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Special Topic: Mean-Field Game and Control of Large Population Systems: From Theory to Practice

Mean-field team in backward linear-quadratic control problems with model uncertainty

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Abstract This paper studies a class of LQ mean-field team problems driven by backward stochastic differential equations (BSDEs) with drift uncertainty. In this framework, agents cooperate through state-average to minimize a shared social cost functional. A notable innovation of this work is modeling agent state dynamics using BSDEs with uncertain generators. Unlike standard social optimal control frameworks that rely on forward stochastic differential equations, we model agent's states dynamics using BSDEs, in which the terminal conditions are specified. Moreover, we consider the model uncertainty in the decision-making process. Accordingly, we construct the backward mean-field team problem under model uncertainty. We derive the worst-case disturbance and formulate the related social cost. Applying a forward backward version of person-by-person optimality, we construct an auxiliary control problem for each agent under the worst scenario and establish the robust decentralized social strategy. The well-posedness of such consistency condition system is obtained by the Riccati decoupling method. The related asymptotic social optimality is also verified.

Keywords mean-field team, model uncertainty, backward stochastic differential equation, forward backward stochastic differential equations, Riccati equation

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1 Introduction

Large population systems are complex dynamical systems composed of numerous individual agents that interact according to predefined rules, thereby giving rise to emergent collective behaviors. These systems have extensive applications across various domains, including economics, finance, engineering, epidemiology, traffic flow management, and energy networks. Notable examples include investor groups in financial markets, vehicle coordination in intelligent transportation systems, and information dissemination on social media. Mean-field game theory, independently developed by Huang, Malhamé and Caines [1] and Lasry and Lions [2], provides a powerful theoretical framework for analyzing such large-scale interactive systems. For further references on mean-field game, see [3–9].

Agents within large population systems are typically coupled through the mean-field terms embedded in their state dynamics or performance functions. When individual agents adhere to externally enforced commitments—such as contracts or agreements—and consequently strive to achieve a collectively optimal return, the resulting framework is referred to as a mean-field team, also known as the social optimum problem. In this setting, all agents act cooperatively, seeking a unified socially optimal strategy that minimizes the overall social cost. Fundamentally, this constitutes a cooperative game, wherein each agent must strike a delicate balance between reducing its own cost and contributing to the minimization of the aggregate social cost. Huang, Caines, and Malhamé [10] studied the social optima linear-quadratic (LQ) problem with N decision makers, providing a theoretical foundation for large-population cooperative control in stochastic dynamic systems. Wang and Zhang [11] analyzed social optima in mean-field linear-quadratic-Gaussian models with Markov jump parameters. For additional references, see [12–15].

The aforementioned research focuses on cooperative games driven by forward systems. When each agent sets a predefined objective, describing such interactions using traditional forward systems becomes

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relatively complex. In contrast, backward stochastic differential equations (BSDEs), with their inherent mathematical structure defined by terminal conditions, provide a more natural formulation for these problems. Bismut [16] initially introduced BSDEs within a linear framework for stochastic optimal control problems. Subsequently, Pardoux and Peng [17] further advanced the theoretical framework of BSDEs, extending it to the nonlinear setting, thereby laying the foundation for subsequent research in the field. Additionally, Duffie and Epstein [18] introduced BSDEs in the context of economics, proposing a stochastic differential representation of recursive utility, which extended the standard additive utility. The LQ optimal control problem for BSDEs with deterministic coefficients and weighting matrices was first investigated by Lim and Zhou [19]. Buckdahn et al. [20] expanded BSDE theory to mean-field settings through asymptotic methods, facilitating its broader application in the analysis of large-scale systems. In recent years, the BSDE theory has been continuously refined and has become an essential tool in game theory, optimal control, and financial mathematics. Readers interested in this topic may refer to [21–26].

In the study of stochastic control and differential games, mathematical models typically assume that system dynamics and cost functions possess specific structural properties, with fixed and precisely estimable parameters. However, the inherent complexity and uncertainty of financial markets challenge the validity of such assumptions. Even with high-frequency data sampling, estimation errors remain inevitable. Market environments are influenced by a multitude of unpredictable factors, making the development of precise and reliable models particularly challenging in practice. Consequently, when analyzing LQ systems and related control problems, incorporating model uncertainty is crucial for enhancing the robustness and practical applicability of the model. Uncertainty was first introduced in [27] and later extended through the Ellsberg paradox [28], having long been recognized as a fundamental factor in decision-making processes. To address model uncertainty, a soft-constraint approach is commonly employed, integrating penalty terms into the objective function to achieve a trade-off between constraint enforcement and solution flexibility. This method permits mild constraint violations while adjusting penalty parameters to modulate sensitivity to uncertainty, thereby enhancing the robustness of the model. In dynamic systems, uncertainty typically manifests in the drift or diffusion terms, referred to as drift uncertainty and volatility uncertainty, respectively. For drift uncertainty, relevant studies include [29] for the mean-field LQ game, [30] for social optimal control of the mean field LQ model, and [31] for stochastic Stackelberg LQ differential game. For volatility uncertainty, Ref. [32] provided a reference for mean field social optimum control, while Ref. [33] examined its impact within the framework of mean-field game. More studies can be referred to [34–38].

Inspired by the above, we investigate a class of LQ mean-field team problems driven by BSDEs with drift uncertainty. In this setting, agents cooperate and are coupled through state-average, collectively minimizing a shared social cost functional. A key novelty of this work lies in the agents' state dynamics, which are governed by BSDEs with uncertain generators. This formulation fundamentally distinguishes our framework from existing studies on robust socially optimal control, where system dynamics is typically described by forward stochastic differential equations (SDEs). Unlike forward SDEs, BSDEs are formulated with a prescribed terminal condition, and their solutions consist of a pair of adapted processes $(y(\cdot), z(\cdot))$, which depend exclusively on past and present information, without reliance on future data. We introduce a stochastic process f to represent all unspecified external influences on drift evolution, serving as a critical factor in the decision-making process. In this setting, f can be interpreted as an adversarial player acting against all agents, aiming to exacerbate the overall social cost. Therefore, a soft-constraint analysis is adopted by incorporating a negative quadratic penalty on f in the cost functional to characterize the impact of uncertainty on the system. The principal contributions of this work are summarized as follows.

