor cycle $\mathcal{G}_{n,k}$, we have

$$v^* = k + \left\lfloor \frac{n-k}{k+1} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ denotes the largest integer not greater than the given real number.

Proof. We first prove that $v^* \geqslant k + \lfloor\frac{n-k}{k+1}\rfloor$.

Let $\mathcal{V}_1$ be the set of agents with prejudice that enables the system (2) to reach consensus. If the agent $i(i\in\mathcal{V}_2)$ has at least $k+1$ neighbors belonging to $\mathcal{V}_2$, by Lemma 2 we see that $\mathcal{V}_2$ is a cohesive set. Then, by Lemma 1, the system described by (2) cannot reach consensus. This contradicts the assertion that the system (2) achieves consensus when the agents in the set $\mathcal{V}_1$ have the prejudiced opinion. So there exists $i^*\in\mathcal{V}_2$ such that i^* has at most k neighbors belonging to $\mathcal{V}_2$. Since each agent has $2k+1$ neighbors, i^* has at least $k+1$ neighbors belonging to $\mathcal{V}_1$. Without loss of generality, we assume $i^*=1$, then at least $k+1$ agents in $\mathcal{N}_1=\{1,\dots,k+1\}\cup\{n-k+1,\dots,n\}$ are prejudiced.

In the graph $\mathcal{G}_{n,k}$, the $k+1$ consecutive agents are neighbors of each other. By Lemma 2, the $k+1$ consecutive agents in $\{k+2,\dots,n-k\}$ form a cohesive set. By Lemma 1, in order to ensure that the system (2) achieves consensus, at least one of the $k+1$ consecutive agents in $\{k+2,\dots,n-k\}$ belongs to $\mathcal{V}_1$. With each cluster consisting of $k+1$ agents, the n agents are grouped into $\lfloor\frac{(n-k)-(k+2)+1}{k+1}\rfloor$ clusters. Thus, in order to ensure that the system reaches consensus, at least $\lfloor\frac{(n-k)-(k+2)+1}{k+1}\rfloor$ agents in $\{k+2,\dots,n-k\}$ are prejudiced. By the structure of $\mathcal{G}_{n,k}$, we see that the cardinality of $\mathcal{V}_1$ satisfies

$$|\mathcal{V}_1| \geqslant k+1+\left\lfloor\frac{(n-k)-(k+2)+1}{k+1}\right\rfloor = k+\left\lfloor\frac{n-k}{k+1}\right\rfloor. \tag{5}$$

Thus, we have

$$v^* \geqslant k+\left\lfloor\frac{n-k}{k+1}\right\rfloor.$$

Next, we construct a set $\mathcal{V}_1$ whose cardinality is exactly $k+\lfloor\frac{n-k}{k+1}\rfloor$ such that the system (2) can reach consensus. Let agents $1,2,\ldots,k$ be prejudiced. Then for any agent $i\in\{k+1,\ldots,n\}$, it is prejudiced if and only if $(k+1)\mid(i-k)$. Let $\mathcal{V}_1$ be composed of these agents, i.e.,

$$\mathcal{V}_1=\{1,\ldots,k\}\cup\{i\in\mathcal{V}:k+1\leqslant i\leqslant n,(k+1)\mid(i-k)\}.$$

Then $|\mathcal{V}_1|=k+\lfloor\frac{n-k}{k+1}\rfloor$.

We use proof by contradiction to show that there does not exist a non-empty cohesive set $\mathcal{M}\subset\mathcal{V}_2=\mathcal{V}\setminus\mathcal{V}_1$. Assume there exists a non-empty cohesive set $\mathcal{M}\subset\mathcal{V}_2$. We will prove that $\{1,\ldots,n\}\cap\mathcal{M}=\emptyset$ by induction. Since agents $1,\ldots,k$ are prejudiced, then $\{1,\ldots,k\}\cap\mathcal{M}=\emptyset$. Assume that $\{1,\ldots,i^*-1\}\cap\mathcal{M}=\emptyset$ where $k\leqslant i^*-1\leqslant n-1$. By the structure of $\mathcal{G}_{n,k}$ and $\mathcal{V}_1$, the agent i^* has $k+1$ neighbors belonging to $\mathcal{V}_1\cup\{1,\ldots,i^*-1\}$, they are agents $i^*-k,\ldots,i^*-1,i^*+d$, where $0\leqslant d\leqslant k$ and $(k+1)|(i^*+d-k)$. In addition, since $\mathcal{M}\cap\mathcal{V}_1=\emptyset,\mathcal{M}\cap\{1,\ldots,i^*-1\}=\emptyset$ and $\mathcal{M}\cup\mathcal{V}_1\cup\{1,\ldots,i^*-1\}\subset\mathcal{V}$, we have

$$\begin{aligned}\sum_{j\in\mathcal{M}}w_{i^*j}&=1-\sum_{j\notin\mathcal{M}}w_{i^*j}\\&\leqslant 1-\sum_{j\in\mathcal{V}_1\cup\{1,\ldots,i^*-1\}}w_{i^*j}\\&\leqslant 1-\frac{k+1}{2k+1}<\frac{1}{2}.\end{aligned}$$

It is clear that the $i^*\notin\mathcal{M}$. The induction is complete. Thus, the set $\mathcal{M}$ is empty, it means that there is no cohesive set that contains only unprejudiced agents. By Lemma 1, we can deduce that the prejudiced set $\mathcal{V}_1$ can ensure that the system (2) achieves consensus. By the above analysis, we can conclude that

$$v^*=k+\left\lfloor\frac{n-k}{k+1}\right\rfloor.$$

This completes the proof of the theorem.

Theorem 1 not only establishes the minimum number of prejudiced agents needed to intervene in the weighted median opinion dynamics, but also provides a method for selecting prejudiced agents with a minimal number to ensure the consensus of the system.

Remark 2. The small-world networks can be generated from k-nearest-neighbor cycles by randomly adding or rewiring edges, but the presence of random long-range connections disrupts the symmetry of the network. This asymmetry and randomness make it difficult to identify the cohesive sets, thus making the problem of characterizing the minimum number of prejudiced agents challenging.