• To our best knowledge, this article is the first research endeavor to formulate the robust mean-field team optimization problem in a backward setting. Through a soft-constraint analysis, we address drift uncertainty and reformulate the problem as a social optimal control problem governed by a system of forward-backward stochastic differential equations (FBSDEs). Furthermore, we construct an auxiliary control problem with FBSDEs as the dynamics and derive a robust decentralized strategy based on consistency conditions and an approximation scheme.

• With drift uncertainty embedded in the system, the resulting robust decentralized strategy is governed by two forward and two backward optimal equations, which are coupled through initial values. It turns out that the asymptotic analysis of the decentralized strategy involves coupled FBSDEs and is much more challenging than that of [24]. By employing discounting techniques and Riccati decoupling methods, we establish relevant estimates for FBSDEs and verify that the derived decentralized strategy is asymptotically optimal.

Mean-field team models with BSDEs provide a robust framework for analyzing cooperative decision-making under model uncertainty in finance. It has wide applications in financial domains. A key application lies in insurance premium pricing, where insurers must set optimal premiums while accounting for uncertain claim distributions and collective risk exposure [39, 40]. The BSDE state process naturally aligns with terminal liabilities, allowing insurers to solve robust optimization problems under worst-case scenarios [41]; similarly, in portfolio management, investors can use this framework to coordinate strategies under fluctuating market conditions, ensuring that wealth targets are met despite the uncertainty of the parameters [42–44]; another relevant setting is pension fund regulation, where pension funds must ensure solvency over long horizons amid uncertain investment returns and longevity trends [45]. The backward structure naturally models terminal funding requirements, while the mean-field component captures how individual contribution decisions aggregate across the system [46, 47]. This approach enables fund managers to develop robust strategies that account for worst-case scenarios, such as prolonged market downturns or unexpected increases in longevity. The advantage of this model is its unified treatment of cooperative optimization among agents and stochastic control under uncertainty. This combination makes it particularly suitable for real-world financial systems characterized by interdependent decision makers and incomplete information. From an implementation perspective, the high-dimensional nature of these problems typically requires advanced numerical methods [48, 49]. For future research, we can explore the integration of machine learning techniques with BSDE to enhance the practical applicability of the model.

The structure of this paper is outlined as follows. Section 2 introduces the preliminary concepts and formulates the backward mean-field team problem under model uncertainty. In Section 3, we employ a soft-constraint approach to derive the worst-case disturbance. Section 4 addresses a social optimal control problem under the worst scenario and establishes the robust decentralized strategy. Section 5 further presents consistency conditions and their solvability, identifying the limit term. In Section 6, we further demonstrate that the decentralized strategy obtained is an ϵ-Nash equilibrium of the original problem. Finally, Section 7 concludes the paper by summarizing the main results.

2 Problem formulation

Consider a finite time horizon $[0,T]$ for fixed $T>0$. Assume that $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leqslant t\leqslant T},\mathbb{P})$ is a complete filtered probability space satisfying the usual condition and $\{W_i(t),1\leqslant i\leqslant N\}_{0\leqslant t\leqslant T}$ is an N-dimensional Brownian motion on this space. Let $\mathcal{F}_t$ be the filtration generated by $\{W_i(s),1\leqslant i\leqslant N\}_{0\leqslant s\leqslant t}$ and augmented by $\mathcal{N}_{\mathbb{P}}$ (the class of all $\mathbb{P}$-null sets of $\mathcal{F}$). Let $\mathcal{F}_t^i$ be the augmentation of $\sigma\{W_i(s),0\leqslant s\leqslant t\}$ by $\mathcal{N}_{\mathbb{P}}$.

Let $\langle\cdot,\cdot\rangle$ denote the standard Euclidean inner product. $x^\top$ denotes the transpose of a vector (or matrix) x. $\mathbb{S}^n$ denotes the set of symmetric $n\times n$ matrices with real elements. $M>(\geqslant)0$ denotes that $M\in\mathbb{S}^n$ which is positive (semi)definite, while $M\gg 0$ denotes that, $\exists\ \varepsilon>0$, $M-\varepsilon I\geqslant 0$. We introduce the following spaces which will be used in the paper:

- $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n):=\{\eta:\Omega\to\mathbb{R}^n|\eta$ is $\mathcal{F}_T$-measurable such that $\mathbb{E}|\eta|^2<\infty\}$;
- $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n):=\{\zeta(\cdot):[0,T]\times\Omega\to\mathbb{R}^n|\zeta(\cdot)$ is $\mathcal{F}_t$-progressively measurable process such that $\mathbb{E}\int_0^T|\zeta(t)|^2dt<\infty\}$;
- $L^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R}^n)):=\{\zeta(\cdot):[0,T]\times\Omega\to\mathbb{R}^n|\zeta(\cdot)$ is $\mathcal{F}_t$-adapted, continuous, such that $\mathbb{E}\left[\sup_{s\in[0,T]}|\zeta(s)|^2\right]<\infty$;
- $L^\infty(0,T;\mathbb{R}^{n\times n}):=\{\zeta(\cdot):[0,T]\to\mathbb{R}^{n\times n}\mid\zeta(\cdot)$ is uniformly bounded$\}$.

We consider a weakly coupled large population system with agent $\{\mathcal{A}_i,1\leqslant i\leqslant N\}$. The dynamics of the agents is given by a system of linear backward stochastic differential equations with mean-field coupling. For $1\leqslant i\leqslant N$,

$$\begin{cases} dy_i(t)=-\Big[A(t)y_i(t)+B(t)u_i(t)+C(t)y^{(N)}(t)+f(t)\Big]dt+z_i(t)dW_i(t)+\displaystyle\sum_{j=1,j\neq i}^{N}z_{ij}(t)dW_j(t),\\ y_i(T)=\xi_i,\end{cases}\tag{1}$$

where $\xi_i \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ is independent and identical distributed, $f(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is some unknown random disturbance to denote impact from possible modeling uncertainty and $y^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N y_i(t)$ is the average state of the agents. Let $u(\cdot) = (u_1(\cdot),\cdots,u_N(\cdot))$ be the set of strategies of all N agents and $u_{-i}(\cdot) = (u_1(\cdot),\cdots,u_{i-1}(\cdot),u_{i+1}(\cdot),\cdots,u_N(\cdot))$, $1 \leqslant i \leqslant N$. The cost functional for $\mathcal{A}_i$, $1 \leqslant i \leqslant N$, is given by

$$\mathcal{J}_i(u_i(\cdot),u_{-i}(\cdot);f(\cdot)) = \frac{1}{2}\mathbb{E}\Big[\int_0^T\Big(\|u_i(t)\|^2_{R_1}+\|y_i(t)-\Gamma_1(t)y^{(N)}(t)\|^2_Q+\|z_i(t)\|^2_H\Big)dt+\|y_i(0)-\Gamma_2 y^{(N)}(0)\|^2_G\Big]. \tag{2}$$

Since $f(\cdot)$ is unknown to the agents and each agent regards the disturbance as an adversarial player (e.g., [29]). These agents may adopt some soft-constraint analysis and the cost functionals should be revised as

$$\begin{aligned} &J_i(u_i(\cdot),u_{-i}(\cdot);f(\cdot)) \\ =&\frac{1}{2}\mathbb{E}\Big[\int_0^T\Big(\|u_i(t)\|^2_{R_1}+\|y_i(t)-\Gamma_1(t)y^{(N)}(t)\|^2_Q+\|z_i(t)\|^2_H-\|f(t)\|^2_{R_0}\Big)dt+\|y_i(0)-\Gamma_2 y^{(N)}(0)\|^2_G\Big]. \end{aligned} \tag{3}$$

Accordingly, the aggregate team functional of N agents is

$$J_{soc}(u(\cdot);f(\cdot)) = \sum_{i=1}^N J_i(u_i(\cdot),u_{-i}(\cdot);f(\cdot)). \tag{4}$$

We impose the following assumptions on the coefficients.

Assumption 1. $A(\cdot),B(\cdot),C(\cdot)$ are matrix-valued functions satisfying

$$A(\cdot),C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}),\ B(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}).$$

Assumption 2. $G,\Gamma_2 \in \mathbb{R}^{n\times n}$, $R_1(\cdot),\Gamma_1(\cdot),Q(\cdot),H(\cdot),R_0(\cdot)$ are matrix-valued functions satisfying

$$R_1(\cdot) \in L^\infty(0,T;\mathbb{S}^m),\ Q(t),H(\cdot),R_0(\cdot) \in L^\infty(0,T;\mathbb{S}^n),\ \Gamma_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}).$$

Assumption 3. $Q \geqslant 0, G \geqslant 0, H \geqslant 0,\ R_1(\cdot),R_0(\cdot) \gg 0$.

For $i = 1,\cdots,N$, the centralized admissible strategy set for the i-th agent is given by

$$\mathcal{U}_i^c = \Big\{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\Big\}.$$

Correspondingly, the decentralized admissible strategy set for the i-th agent is given by

$$\mathcal{U}_i^d = \Big\{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathcal{F}^i}(0,T;\mathbb{R}^m)\Big\}.$$

According to the minimax control problem, we need to consider the possibility of the worst case scenario. Thus, the social cost under the worst-case disturbance is defined as

$$J^{wo}_{soc}(u(\cdot)) = \sup_{f\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)} J_{soc}(u(\cdot);f).$$

Then we can propose the following optimal control problem.

Problem 1. Find a strategy set $\bar{u} = (\bar{u}_1,\cdots,\bar{u}_N) \in \mathcal{U}^c = \bigotimes_{i=1}^N \mathcal{U}_i^c$, such that

$$J^{wo}_{soc}(\bar{u}(\cdot)) = \inf_{u\in\mathcal{U}^c} J^{wo}_{soc}(u(\cdot)). \tag{5}$$

Due to the framework of a large population system, it is intractable for the agents to find the optimal centralized strategies. Therefore, we introduce our robust BLQ-MFT problem which focuses on the decentralized strategies.

Problem 2. Find a strategy set $\widetilde{u} = (\widetilde{u}_1,\cdots,\widetilde{u}_N)$, where $\widetilde{u}_i \in \mathcal{U}_i^d$, $1 \leqslant i \leqslant N$ such that

$$\frac{1}{N}\Big(J^{wo}_{soc}(\widetilde{u}(\cdot)) - \inf_{u(\cdot)\in\mathcal{U}^c} J^{wo}_{soc}(u(\cdot))\Big) \leqslant \varepsilon,$$

where $\varepsilon = \varepsilon(N) > 0$, $\lim_{N\to\infty}\varepsilon(N) = 0$.

3 Mean-field BLQ problem for the disturbance

In this section, we will first seek the worst-case disturbance $\widetilde{f}(\cdot)$; i.e., for any $u_i \in \mathcal{U}_i^c$, find $\widetilde{f}(\cdot)$ such that

$$J_{soc}(u(\cdot);\widetilde{f}(\cdot)) = \sup_{f\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)} J_{soc}(u(\cdot);f(\cdot)).$$

Clearly, the above problem is equivalent to minimize $-J_{soc}(u(\cdot);f(\cdot))$ over $f(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, where

$$\begin{aligned}
-J_{soc}(u(\cdot);f(\cdot)) &= -\sum_{i=1}^N J_i(u_i(\cdot),u_{-i}(\cdot);f(\cdot))\\
&= -\frac{1}{2}\sum_{i=1}^N \mathbb{E}\Big[\int_0^T\Big(\|u_i\|^2_{R_1}+\|y_i-\Gamma_1 y^{(N)}\|^2_Q+\|z_i\|^2_H-\|f\|^2_{R_0}\Big)dt+\|y_i(0)-\Gamma_2 y^{(N)}(0)\|^2_G\Big].
\end{aligned}$$

For further analysis, we make the following assumption.

Assumption 4. $-J_{soc}(u(\cdot);f(\cdot))$ is uniformly convex respect to f.