4 Minimum number of prejudiced agents over grid graph

We first present some notations on the grid graph. Let $m,n\in\mathbb{N}$, $m\geqslant 3$, $n\geqslant 3$. A grid graph $\mathcal{G}_{m\times n}=(\mathcal{V},\mathcal{E},W)$ (see Figure 2) is a graph with vertex set $\mathcal{V}=\{1,\ldots,mn\}$. The mn vertices are distributed in the grid $(\mathbb{N}\times\mathbb{N})\cap([1,n]\times[1,m])$ from left to right and from bottom to top. It is clear that grid graphs have a natural and intuitive spatial structure, and can be related to the physical or geographical layout of a community. Let $\boldsymbol{z}_i=(z_{i1},z_{i2})\in(\mathbb{N}\times\mathbb{N})\cap([1,n]\times[1,m])$ be the position of the vertex i. The edge set $\mathcal{E}=\{(i,j):\|\boldsymbol{z}_i-\boldsymbol{z}_j\|=1\}\cup\{(i,i):i\in\mathcal{V}\}$. The elements of the weight matrix W of $\mathcal{G}_{n,k}$ are taken as

$$w_{ij}=\begin{cases}\dfrac{1}{N_i}, & \text{if } (i,j)\in\mathcal{E},\\ 0, & \text{if } (i,j)\notin\mathcal{E},\end{cases}\tag{6}$$

where $N_i\in\{3,4,5\}$ is the number of neighbors of the agent i and its value is determined by the position of i.

For the given prejudiced and unprejudiced set $\mathcal{V}_1$ and $\mathcal{V}_2$ in the grid graph, let $\mathcal{G}_2=(\mathcal{V}_2,\mathcal{E}_2)$ denote a graph with $\mathcal{V}_2=\mathcal{V}\setminus\mathcal{V}_1$ and $\mathcal{E}_2=\{(i,j)\in\mathcal{E}:i,j\in\mathcal{V}_2\}$. In order to proceed with our analysis, we first give a lemma on the graph $\mathcal{G}_2$.

Lemma 4. For the graph $\mathcal{G}_2$ in the grid graph with $|\mathcal{V}_2|\geqslant 3$, if $|\mathcal{E}_2|\geqslant 2|\mathcal{V}_2|$, then $\mathcal{G}_2$ has a cycle.

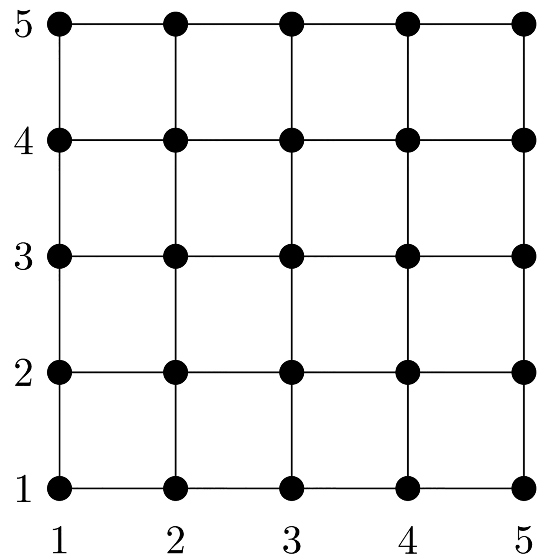

Figure 2 A 5×5 grid.

Proof. We first consider the case where $\mathcal{G}_2$ is connected. According to the structure of the grid graph, we have $N_i \geqslant 2$ for any $i \in \mathcal{V}_2$. We use induction to prove that $\mathcal{G}_2$ has a cycle. If $|\mathcal{V}_2| = 3$, the graph $\mathcal{G}_2$ is a complete graph with three loops. Thus, the graph $\mathcal{G}_2$ must contain a cycle. Assume that the lemma holds for $|\mathcal{V}_2| = l, l \geqslant 3$. When $|\mathcal{V}_2| = l + 1$, if the number of neighbors satisfies $N_i \geqslant 3$ for any $i \in \mathcal{V}_2$, i.e., each agent i has at least two neighbors except for itself, then by the Subsection 1.7 of [36], $\mathcal{G}_2$ has a cycle. If there exists an agent $i \in \mathcal{V}$ such that $N_i = 2$, then we remove the vertex i from the graph $\mathcal{G}_2$ then the remaining part of the graph has a cycle according to the condition of this lemma and our induction assumption. Thus, the induction argument is completed.

Next, we consider the case where $\mathcal{G}_2$ is not connected. By the assumption that $|\mathcal{E}_2| \geqslant 2|\mathcal{V}_2|$ and the structure of the graph, we see that there exists at least one connected component $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ of $\mathcal{G}$ satisfying $|\mathcal{E}^*| \geqslant 2|\mathcal{V}^*|$. By the above analysis, we see that $\mathcal{G}^*$ has a cycle.

This completes the proof of the lemma.

By Definition 1, we have the following lemma on the cohesive set of $\mathcal{G}_{m\times n}$.

Lemma 5. For any cycle $\mathcal{C} = (\mathcal{V}_c, \mathcal{E}_c)$ of $\mathcal{G}_{m\times n}$, $\mathcal{V}_c$ is a cohesive set.

Proof. For any $i \in \mathcal{V}_c$, we have $|\mathcal{N}_i \cap \mathcal{V}_c| = 3$. Thus, for $i \in \mathcal{V}_c$, we have $\sum_{j\in\mathcal{V}_c} w_{ij} = \frac{3}{N_i} \geqslant \frac{3}{5} > \frac{1}{2}$. Then by Definition 1, we see that $\mathcal{V}_c$ is a cohesive set.

By Lemmas 1 and 5, in order for the system (2) to reach consensus, the grid graph $\mathcal{G}_{m\times n}$ cannot contain a cycle consisting of unprejudiced agents only.

Based on the above analysis, we provide a lower bound for the minimum number v^* such that the system (2) achieves consensus over the grid graph $\mathcal{G}_{m\times n}$.

Theorem 2. For the system (2) over the grid graph $\mathcal{G}_{m\times n}$, if the number of the prejudiced set $\mathcal{V}_1$ satisfies

$$|\mathcal{V}_1| \leqslant \frac{mn - m - n}{3}, \tag{7}$$

then the system (2) cannot achieve consensus.

Proof. According to the structure of the grid graph, we see that each agent has at most 5 neighbors. Thus, we have

$$|\mathcal{V}_2| = |\mathcal{V}| - |\mathcal{V}_1| = mn - |\mathcal{V}_1|, \tag{8}$$

$$|\mathcal{E}_2| \geqslant |\mathcal{E}| - 5|\mathcal{V}_1| = 3mn - m - n - 5|\mathcal{V}_1|. \tag{9}$$

By (7)–(9), we have $|\mathcal{E}_2| \geqslant 2|\mathcal{V}_2|$. Then by Lemma 4, we see that the graph $\mathcal{G}_2$ has a cycle. Thus, the graph $\mathcal{G}_{m\times n}$ has a cycle consisting of unprejudiced agents only. By Lemmas 1 and 5, the system (2) can not achieve consensus. This completes the proof of the theorem.