We will apply the variation analysis to construct the worst disturbance $\widetilde{f}(\cdot)$. For this, let $(\widetilde{y}_i,\widetilde{z}_{ij})$ and $(y_i^\theta,z_{ij}^\theta)$ be the states corresponding to $\widetilde{f}(\cdot)$ and $\widetilde{f}(\cdot)+\theta f$, respectively, where $f\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and $\theta\in\mathbb{R}$. That is,

$$\begin{cases}
d\widetilde{y}_i = -\Big(A\widetilde{y}_i+Bu_i+C\widetilde{y}^{(N)}+\widetilde{f}\Big)dt+\widetilde{z}_i dW_i(t)+\displaystyle\sum_{j=1,j\neq i}^N \widetilde{z}_{ij}dW_j(t),\\
\widetilde{y}_i(T)=\xi_i
\end{cases}$$

and

$$\begin{cases}
dy_i^\theta = -\Big(Ay_i^\theta+Bu_i+Cy^{(N),\theta}+\widetilde{f}+\theta f\Big)dt+z_i^\theta dW_i(t)+\displaystyle\sum_{j=1,j\neq i}^N z_{ij}^\theta dW_j(t),\\
y_i^\theta(T)=\xi_i,
\end{cases}$$

where $y^{(N),\theta}=\frac{1}{N}\sum_{i=1}^N y_i^\theta$, $\widetilde{y}^{(N)}=\frac{1}{N}\sum_{i=1}^N\widetilde{y}_i$. Therefore,

$$\begin{aligned}
&-J_{\rm soc}(u(\cdot);\widetilde{f}+\theta f)+J_{\rm soc}(u(\cdot);\widetilde{f})\\
&=-\frac{1}{2}\sum_{i=1}^N\mathbb{E}\Big[\int_0^T\Big(\|y_i^\theta-\Gamma_1 y^{(N),\theta}\|^2_Q-\|\widetilde{y}_i-\Gamma_1\widetilde{y}^{(N)}\|^2_Q+\|z_i^\theta\|^2_H-\|\widetilde{z}_i\|^2_H-\|\widetilde{f}+\theta f\|^2_{R_0}+\|\widetilde{f}\|^2_{R_0}\Big)dt\\
&\quad+\|y_i^\theta(0)-\Gamma_2 y^{(N),\theta}(0)\|^2_G-\|\widetilde{y}_i(0)-\Gamma_2\widetilde{y}^{(N)}(0)\|^2_G\Big]\\
&=-\frac{1}{2}\sum_{i=1}^N\Big\{\Big[\mathbb{E}\int_0^T\Big(\|Y_i-\Gamma_1Y^{(N)}\|^2_Q+\|Z_i\|^2_H-\|f\|^2_{R_0}\Big)dt+\|Y_i(0)-\Gamma_2Y^{(N)}(0)\|^2_G\Big]\theta^2\\
&\quad+2\Big[\mathbb{E}\int_0^T\Big(\Big\langle Q(Y_i-\Gamma_1Y^{(N)}),\widetilde{y}_i-\Gamma_1\widetilde{y}^{(N)}\Big\rangle+\langle HZ_i,\widetilde{z}_i\rangle-\langle R_0 f,\widetilde{f}\rangle\Big)dt\\
&\quad+\Big\langle G(Y_i(0)-\Gamma_2Y^{(N)}(0)),\widetilde{y}_i(0)-\Gamma_2\widetilde{y}^{(N)}(0)\Big\rangle\Big]\theta\Big\},
\end{aligned}\tag{6}$$

where

$$Y_i=\frac{y_i^\theta-\widetilde{y}_i}{\theta},\ Z_i=\frac{z_i^\theta-\widetilde{z}_i}{\theta},\ Z_{ij}=\frac{z_{ij}^\theta-\widetilde{z}_{ij}}{\theta},\ Y^{(N)}(t)=\frac{1}{N}\sum_{i=1}^N Y_i(t)$$

and

$$\begin{cases}
dY_i=-\Big(AY_i+CY^{(N)}+f\Big)dt+Z_idW_i(t)+\displaystyle\sum_{j=1,j\neq i}^N Z_{ij}dW_j(t),\\
Y_i(T)=0.
\end{cases}$$

Introducing the following adjoint equation:

$$\begin{cases} dp_i = \alpha_i dt + \beta_i dW_i(t), \\ p_i(0) = -G\widetilde{y}_i(0) + (G\Gamma_2 - \Gamma_2^\top G\Gamma_2 + \Gamma_2^\top G)\widetilde{y}^{(N)}(0) \end{cases}$$

and $p^{(N)} = \frac{1}{N}\sum_{i=1}^N p_i$. Applying Itô's formula to $\langle Y_i, p_i\rangle$ and combining with (6), we have

$$\begin{aligned} &- J_{soc}(u(\cdot); \widetilde{f} + \theta f) + J_{soc}(u(\cdot); \widetilde{f}) \\ &= -\frac{1}{2}\sum_{i=1}^N \Big[\mathbb{E}\int_0^T \Big(\|Y_i - \Gamma_1 Y^{(N)}\|_Q^2 + \|Z_i\|_H^2 - \|f\|_{R_0}^2\Big)dt + \|Y_i(0) - \Gamma_2 Y^{(N)}(0)\|_G^2\Big]\theta^2 \\ &\quad - \sum_{i=1}^N \Big[\mathbb{E}\int_0^T \Big(\langle Q(Y_i - \Gamma_1 Y^{(N)}), \widetilde{y}_i - \Gamma_1\widetilde{y}^{(N)}\rangle + \langle HZ_i, \widetilde{z}_i\rangle - \langle R_0 f, \widetilde{f}\rangle\Big)dt \\ &\quad + \Big\langle G(Y_i(0) - \Gamma_2 Y^{(N)}(0)), \widetilde{y}_i(0) - \Gamma_2\widetilde{y}^{(N)}(0)\Big\rangle\Big]\theta \\ &= -\frac{1}{2}\sum_{i=1}^N \Big[\mathbb{E}\int_0^T \Big(\|Y_i - \Gamma_1 Y^{(N)}\|_Q^2 + \|Z_i\|_H^2 - \|f\|_{R_0}^2\Big)dt + \|Y_i(0) - \Gamma_2 Y^{(N)}(0)\|_G^2\Big]\theta^2 \\ &\quad - \sum_{i=1}^N \Big[\mathbb{E}\int_0^T \Big(\langle QY_i, \widetilde{y}_i - \Gamma_1\widetilde{y}^{(N)}\rangle - \langle Y_i, \Gamma_1^\top Q(I - \Gamma_1)\widetilde{y}^{(N)}\rangle + \langle HZ_i, \widetilde{z}_i\rangle - \langle R_0 f, \widetilde{f}\rangle - \langle AY_i, p_i\rangle \\ &\quad - \langle CY_i, p^{(N)}\rangle - \langle f, p_i\rangle + \langle Y_i, \alpha_i\rangle + \langle \beta_i, Z_i\rangle\Big)dt\Big]\theta. \end{aligned}$$

Let

$$\alpha_i = A^\top p_i + C^\top p^{(N)} - Q\widetilde{y}_i + (Q\Gamma_1 - \Gamma_1^\top Q\Gamma_1 + \Gamma_1^\top Q)\widetilde{y}^{(N)},\ \ \beta_i = -H\widetilde{z}_i.$$

We have

$$\begin{aligned} -J_{soc}(u(\cdot); \widetilde{f} + \theta f) + J_{soc}(u(\cdot); \widetilde{f}) =& \frac{1}{2}\sum_{i=1}^N \Big[\mathbb{E}\int_0^T \Big(-\|Y_i - \Gamma_1 Y^{(N)}\|_Q^2 - \|Z_i\|_H^2 + \|f\|_{R_0}^2\Big)dt \\ & - \|Y_i(0) - \Gamma_2 Y^{(N)}(0)\|_G^2\Big]\theta^2 + \sum_{i=1}^N \Big[\mathbb{E}\int_0^T \langle f, R_0\widetilde{f} + p_i\rangle dt\Big]\theta. \end{aligned}$$

Then $\widetilde{f}$ is a minimizer of $-J_{soc}(u(\cdot); f)$ if

$$\sum_{i=1}^N \Big[\mathbb{E}\int_0^T \Big(-\|Y_i - \Gamma_1 Y^{(N)}\|_Q^2 - \|Z_i\|_H^2 + \|f\|_{R_0}^2\Big)dt - \|Y_i(0) - \Gamma_2 Y^{(N)}(0)\|_G^2\Big] \geqslant 0$$

and

$$\sum_{i=1}^N \mathbb{E}\int_0^T \langle f, R_0\widetilde{f} + p_i\rangle dt = 0.$$

Therefore, $\widetilde{f} = -\frac{1}{N}R_0^{-1}\sum_{i=1}^N p_i = -R_0^{-1}p^{(N)}$, where

$$\begin{cases} d\widetilde{y}_i = -\Big(A\widetilde{y}_i + Bu_i + C\widetilde{y}^{(N)} - R_0^{-1}p^{(N)}\Big)dt + \widetilde{z}_i dW_i(t) + \sum\limits_{j=1, j\neq i}^N \widetilde{z}_{ij} dW_j(t), \\ dp_i = \Big(A^\top p_i + C^\top p^{(N)} - Q\widetilde{y}_i + (Q\Gamma_1 - \Gamma_1^\top Q\Gamma_1 + \Gamma_1^\top Q)\widetilde{y}^{(N)}\Big)dt - H\widetilde{z}_i dW_i(t), \\ \widetilde{y}_i(T) = \xi_i, p_i(0) = -G\widetilde{y}_i(0) + (G\Gamma_2 - \Gamma_2^\top G\Gamma_2 + \Gamma_2^\top G)\widetilde{y}^{(N)}(0). \end{cases}$$

4 Stochastic optimal control problem for the agents $\mathcal{A}_i$

For simplicity, let

$$\bar{\Gamma}_1 = Q\Gamma_1 - \Gamma_1^\top Q\Gamma_1 + \Gamma_1^\top Q,\ \bar{\Gamma}_2 = G\Gamma_2 - \Gamma_2^\top G\Gamma_2 + \Gamma_2^\top G.$$

In Section 3, the robust disturbance is determined. Then the state equation of the agent can be written as

$$\begin{cases} dy_i = -\Big(Ay_i + Bu_i + Cy^{(N)} - R_0^{-1}p^{(N)}\Big)dt + z_i dW_i(t) + \displaystyle\sum_{j=1,j\neq i}^{N} z_{ij}dW_j(t), y_i(T) = \xi_i, \\ dp_i = \Big(A^\top p_i + C^\top p^{(N)} - Qy_i + \bar{\Gamma}_1 y^{(N)}\Big)dt - Hz_i dW_i(t), p_i(0) = -Gy_i(0) + \bar{\Gamma}_2 y^{(N)}(0), \end{cases} \tag{7}$$

and the corresponding cost functionals become

$$J_i(u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(\|u_i\|_{R_1}^2 + \|y_i - \Gamma_1 y^{(N)}\|_Q^2 + \|z_i\|_H^2 - \|p^{(N)}\|_{R_0^{-1}}^2\Big)dt + \|y_i(0) - \Gamma_2 y^{(N)}(0)\|_G^2\Big]$$

and

$$J_{soc}^{wo}(u(\cdot)) = \sum_{i=1}^{N} J_i(u_i(\cdot), u_{-i}(\cdot)).$$

Since the agents focus on minimizing the social cost instead of their individual costs, a variational analysis is essential to quantify the total variation in social cost, $\delta J_{\rm soc}^{\rm wo}(\delta u_i)$, triggered by individual variation δu_i of a generic agent $\mathcal{A}_i$. This type of analysis is not necessary in a cooperative framework and highlights the key distinction between MFG and MFT.

Remark 1. Based on Assumptions 1–4 and referring to Lemma 2.1 in [30], we can further establish the uniform convexity of the social cost functional $J_{soc}^{wo}(u(\cdot))$ with respect to u.

4.1 Person-by-person optimality

Let $\{\bar{u}_i, \bar{u}_i \in \mathcal{U}_i^c\}_{i=1}^N$ be centralized optimal strategy of all agents. In order to quantify (total) variation $\delta J_{soc}^{wo}(\delta u_i)$ owing to basic δu_i by a generic agent $\mathcal{A}_i$, we need to consider the perturbation that the agent $\mathcal{A}_i$ uses the strategy $u_i \in \mathcal{U}_i^c$ and all the other agents use the strategy $\bar{u}_{-i} = (\bar{u}_1, \cdots, \bar{u}_{i-1}, \bar{u}_{i+1}, \cdots, \bar{u}_N)$. The realized states (7) corresponding to $(u_i, \bar{u}_{-i})$ and $(\bar{u}_i, \bar{u}_{-i})$ are defined as $(y_1, \cdots, y_N)$ and $(\bar{y}_1, \cdots, \bar{y}_N)$ respectively. For $j = 1, \cdots, N$, define the perturbation

$$\delta u_j = u_j - \bar{u}_j,\ \delta y_j = y_j - \bar{y}_j,\ \delta z_j = z_j - \bar{z}_j,\ \delta z_{ij} = z_{ij} - \bar{z}_{ij},\ \delta p_j = p_j - \bar{p}_j,\ \delta\mathcal{J}_j = J_j(u_i, \bar{u}_{-i}) - J_j(\bar{u}_i, \bar{u}_{-i}).$$