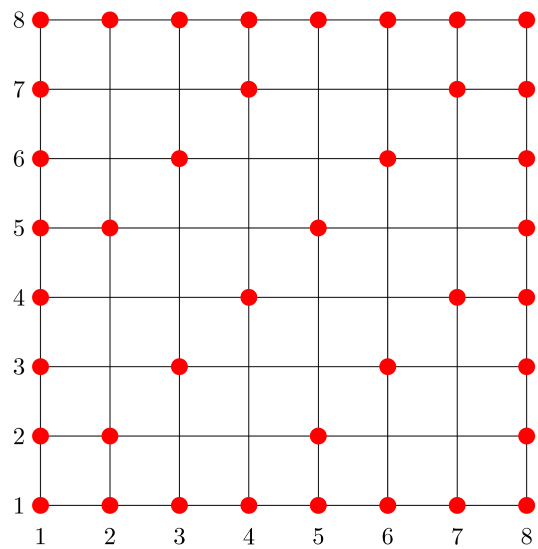

Figure 3 Illustration of choosing the prejudiced agents over $\mathcal{G}_{8\times 8}$, where the prejudiced agents are marked by red points.

In the following, we consider the sufficient condition for the minimum number of prejudiced agents needed for consensus of the system (2).

Theorem 3. For the system (2) over grid graph $\mathcal{G}_{m\times n}$, the minimal number of prejudiced agent satisfies

$$v^* \leqslant \frac{mn}{3} + 2m + 2n - 4. \tag{10}$$

Proof. In order to prove the theorem, we first provide a method to choose the set of prejudiced agents $\mathcal{V}_1$ over the grid graph $\mathcal{G}_{m\times n}$ such that the system (2) can reach consensus on the prejudice.

For the agent $i \in \mathcal{V}$, if i is on the boundary of $\mathcal{G}_{m\times n}$ then it is selected to be prejudiced agent; otherwise if the position $\boldsymbol{z}_i = (z_{i1}, z_{i2})$ of the agent i satisfies $3 \mid (z_{i1} - z_{i2})$ then it is prejudiced. This method is illustrated in Figure 3.

In the above method, the boundary of the grid graph $\mathcal{G}_{m\times n}$ has $2m+2n-4$ prejudiced agents. Apart from the boundaries, the grid graph $\mathcal{G}_{m\times n}$ has $m-2$ rows and each row has at most $\frac{n}{3}$ prejudiced agents. Thus, cardinality of the prejudiced set $\mathcal{V}_1$ satisfies

$$|\mathcal{V}_1| \leqslant \frac{n}{3}\cdot(m-2) + 2m + 2n - 4 < \frac{mn}{3} + 2m + 2n - 4.$$

In the following, we will prove that the system (2) reaches consensus on the prejudice u over the grid graph if the set of prejudiced agents $\mathcal{V}_1$ is chosen according to the method given above. For this purpose, we use induction to prove that for any $1 \leqslant l \leqslant m$, all agents in the lth row do not belong to any unprejudiced cohesive set. As a result, the graph contains no unprejudiced cohesive sets. According to Lemma 1, this implies that the system (2) reaches consensus.

First, for $l = 1$ and $l = m$, the agents in the 1st and mth rows do not belong to any unprejudiced cohesive set since all of them are prejudiced. We assume that for $2 \leqslant l < l^*(\leqslant m-1)$, all agents in the lth row do not belong to any unprejudiced cohesive set. Denote the agents in the l^*th row as $i_1, \ldots, i_n$ from left to right, with positions $\boldsymbol{z}_{i_1} = (1, l^*), \ldots, \boldsymbol{z}_{i_n} = (n, l^*)$.

The agent i_1 is prejudiced since it lies at the boundary of $\mathcal{G}_{m\times n}$. Assume that for any k with $2 \leqslant k < k^*(\leqslant n-1)$, the agent i_k does not belong to any unprejudiced cohesive set. If the agent i_{k^*} is prejudiced, then it does not belong to any unprejudiced cohesive set. Otherwise, by the method to choose the prejudiced agents, we have $3 \nmid (l^* - k^*)$. Thus, we have $3 \mid (l^* - k^* - 1)$ or $3 \mid (l^* - k^* + 1)$. By these two equations, we can deduce that one of the neighbors on the right or top of the agent i_{k^*} is prejudiced. Moreover, by the induction assumption, the neighbors on the left and bottom of the agent i_{k^*} do not belong to any unprejudiced cohesive set. Note that for any unprejudiced cohesive set $\mathcal{M} \subset \mathcal{V}_2$, we have

$$\sum_{j\in\mathcal{M}} w_{i_{k^*}j} = 1 - \sum_{j\notin\mathcal{M}} w_{i_{k^*}j} \leqslant 1 - \frac{3}{N_{i_{k^*}}} < \frac{1}{2}.$$

Thus, the agent i_{k^*} does not belong to any unprejudiced cohesive set. By the above analysis, we conclude that all agents in the l^*th row do not belong to any unprejudiced cohesive set. The induction is complete.

By Lemma 1, the set $\mathcal{V}_1$ of prejudiced agents chosen according to the above method can ensure that the system (2) reaches consensus on the prejudice, that is, the minimum number v^* satisfies (10). This completes the proof of the theorem.

Remark 3. Theorem 3 provides a specific selection strategy of prejudiced agents that ensures the system (2) reaches consensus. By Theorems 2 and 3, we see that the minimum number v^* satisfies

$$\frac{mn-m-n}{3} \leqslant v^* \leqslant \frac{mn}{3} + 2m + 2n - 4.$$

When both m and n tend to infinity, the orders of magnitude of the upper and lower bounds are essentially the same, which means that the ratio of the upper and lower bounds tends to 1.

5 Minimum number of prejudiced agents over random geometric graph

Compared with grid graphs or other regular lattices, random geometric graphs allow agents to be placed randomly within a spatial domain (e.g., a plane or a cube) and provide a more practical framework in opinion dynamics, as they are isotropic, allow flexible average degrees, and more accurately capture local interactions based on spatial proximity among agents [37, 38].