Defining $\delta y^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta y_i, \delta p^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta p_i$, the variation of the state for $\mathcal{A}_i$ is given by

$$\begin{cases} d\delta y_i = -\Big(A\delta y_i + B\delta u_i + C\delta y^{(N)} - R_0^{-1}\delta p^{(N)}\Big)dt + \delta z_i dW_i(t) + \displaystyle\sum_{k=1,k\neq i}^{N} \delta z_{ik}dW_k(t), \\ d\delta p_i = \Big(A^\top \delta p_i + C^\top \delta p^{(N)} - Q\delta y_i + \bar{\Gamma}_1 \delta y^{(N)}\Big)dt - H\delta z_i dW_i(t), \\ \delta y_i(T) = 0, \quad \delta p_i(0) = -G\delta y_i(0) + \bar{\Gamma}_2 \delta y^{(N)}(0), \end{cases} \tag{8}$$

and for $\mathcal{A}_j$, $j \neq i$,

$$\begin{cases} d\delta y_j = -\Big(A\delta y_j + C\delta y^{(N)} - R_0^{-1}\delta p^{(N)}\Big)dt + \delta z_j dW_j(t) + \displaystyle\sum_{k=1,k\neq j}^{N} \delta z_{jk}dW_k(t), \\ d\delta p_j = \Big(A^\top \delta p_j + C^\top \delta p^{(N)} - Q\delta y_j + \bar{\Gamma}_1 \delta y^{(N)}\Big)dt - H\delta z_j dW_j(t), \\ \delta y_j(T) = 0, \quad \delta p_j(0) = -G\delta y_j(0) + \bar{\Gamma}_2 \delta y^{(N)}(0). \end{cases}$$

Defining $\delta y_{-i}=\sum_{j=1,j\neq i}^{N}\delta y_j$ and $\delta p_{-i}=\sum_{j=1,j\neq i}^{N}\delta p_j$, thus

$$\begin{cases} d\delta y_{-i}=-\Big(A\delta y_{-i}+(N-1)C\delta y^{(N)}-(N-1)R_0^{-1}\delta p^{(N)}\Big)dt+\sum\limits_{j=1,j\neq i}^{N}\delta z_j dW_j(t)+\sum\limits_{j=1,j\neq i}^{N}\sum\limits_{k=1,k\neq j}^{N}\delta z_{jk}dW_k(t),\\ d\delta p_{-i}=\Big(A^\top\delta p_{-i}+(N-1)C^\top\delta p^{(N)}-Q\delta y_{-i}+(N-1)\bar\Gamma_1\delta y^{(N)}\Big)dt-\sum\limits_{j=1,j\neq i}^{N}H\delta z_j dW_j(t),\\ \delta y_{-i}(T)=0,\quad \delta p_{-i}(0)=-G\delta y_{-i}(0)+(N-1)\bar\Gamma_2\delta y^{(N)}(0). \end{cases} \tag{9}$$

By some elementary calculations, we can further obtain the variation of the cost functional of $\mathcal{A}_i$ as

$$\begin{aligned} \delta J_i=\mathbb{E}\Big[\int_0^T\Big(&\langle R_1\bar u_i,\delta u_i\rangle+\langle Q(\bar y_i-\Gamma_1\bar y^{(N)}),\delta y_i-\Gamma_1\delta y^{(N)}\rangle+\langle H\bar z_i,\delta z_i\rangle-\langle R_0^{-1}\bar p^{(N)},\delta p^{(N)}\rangle\Big)dt\\ &+\langle G(\bar y_i(0)-\Gamma_2\bar y^{(N)}(0)),\delta y_i(0)-\Gamma_2\delta y^{(N)}(0)\rangle\Big]. \end{aligned}$$

For $j\neq i$, the variation of the cost functional of $\mathcal{A}_j$ is given by

$$\begin{aligned} \delta J_j=\mathbb{E}\Big[\int_0^T\Big(&\langle Q(\bar y_j-\Gamma_1\bar y^{(N)}),\delta y_j-\Gamma_1\delta y^{(N)}\rangle+\langle H\bar z_j,\delta z_j\rangle-\langle R_0^{-1}\bar p^{(N)},\delta p^{(N)}\rangle\Big)dt\\ &+\langle G(\bar y_j(0)-\Gamma_2\bar y^{(N)}(0)),\delta y_j(0)-\Gamma_2\delta y^{(N)}(0)\rangle\Big]. \end{aligned}$$

Therefore, the variation of the social cost satisfies

$$\begin{aligned} \delta\mathcal{J}_{soc}^{wo}=\mathbb{E}\Big[\int_0^T\Big(&\langle R_1\bar u_i,\delta u_i\rangle+\sum_{j=1}^{N}\langle Q(\bar y_j-\Gamma_1\bar y^{(N)}),\delta y_j-\Gamma_1\delta y^{(N)}\rangle+\sum_{j=1}^{N}\langle H\bar z_j,\delta z_j\rangle\\ &-N\langle R_0^{-1}\bar p^{(N)},\delta p^{(N)}\rangle\Big)dt+\sum_{j=1}^{N}\langle G(\bar y_j(0)-\Gamma_2\bar y^{(N)}(0)),\delta y_j(0)-\Gamma_2\delta y^{(N)}(0)\rangle\Big]. \end{aligned} \tag{10}$$

Replacing $(\bar y^{(N)},\bar p^{(N)})$ in (10) by some mean-field term $(\hat y,\hat p)$ which will be determined later,

$$\begin{aligned} \delta\mathcal{J}_{soc}^{(N)}=\mathbb{E}\Big[\int_0^T\Big(&\langle R_1\bar u_i,\delta u_i\rangle+\langle Q(\bar y_i-\Gamma_1\bar y^{(N)}),\delta y_i-\Gamma_1\delta y^{(N)}\rangle+\sum_{j\neq i}\langle Q(\bar y_j-\Gamma_1\bar y^{(N)}),\delta y_j-\Gamma_1\delta y^{(N)}\rangle\\ &+\langle H\bar z_i,\delta z_i\rangle+\sum_{j\neq i}\langle H\bar z_j,\delta z_j\rangle-N\langle R_0^{-1}\bar p^{(N)},\delta p^{(N)}\rangle\Big)dt+\langle G(\bar y_i(0)-\Gamma_2\bar y^{(N)}(0)),\delta y_i(0)-\Gamma_2\delta y^{(N)}(0)\rangle\\ &+\sum_{j\neq i}\langle G(\bar y_j(0)-\Gamma_2\bar y^{(N)}(0)),\delta y_j(0)-\Gamma_2\delta y^{(N)}(0)\rangle\Big]\\ =\mathbb{E}\Big[\int_0^T\Big(&\langle R_1\bar u_i,\delta u_i\rangle+\langle Q\bar y_i,\delta y_i\rangle+\langle(-Q\Gamma_1-\Gamma_1^\top Q+\Gamma_1^\top Q\Gamma_1)\hat y,\delta y_i\rangle+\frac{1}{N}\sum_{j\neq i}\langle Q\bar y_j,N\delta y_j\rangle\\ &+\sum_{j\neq i}\langle(-Q\Gamma_1-\Gamma_1^\top Q+\Gamma_1^\top Q\Gamma_1)\hat y,\delta y_j\rangle+\langle H\bar z_i,\delta z_i\rangle+\sum_{j\neq i}\langle H\bar z_j,\delta z_j\rangle-N\langle R_0^{-1}\hat p,\delta p^{(N)}\rangle\Big)dt\\ &+\langle G\bar y_i(0),\delta y_i(0)\rangle+\langle(-G\Gamma_2-\Gamma_2^\top G+\Gamma_2^\top G\Gamma_2)\hat y(0),\delta y_i(0)\rangle\\ &+\frac{1}{N}\sum_{j\neq i}\langle G\bar y_j(0),N\delta y_j(0)\rangle+\sum_{j\neq i}\langle(-G\Gamma_2-\Gamma_2^\top G+\Gamma_2^\top G\Gamma_2)\hat y(0),\delta y_j(0)\rangle\Big]+\sum_{l=1}^{3}\varepsilon_l, \end{aligned}$$

where

$$\begin{cases} \varepsilon_1=\mathbb{E}\displaystyle\int_0^T\langle(Q\Gamma_1+\Gamma_1^\top Q-\Gamma_1^\top Q\Gamma_1)(\hat y-\bar y^{(N)}),N\delta y^{(N)}\rangle dt,\\ \varepsilon_2=\mathbb{E}\displaystyle\int_0^T\langle R_0^{-1}(\hat p-\bar p^{(N)}),N\delta p^{(N)}\rangle dt,\\ \varepsilon_3=\mathbb{E}\langle(G\Gamma_2+\Gamma_2^\top G-\Gamma_2^\top G\Gamma_2)(\hat y(0)-\bar y^{(N)}(0)),N\delta y^{(N)}(0)\rangle. \end{cases}$$

Introduce the limit (y^{**}, p^{**}) to replace $(\delta y_{-i}, \delta p_{-i})$, and for $j \neq i$, introduce the limit (y_j^*, z_j^*, z_{jk}^*) to replace $(N\delta y_j, N\delta z_j, N\delta z_{jk})$, where

$$\begin{cases} dy^{**} = -\Big[(A+C)y^{**} - R_0^{-1}p^{**} + C\delta y_i - R_0^{-1}\delta p_i\Big]dt + \sum_{k=1}^{N} z_k^{**} dW_k(t), \\ dp^{**} = \Big[(A+C)^\top p^{**} + (\bar{\Gamma}_1 - Q)y^{**} + C^\top \delta p_i + \bar{\Gamma}_1 \delta y_i\Big]dt, \\ dy_j^* = -\Big[Ay_j^* + Cy^{**} - R_0^{-1}p^{**} + C\delta y_i - R_0^{-1}\delta p_i\Big]dt + z_j^* dW_j(t) + \sum_{k=1, k\neq j}^{N} z_{jk}^* dW_k(t), \\ y^{**}(T) = 0, \quad p^{**}(0) = -Gy^{**}(0) + \bar{\Gamma}_2 \delta y_i(0) + \bar{\Gamma}_2 y^{**}(0), \quad y_j^*(T) = 0. \end{cases} \tag{11}$$

Therefore,

$$\begin{aligned} \delta \mathcal{J}_{soc}^{(N)} = \mathbb{E}\Big[\int_0^T \Big(& \langle R_1 \bar{u}_i, \delta u_i\rangle + \langle Q\bar{y}_i, \delta y_i\rangle + \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, \delta y_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle Q\bar{y}_j, N\delta y_j\rangle \\ &+ \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, \delta y_{-i}\rangle + \langle H\bar{z}_i, \delta z_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle H\bar{z}_j, N\delta z_j\rangle - \langle R_0^{-1}\hat{p}, \delta p_i\rangle \\ &- \sum_{j\neq i}\langle R_0^{-1}\hat{p}, \delta p_j\rangle\Big)dt + \langle G\bar{y}_i(0), \delta y_i(0)\rangle + \langle(-G\Gamma_2 - \Gamma_2^\top G + \Gamma_2^\top G\Gamma_2)\hat{y}(0), \delta y_i(0)\rangle \\ &+ \frac{1}{N}\sum_{j\neq i}\langle G\bar{y}_j(0), N\delta y_j(0)\rangle + \langle(-G\Gamma_2 - \Gamma_2^\top G + \Gamma_2^\top G\Gamma_2)\hat{y}(0), \delta y_{-i}(0)\rangle\Big] + \sum_{l=1}^{3}\varepsilon_l \\ = \mathbb{E}\Big[\int_0^T \Big(& \langle R_1 \bar{u}_i, \delta u_i\rangle + \langle Q\bar{y}_i, \delta y_i\rangle + \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, \delta y_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle Q\bar{y}_j, y_j^*\rangle, \\ &+ \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, y^{**}\rangle + \langle H\bar{z}_i, \delta z_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle H\bar{z}_j, z_j^*\rangle - \langle R_0^{-1}\hat{p}, \delta p_i\rangle \\ &- \langle R_0^{-1}\hat{p}, p^{**}\rangle\Big)dt + \langle G\bar{y}_i(0), \delta y_i(0)\rangle + \langle(-G\Gamma_2 - \Gamma_2^\top G + \Gamma_2^\top G\Gamma_2)\hat{y}(0), \delta y_i(0)\rangle \\ &+ \frac{1}{N}\sum_{j\neq i}\langle G\bar{y}_j(0), y_j^*(0)\rangle + \langle(-G\Gamma_2 - \Gamma_2^\top G + \Gamma_2^\top G\Gamma_2)\hat{y}(0), y^{**}(0)\rangle\Big] + \sum_{l=1}^{9}\varepsilon_l, \end{aligned}$$