We first give some notations on a random geometric graph. Let $n \in \mathbb{N}$ and $r_n \geqslant 0$ be the interaction radius. Assume that there are n vertices with positions uniformly and independently distributed in the unit square $[0,1]^2$. Let $\boldsymbol{z}_i = (z_{i1}, z_{i2}) \in [0,1]^2$ be the position of the vertex i. A random geometric graphs $\mathcal{G}(n, r_n)$ is a graph with the vertex set $\mathcal{V} = \{1, \ldots, n\}$, and the edge set $\mathcal{E} = \{(i,j) : \|\boldsymbol{z}_i - \boldsymbol{z}_j\| \leqslant r_n\}$. The neighbors of the agent i are those agents lying within a circle centered at i's position with the radius r_n, i.e., $\mathcal{N}_i = \{j : \|\boldsymbol{z}_i - \boldsymbol{z}_j\| \leqslant r_n\}$.

In this section, we investigate consensus of the system (2) with a large population size n, and provide sufficient conditions for consensus on the prejudice. For this purpose, we will construct a prejudiced set such that the system (2) can reach consensus. First, we will give some preliminary results on the distribution of prejudiced and unprejudiced agents.

Divide the unit square $[0,1]^2$ into $M_n = \lfloor \sqrt{n} + 1 \rfloor^2$ equally small squares with the length of each side equal to $a_n = \frac{1}{\lfloor \sqrt{n}+1 \rfloor}$, labeled from left to right and from bottom to top as $\mathcal{S}_{i,j}, i, j = 1, \ldots, \frac{1}{a_n}$. Let $\boldsymbol{z}_{i,j}$ be the position of the center of small square $\mathcal{S}_{i,j}$. Let $\mathcal{B}(\boldsymbol{z}, r)$ denote the circle centered at $\boldsymbol{z}$ with radius r. For any agent $k \in \mathcal{S}_{i,j}$,

$$\|\boldsymbol{z}_{i,j} - \boldsymbol{z}_k\| \leqslant \frac{\sqrt{2}}{2} a_n.$$

Then for any $\boldsymbol{z} \in \mathcal{B}(\boldsymbol{z}_{i,j}, r_n - a_n)$ and any agent $k \in \mathcal{S}_{i,j}$, we have

$$\|\boldsymbol{z} - \boldsymbol{z}_k\| \leqslant \|\boldsymbol{z} - \boldsymbol{z}_{i,j}\| + \|\boldsymbol{z}_{i,j} - \boldsymbol{z}_k\| \leqslant r_n - a_n + \frac{\sqrt{2}}{2} a_n < r_n.$$

Thus, for any agent k in $\mathcal{S}_{i,j}$, all agents in $\mathcal{B}(\boldsymbol{z}_{i,j}, r_n - a_n)$ are its neighbors. Denote the line determined by the bottom two vertices of the square $\mathcal{S}_{i,j}$ as $l_{i,j}$, and the regions of the circle $\mathcal{B}(\boldsymbol{z}_{i,j}, r_n - a_n)$ below and above the line $l_{i,j}$ as $\mathcal{B}^1_{i,j}$ and $\mathcal{B}^2_{i,j}$ (see Figure 4).

Let $\mathcal{Z} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ be the set of positions of all agents in $\mathcal{G}(n, r_n)$. Let $N^1_{i,j} = |\mathcal{Z} \cap \mathcal{B}^1_{i,j}|$, where $\mathcal{Z} \cap \mathcal{B}^1_{i,j}$ represents the set of positions of the agents in $\mathcal{B}^1_{i,j}$. Let $\{g_n\}_{n \in \mathbb{N}}$ be any positive sequence satisfying

$$1 \ll g_n \ll \frac{\sqrt{n} r_n}{\log n}. \tag{11}$$

Let $b_n = \lfloor \frac{r_n}{a_n} \rfloor$. We have

$$b_n \leqslant \frac{r_n}{a_n} \leqslant b_n + 1, \tag{12}$$

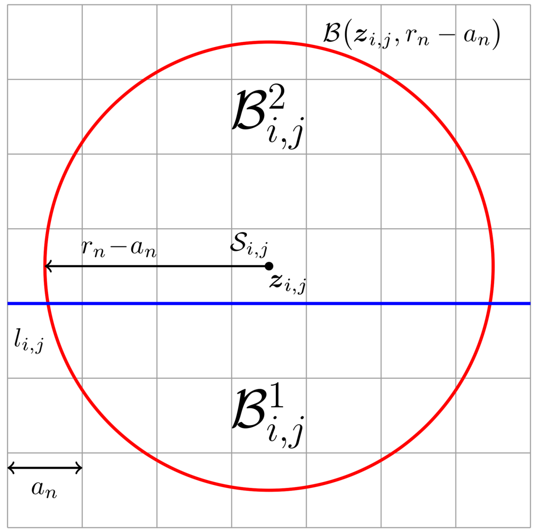

Figure 4 $\mathcal{B}(\boldsymbol{z}_{i,j}, r_n - a_n)$ (red line) and $l_{i,j}$ (blue line).

$$1 \ll g_n \ll \frac{b_n}{\log n}. \tag{13}$$

Then, we have the following estimates for $N^1_{i,j}$.

Lemma 6. Suppose that the radius r_n satisfies $\frac{\log n}{\sqrt{n}} = o(r_n)$[1]. Then the probability of the event $A_1 = \{\omega \in \Omega : N^1_{i,j} \geqslant n(\frac{\pi}{2}(r_n - a_n)^2 - r_n a_n)(1 + o(1)), b_n + 2 \leqslant i, j \leqslant \frac{1}{a_n} - b_n - 1\}$ satisfies

$$P\{A_1\} \geqslant 1 - O(n^{-g_n}). \tag{14}$$

Proof. For any $b_n + 2 \leqslant i, j \leqslant \frac{1}{a_n} - b_n - 1$, by the definition of the set $\mathcal{B}^1_{i,j}$ and line $l_{i,j}$, we have for sufficiently large n

$$\begin{aligned} S(\mathcal{B}^1_{i,j}) &\geqslant \frac{S\Big(\mathcal{B}(\boldsymbol{z}_{i,j}, r_n - a_n)\Big)}{2} - \frac{1}{2} \cdot 2r_n \cdot a_n \\ &= \frac{\pi}{2}(r_n - a_n)^2 - r_n a_n, \end{aligned} \tag{15}$$

where $S(\cdot)$ denotes the area of the corresponding region.

Let ξ^l_{ij}, $1 \leqslant l \leqslant n$ be the indicator function defined as follows:

$$\xi^l_{ij} = \begin{cases} 1, & \text{if } \boldsymbol{z}_1 \in \mathcal{B}^1_{i,j}, \\ 0, & \text{if } \boldsymbol{z}_l \notin \mathcal{B}^1_{i,j}. \end{cases}$$

Then ξ^l_{ij} are i.i.d. Bernoulli random variables with $P(\xi^l_{ij} = 1) = S(\mathcal{B}^1_{i,j}) \triangleq p$. It is clear that $N^1_{i,j} = \sum_{l=1}^n \xi^l_{ij}$. By the Chernoff bound (cf., [39]), we have the following inequality for any $\epsilon \in (0, 1)$:

$$P\Big\{|N^1_{i,j} - np| > \epsilon np\Big\} \leqslant 2\exp\left(-\frac{\epsilon^2 np}{3}\right).$$

Since $\{N^1_{i,j}\}$ are identically distributed random variables, then we have

$$\begin{aligned} &P\left\{\max_{b_n+2\leqslant i,j\leqslant \frac{1}{a_n}-b_n-1} |N^1_{i,j} - np| \leqslant \epsilon np\right\} \\ &\geqslant 1 - \sum_{b_n+2\leqslant i,j\leqslant \frac{1}{a_n}-b_n-1} P\Big\{|N^1_{i,j} - np| > \epsilon np\Big\} \end{aligned}$$

1) For two positive sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, $a_n = O(b_n)$ means that there exists a constant $C > 0$ independent of n, such that $a_n \leqslant Cb_n$ for any $n \in \mathbb{N}$; $a_n = o(b_n)$ or $a_n \ll b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

$$\geqslant 1 - 2M_n \exp\left(-\frac{\epsilon^2 np}{3}\right)$$
$$\geqslant 1 - 3n \exp\left(-\frac{\epsilon^2 np}{3}\right).$$

Take

$$\epsilon = \epsilon_n = \sqrt{\frac{3(g_n \log n + \log n)}{np}} = o(1),$$

where g_n satisfies $1 \ll g_n \ll \frac{r_n}{\sqrt{n}\log n}$. Then for sufficiently large n, we have

$$P\left\{\max_{b_n+2\leqslant i,j\leqslant \frac{1}{a_n}-b_n-1} |N_{i,j}^1 - np| \leqslant \epsilon_n np\right\}$$
$$\geqslant 1 - 3\exp\left(\log n - \frac{1}{3}\cdot\frac{3(g_n\log n + \log n)}{np}\cdot np\right)$$
$$\geqslant 1 - O\left(\frac{1}{n^{g_n}}\right).$$

Thus,

$$P\left\{N_{i,j}^1 = np(1+o(1)), b_n + 2 \leqslant i, j \leqslant \frac{1}{a_n} - b_n - 1\right\} \geqslant 1 - O(n^{-g_n}).$$

Substituting (15) and $p = S(\mathcal{B}_{i,j}^1)$ into the above inequality, we can deduce the inequality (14). This completes the proof of the lemma.

In the proof of the above lemma, we see that the condition $\frac{\log n}{\sqrt{n}} = o(r_n)$ is important for the estimate of the $N_{i,j}^1$, which guarantees that each agent has a sufficiently large number of neighbors. It is also used in the following two lemmas and the main theorems to obtain the asymptotic results.

Following the proof of the above lemma, we can estimate the number of neighbors of each agent, see the following lemma.

Lemma 7. Suppose that the radius r_n satisfies $\frac{\log n}{\sqrt{n}} = o(r_n)$. Then the probability of the event $A_2 = \{\omega \in \Omega : N_k < n\pi(r_n + a_n)^2(1 + o(1)),\ k = 1, \ldots, n\}$ satisfies

$$P\{A_2\} \geqslant 1 - O(n^{-g_n}). \tag{16}$$

Let $\tilde{\mathcal{S}}_{i,j}$ denote the set of all small squares $\mathcal{S}_{i,j}$ whose indexes i and j satisfy

$$\min_{k\in\mathbb{Z}}\left\{\left|i - j + k\cdot b_n\right|\right\} \leqslant 2. \tag{17}$$

In the following proof of the main theorem, we will choose the agents located in these small squares $\tilde{\mathcal{S}}_{i,j}$ as prejudice agent. Let $\tilde{\mathcal{B}}_{i,j}^2$ denote the overlapping area of $\mathcal{B}_{i,j}^2$ and $\tilde{\mathcal{S}}_{i,j}$, and $N_{i,j}^2 = |\mathcal{Z} \cap \tilde{\mathcal{B}}_{i,j}^2|$. We have the following result on the estimate of $N_{i,j}^2$.

Lemma 8. Suppose that the radius r_n satisfies $\frac{\log n}{\sqrt{n}} = o(r_n)$. Then the probability of the event $A_3 = \{\omega \in \Omega : N_{i,j}^2 \geqslant (2\pi + 1)nr_na_n(1 + o(1)),\ b_n + 2 \leqslant i, j \leqslant \frac{1}{a_n} - b_n - 1\}$ satisfies

$$P\{A_3\} \geqslant 1 - O(n^{-g_n}). \tag{18}$$

Proof. For $b_n + 2 \leqslant i, j \leqslant \frac{1}{a_n} - b_n - 1$, by (17) we have for sufficiently large n,

$$\begin{aligned} S(\tilde{\mathcal{B}}_{i,j}^2) &= \frac{5}{b_n}(1 + o(1))S(\mathcal{B}_{i,j}^2) \\ &\geqslant \frac{5S\Big(\mathcal{B}(\boldsymbol{z}_{i,j}, r_n - a_n)\Big)}{2b_n} \\ &= \frac{5\pi}{2b_n}(r_n - a_n)^2 \geqslant (2\pi + 1)r_na_n. \end{aligned}$$

The rest of the proof is similar to that of Lemma 6. Here we omit the proof details. This completes the proof of the lemma.

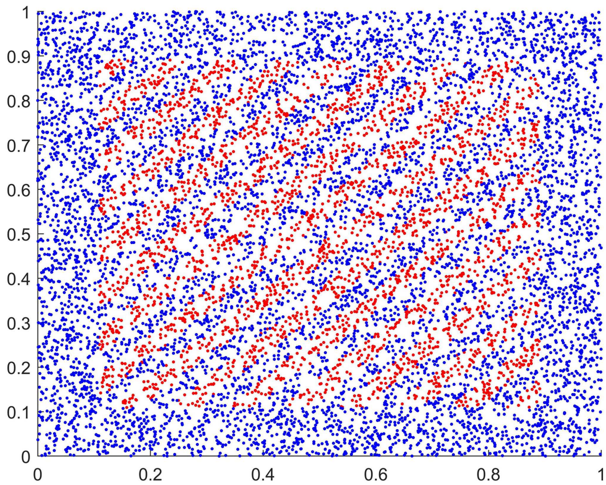

Figure 5 Illustration of choosing the prejudiced agents over $\mathcal{G}(n, r_n)$ with $n = 10000$ and $r_n = 0.1$. The blue and red points represent prejudiced and unprejudiced agents, respectively.

Now, we present the main theorem concerning sufficient conditions for the consensus of the weighted median opinion dynamics with prejudice. We prove this theorem by providing a specific strategy for selecting prejudiced agents.

Theorem 4. For the system (2) over random geometric graphs $\mathcal{G}(n, r_n)$. Suppose that the radius r_n satisfies $\frac{\log n}{\sqrt{n}} = o(r_n)$. Then there exists a set of prejudiced agents whose cardinality satisfies

$$|\mathcal{V}_1| < \left(4r_n n + \frac{6\sqrt{n}}{r_n}\right)\big(1 + o(1)\big), \tag{19}$$

such that the system (2) reaches consensus with a probability not less than $1 - O(n^{-g_n})$.

Proof. First, we provide a method to choose the set of prejudiced agents over the random geometric graphs $\mathcal{G}(n, r_n)$ such that the system (2) can reach consensus on the prejudice. For small square $\mathcal{S}_{i,j}$, if $\mathcal{S}_{i,j}$ satisfies

$$\max\left\{\left|i \cdot a_n - \frac{1}{2}\right|, \left|j \cdot a_n - \frac{1}{2}\right|\right\} \geqslant \frac{1}{2} - (b_n + 1)a_n, \tag{20}$$

i.e., $i \notin [b_n + 2, \frac{1}{a_1} - b_n - 1]$ or $j \notin [b_n + 2, \frac{1}{a_1} - b_n - 1]$, or satisfies (17), then all the agents located in these small squares are chosen to be prejudiced. This method on how to choose the prejudiced agents is illustrated in Figure 5.

According to the above method, we see that there are at most $\big(\frac{5}{b_n}\frac{1}{a_n} + 5\big)\frac{1}{a_n}$ small square $\mathcal{S}_{i,j}$ satisfying (17) and $4(b_n + 1)\big(\frac{1}{a_n} - b_n - 1\big)$ small squares $\mathcal{S}_{i,j}$ satisfying (20). Let $\mathcal{B}_2$ denote the area consisting of all small squares which satisfy (17) and (20). Then for sufficiently large n, we have

$$\begin{aligned} S(\mathcal{B}_2) &\leqslant \left(4(b_n + 1)\left(\frac{1}{a_n} - b_n - 1\right) + \left(\frac{5}{b_n}\frac{1}{a_n} + 5\right)\frac{1}{a_n}\right) S(\mathcal{S}_{i,j}) \\ &= \left(4(b_n + 1)\left(\frac{1}{a_n} - b_n - 1\right) + \left(\frac{5}{b_n}\frac{1}{a_n} + 5\right)\frac{1}{a_n}\right) a_n^2 \\ &< 4a_n b_n + \frac{5}{b_n} + 9a_n \\ &< 4r_n + \frac{6}{2r_n\sqrt{n}}. \end{aligned}$$

Similar to the proof of Lemma 6, the probability of the event $A_4 = \big\{\omega \in \Omega : N(n, r_n) < \big(4r_n n + \frac{6\sqrt{n}}{r_n}\big)\big(1 + o(1)\big)\big\}$ satisfies

$$P\big\{A_4\big\} \geqslant 1 - O(n^{-g_n}). \tag{21}$$

By Lemmas 6–8, the probability that the event $A_1 \cap A_2 \cap A_3 \cap A_4$ happens is no less than $1 - O(n^{-g_n})$. The following analysis is on the event $A_1 \cap A_2 \cap A_3 \cap A_4$. We will prove that if the prejudiced agents

Table 1 Minimum number of prejudiced agents under different network structures.

Network structure	Number of agents	Minimum number v^*
K-nearest-neighbor cycle $\mathcal{G}_{n,k}$	n	$v^* = k + \lfloor \frac{n-k}{k+1} \rfloor$
Grid graph $\mathcal{G}_{m\times n}$	mn	$\frac{mn-m-n}{3} \leqslant v^* \leqslant \frac{mn}{3} + 2m + 2n - 4$
Random geometric graphs $\mathcal{G}(n, r_n)$	n	$v^* < \left(4r_n n + \frac{6\sqrt{n}}{r_n}\right)\left(1 + o(1)\right)$

are chosen according to the method given above, then the system (2) reaches consensus on prejudice. For this purpose, we use the induction method to show that for all $1 \leqslant i \leqslant \frac{1}{a_n}$, agents in the square $\mathcal{S}_{i,j}(1 \leqslant j \leqslant \frac{1}{a_n})$ do not belong to any unprejudiced cohesive set.

If i satisfies $i \leqslant b_n$ or $i \geqslant \frac{1}{a_n} - b_n$, then by (20) all agents in $\mathcal{S}_{i,j}$ are prejudiced, and thus they do not belong to any unprejudiced cohesive set. Assume that for $i \leqslant i^* - 1$ with $b_n + 1 \leqslant i^* \leqslant \frac{1}{a_n} - b_n - 1$, all agents in $\mathcal{S}_{i,j}(1 \leqslant j \leqslant \frac{1}{a_n})$ do not belong to any unprejudiced cohesive set. Then for $i = i^*$, by the induction assumption and the definition of $\mathcal{B}^1_{i,j}$ and $\tilde{\mathcal{B}}^2_{i,j}$, we see that all the agents in $\mathcal{B}^1_{i^*,j}(1 \leqslant j \leqslant \frac{1}{a_n})$ do not belong to any unprejudiced cohesive set and all the agents in $\tilde{\mathcal{B}}^2_{i^*,j}(1 \leqslant j \leqslant \frac{1}{a_n})$ are prejudiced. Thus, any agent k in the set $\mathcal{S}_{i^*,j}$ has at least $N^1_{i^*,j} + N^2_{i*,j}$ neighbors do not belong to any unprejudiced cohesive set. By Lemmas 6–8, we have $N^1_{i^*,j} + N^2_{i*,j} > \frac{1}{2}N_k$. Then for agent $k \in \mathcal{S}_{i^*,j}$ and any unprejudiced cohesive set $\mathcal{M}$, we have

$$\sum_{j\in\mathcal{M}} w_{kj} = 1 - \sum_{j\notin\mathcal{M}} w_{i^*j} \leqslant 1 - \frac{N^1_{i^*,j} + N^2_{i*,j}}{N_k} < \frac{1}{2},$$

which means $k \notin \mathcal{M}$. The induction argument is complete. Then according to Lemma 1, the system (2) reaches consensus under the chosen prejudiced set.

Remark 4. According to the definition of cohesive set, if the number of all prejudiced agents is less than half of the minimum number of neighbors of all agents, then no matter how we choose prejudiced agents, all unprejudiced agents form an unprejudiced cohesive set. For such a case, the system (2) over the random geometric graphs $\mathcal{G}(n, r_n)$ cannot reach consensus on the prejudice. Similar to the proof of Lemma 6, the minimum number of neighbors of all agents equals $\frac{\pi}{4}r_n^2 n\big(1 + o(1)\big)$ with a probability no less than $1 - O(n^{-g_n})$ if the radius r_n satisfies $\frac{\log n}{\sqrt{n}} = o(r_n)$. Therefore, the minimum number v^* satisfies $v^* \geqslant \frac{\pi}{8}r_n^2 n\big(1 + o(1)\big)$. Furthermore, by Theorem 4, we derive that the minimum number of the prejudiced agent v^* has the following upper and lower bounds:

$$\frac{\pi}{8}r_n^2 n\big(1 + o(1)\big) \leqslant v^* < \left(4r_n n + \frac{6\sqrt{n}}{r_n}\right)\big(1 + o(1)\big).$$

It is clear that the lower bound is very restrictive, and how to relax the lower bound of v^* needs further investigation on the property of the cohesive sets in the random geometric graph.

In this paper, we have presented the minimum number of prejudiced agents required to achieve consensus under three different network structures. The results can be summarized in Table 1.

6 Simulation results

In this section, we provide some simulation results. Consider the multi-agent systems where all the agents interact over k-nearest-neighbor cycle and grid graph. The opinions of all agents obey the dynamics (2), where the prejudice is set $u = 0$ and the parameter λ_i is chosen randomly and uniformly from $(0, 1]$ for each prejudiced agent i. The initial state $x_i(0)$ is chosen randomly and uniformly from $[-1, 1]$ for each agent i.

First, the interaction graph is taken as the k-nearest-neighbor cycle $\mathcal{G}_{n,k}$, with $n = 17$, $k = 2$. For such a case, $v^* = 7$ agents are informed about the prejudice, and these agents are chosen according to the method in Theorem 1. The evolution of opinions is shown in Figure 6(a). We see that the system (2) can reach consensus on the prejudice. However, when the number of prejudiced agents is taken as 6, which is less than v^*, the evolution of opinions is shown in Figure 6(b) and the system (2) cannot reach consensus.

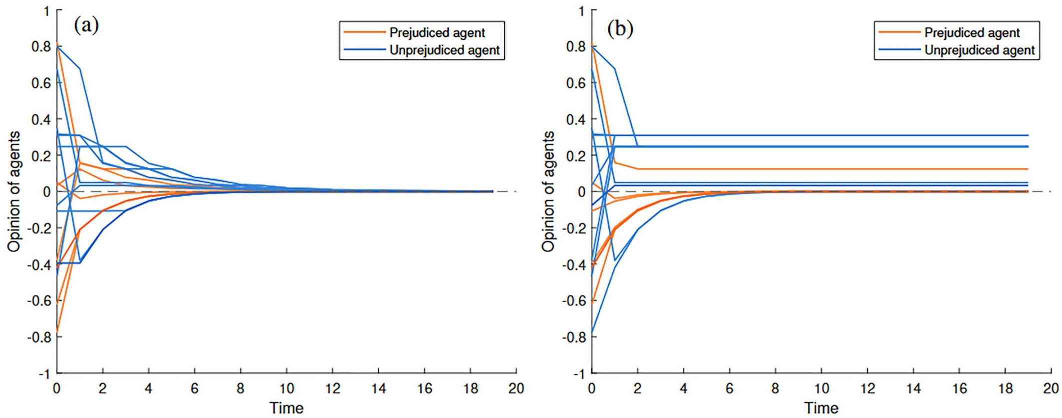

Figure 6 Evolution of all agents' opinions over the k-nearest-neighbor cycle $\mathcal{G}_{17,2}$ where (a) 7 and (b) 6 agents are prejudiced, respectively.

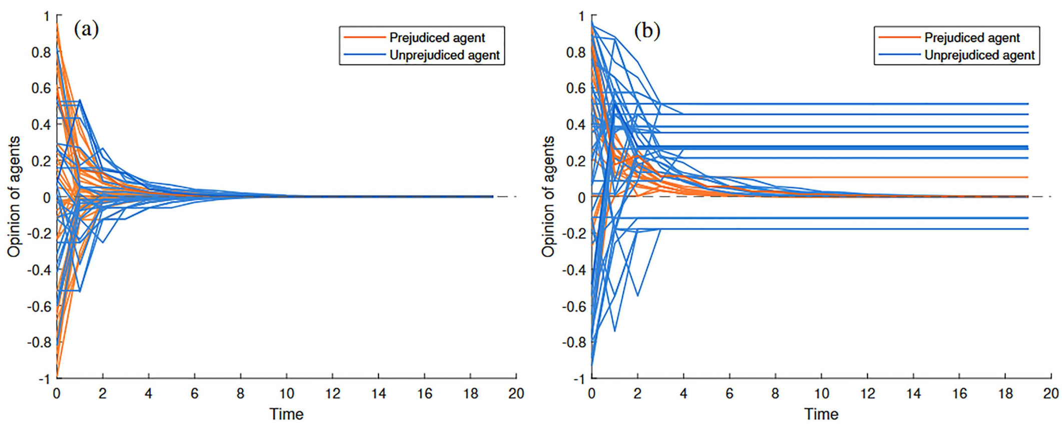

Figure 7 Evolution of all agents' opinions over the grid graph $\mathcal{G}_{8\times 8}$ where (a) 40 and (b) 16 agents are prejudiced, respectively.

Then, the agents interact over the grid graph $\mathcal{G}_{8\times 8}$. 40 agents are informed of the prejudice, and they are chosen according to the method in Theorem 3 (see Figure 3). The evolution of opinions is shown in Figure 7(a), and we see that the system (2) can reach consensus on the prejudice. However, according to Theorem 2, when the number of the prejudiced agents is less than 16, all agents cannot reach consensus as shown in Figure 7(b).

7 Concluding remarks

In this paper, we investigated the consensus problem of the weighted median opinion dynamics where some agents are informed of the prejudice. By analyzing properties of the cohesive sets and the network topology, we established quantitative results for the minimum number of prejudiced agents needed for the expected consensus over three classes of proximity-based graphs. For the k-nearest-neighbor cycle, we provide the exact value for the minimum number of prejudiced agents, and for the grid graph we give the necessary and sufficient conditions for the minimum number of prejudiced agents. Moreover, we give the sufficient condition for the minimum number of prejudiced agents when all agents interact over the random geometric graph. Besides this, we separately construct methods for selecting prejudiced agents to ensure consensus of the whole system. Some interesting problems deserve to be further investigated, e.g., the necessary condition for the minimum number of prejudiced agents to guarantee consensus when agents interact over random geometric graph, and quantitative results for the minimum number of prejudiced agents over small-world network.

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References

1 Becker J, Brackbill D, Centola D. Network dynamics of social influence in the wisdom of crowds. Proc Nat Acad Sci, 2017, 114: 5070–5076
2 Nowak A, Vallacher R R. Nonlinear societal change: the perspective of dynamical systems. Br J Soc Psychol, 2019, 58: 105–128
3 Galesic M, Stein D L. Statistical physics models of belief dynamics: theory and empirical tests. Phys A-Stat Mech its Appl, 2019, 519: 275–294
4 Guess A, Malhotra N. How do social media feed algorithms affect attitudes and behavior in an election campaign? Science, 2023, 381: 398–404
5 Klemm K, Eguíluz V M, Toral R, et al. Role of dimensionality in Axelrod's model for the dissemination of culture. Phys A-Stat Mech its Appl, 2003, 327: 1–5
6 Willaert T, Paul V, Katrien B, et al. Building social media observatories for monitoring online opinion dynamics. Soc Media Soc, 2020, 6: 63–78
7 French J R P. A formal theory of social power. Psychol Rev, 1956, 63: 181–194
8 Degroot M H. Reaching a consensus. J Am Stat Assoc, 1974, 69: 118–121
9 Hegselmann R, Krause U. Opinion dynamics and bounded confidence: models, analysis and simulation. J Artif Soc Soc Simul, 2002, 5: 1–33
10 Deffuant G, Neau D, Amblard F, et al. Mixing beliefs among interacting agents. Advs Complex Syst, 2000, 03: 87–98
11 Dittmer J C. Consensus formation under bounded confidence. Nonlinear Anal-Theor Methods Appl, 2001, 47: 4615–4621
12 Abelson P. Mathematical models of the distribution of attitudes under controversy. In: Contributions to Mathematical Psychology. New York: Holt, Rinehart and Winston, 1964. 141–160
13 Axelrod R. The dissemination of culture. J Conflict Resolution, 1997, 41: 203–226
14 Altafini C. Consensus problems on networks with antagonistic interactions. IEEE Trans Automat Contr, 2012, 58: 935–946
15 Hendrickx J. A lifting approach to models of opinion dynamics with antagonisms. In: Proceedings of the 53rd IEEE Conference on Decision and Control, 2014. 2118–2123
16 Bauso D, Cannon M. Consensus in opinion dynamics as a repeated game. Automatica, 2018, 90: 204–211
17 Yang Y, Dimarogonas D V, Hu X. Opinion consensus of modified Hegselmann-Krause models. Automatica, 2014, 50: 622–627
18 Athey S, Grabarz K, Luca M, et al. Digital public health interventions at scale: the impact of social media advertising on beliefs and outcomes related to covid vaccines. Proc Natl Acad Sci, 2023, 120: e2208110120
19 Metaxas P T, Mustafaraj E. Social media and the elections. Science, 2012, 338: 472–473
20 Dunivin Z O, Yan H Y, Ince J, et al. Black lives matter protests shift public discourse. Proc Natl Acad Sci USA, 2022, 119: e2117320119
21 Im W, Youk S, Park H. Apologies combined with other crisis response strategies: do the fulfillment of individuals' needs to be heard and the timing of response message affect apology appropriateness? Public Relat Rev, 2021, 47: 102002
22 Zhao Z, Shi L, Li T, et al. Opinion dynamics of social networks with intermittent-influence leaders. IEEE Trans Comput Soc Syst, 2022, 10: 1073–1082
23 Friedkin N E, Johnsen E C. Social influence and opinions. J Math Sociology, 1990, 15: 193–206
24 Wang L, Bernardo C, Hong Y, et al. Consensus in concatenated opinion dynamics with stubborn agents. IEEE Trans Automat Contr, 2022, 68: 4008–4023
25 Mei W, Bullo F, Chen G, et al. Micro-foundation of opinion dynamics: rich consequences of the weighted-median mechanism. Phys Rev Res, 2022, 4: 023213
26 Bizyaeva A, Franci A, Leonard N E. Nonlinear opinion dynamics with tunable sensitivity. IEEE Trans Automat Contr, 2023, 68: 1415–1430
27 Matz D C, Wood W. Cognitive dissonance in groups: the consequences of disagreement. J Personality Soc Psychol, 2005, 88: 22–37
28 Mei W, Hendrickx J M, Chen G, et al. Convergence, consensus, and dissensus in the weighted-median opinion dynamics. IEEE Trans Automat Contr, 2024, 69: 6700–6714
29 Zhang R, Liu Z, Chen G, et al. Convergence analysis of weighted-median opinion dynamics with prejudice. IEEE Trans Automat Contr, 2025, 70: 4155–4162
30 Dyer J R G, Johansson A, Helbing D, et al. Leadership, consensus decision making and collective behaviour in humans. Phil Trans R Soc B, 2009, 364: 781–789
31 Valente T W, Pumpuang P. Identifying opinion leaders to promote behavior change. Health Educ Behav, 2007, 34: 881–896
32 Vega C, Suarez O, Sanchez E, et al. Trajectory tracking on complex networks via inverse optimal pinning control. IEEE Trans Autom Control, 2018, 64: 767–774
33 Della Rossa F, Vega C J, de Lellis P. Nonlinear pinning control of stochastic network systems. Automatica, 2023, 147: 110712
34 Li T, Zhang R, Liu Z, et al. Controllability of networked multiagent systems based on linearized Turing's model. Automatica, 2024, 162: 111507
35 Liu Z, Guo L. Synchronization of multi-agent systems without connectivity assumptions. Automatica, 2009, 45: 2744–2753
36 Bondy A, John A, Murty U, et al. Graph Theory with Applications. London: Macmillan, 1976
37 Zhang W, Lim C, Korniss G, et al. Opinion dynamics and influencing on random geometric graphs. Sci Rep, 2014, 4: 55–68
38 Estrada E, Sheerin M. Consensus dynamics on random rectangular graphs. Phys D-Nonlinear Phenomena, 2016, 323-324: 20–26
39 Rajasekaran S. Handbook of Randomized Computing. Berlin: Springer, 2001