where

$$\begin{cases} \varepsilon_4 = \mathbb{E}\int_0^T \langle(Q\Gamma_1 + \Gamma_1^\top Q - \Gamma_1^\top Q\Gamma_1)\hat{y}, y^{**} - \delta y_{-i}\rangle dt, \\ \varepsilon_5 = \mathbb{E}\int_0^T \frac{1}{N}\sum_{j\neq i}\langle Q\bar{y}_j, N\delta y_j - y_j^*\rangle dt, \\ \varepsilon_6 = \mathbb{E}\int_0^T \frac{1}{N}\sum_{j\neq i}\langle H\bar{z}_j, N\delta z_j - z_j^*\rangle dt, \\ \varepsilon_7 = \mathbb{E}\int_0^T \langle R_0^{-1}\hat{p}, p^{**} - \delta p_{-i}\rangle dt, \\ \varepsilon_8 = \mathbb{E}\langle(G\Gamma_2 + \Gamma_2^\top G - \Gamma_2^\top G\Gamma_2)\hat{y}(0), y^{**}(0) - \delta y_{-i}(0)\rangle, \\ \varepsilon_9 = \mathbb{E}\Big(\frac{1}{N}\sum_{j\neq i}\langle G\bar{y}_j(0), N\delta y_j(0) - y_j^*(0)\rangle\Big). \end{cases}$$

For $j \neq i$, introducing the following adjoint equations (x_1^j, x_2, q_2) to replace (y_j^*, y^{**}, p^{**}),

$$\begin{cases} dx_1^j = \alpha_1^j dt + \beta_1^j dW_j(t), & x_1^j(0) = G\bar{y}_j(0), \\ dx_2 = \alpha_2 dt + \displaystyle\sum_{k=1}^K \beta_2^k dW_k(t), & x_2(0) = \bar{\Gamma}_2 \hat{y}(0) + Gq_2(0) - \bar{\Gamma}_2 q_2(0), \\ dq_2 = \gamma_2 dt + \displaystyle\sum_{k=1}^K \eta_2^k dW_k(t), & q_2(T) = 0. \end{cases} \tag{12}$$

Applying Itô's formula to $\langle x_1^j, y_j^* \rangle$, we have

$$\begin{aligned} &- \mathbb{E}\langle G\bar{y}_j(0), y_j^*(0)\rangle \\ &= \mathbb{E}\int_0^T \Big[\langle -A^\top x_1^j + \alpha_1^j, y_j^*\rangle + \langle \beta_1^j, z_j^*\rangle - \langle C^\top x_1^j, y^{**}\rangle + \langle R_0^{-1} x_1^j, p^{**}\rangle - \langle C^\top x_1^j, \delta y_i\rangle + \langle R_0^{-1} x_1^j, \delta p_i\rangle\Big] dt. \end{aligned}$$

Applying Itô's formula to $\langle x_2, y^{**}\rangle$, we have

$$\begin{aligned} &- \mathbb{E}\langle \bar{\Gamma}_2\hat{y}(0) + Gq_2(0) - \bar{\Gamma}_2 q_2(0), y^{**}(0)\rangle \\ &= \mathbb{E}\int_0^T \Big[- \langle x_2, (A+C)y^{**}\rangle + \langle x_2, R_0^{-1}p^{**}\rangle - \langle x_2, C\delta y_i\rangle + \langle x_2, R_0^{-1}\delta p_i\rangle + \langle \alpha_2, y^{**}\rangle + \sum_{k=1}^N \langle \beta_2^k, z_k^{**}\rangle\Big] dt. \end{aligned}$$

Applying Itô's formula to $\langle q_2, p^{**}\rangle$, we have

$$\begin{aligned} &\mathbb{E}\langle q_2(0), Gy^{**}(0) - \bar{\Gamma}_2\delta y_i(0) - \bar{\Gamma}_2 y^{**}(0)\rangle \\ &= \mathbb{E}\int_0^T \Big[\langle q_2, (A+C)^\top p^{**}\rangle + \langle q_2, (\bar{\Gamma}_1 - Q)y^{**}\rangle + \langle q_2, C^\top \delta p_i\rangle + \langle q_2, \bar{\Gamma}_1 \delta y_i\rangle + \langle \gamma_2, p^{**}\rangle\Big] dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \delta\mathcal{J}_{soc}^{(N)} = \mathbb{E}\Big[\int_0^T &\Big(\langle R_1\bar{u}_i, \delta u_i\rangle + \langle Q\bar{y}_i, \delta y_i\rangle + \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, \delta y_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle Q\bar{y}_j, y_j^*\rangle \\ &+ \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, y^{**}\rangle + \langle H\bar{z}_i, \delta z_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle H\bar{z}_j, z_j^*\rangle - \langle R_0^{-1}\hat{p}, \delta p_i\rangle \\ &- \langle R_0^{-1}\hat{p}, p^{**}\rangle\Big) dt + \langle G\bar{y}_i(0), \delta y_i(0)\rangle - \langle \bar{\Gamma}_2\hat{y}(0), \delta y_i(0)\rangle + \frac{1}{N}\sum_{j\neq i}\langle G\bar{y}_j(0), y_j^*(0)\rangle \\ &+ \langle \bar{\Gamma}_2\hat{y}(0), y^{**}(0)\rangle\Big] - \frac{1}{N}\sum_{j\neq i}\mathbb{E}\langle G\bar{y}_j(0), y_j^*(0)\rangle \\ &- \frac{1}{N}\sum_{j\neq i}\mathbb{E}\int_0^T \Big[\langle -A^\top x_1^j + \alpha_1^j, y_j^*\rangle + \langle \beta_1^j, z_j^*\rangle - \langle C^\top x_1^j, y^{**}\rangle + \langle R_0^{-1}x_1^j, p^{**}\rangle - \langle C^\top x_1^j, \delta y_i\rangle \\ &+ \langle R_0^{-1}x_1^j, \delta p_i\rangle\Big] dt + \mathbb{E}\langle \bar{\Gamma}_2\hat{y}(0) + Gq_2(0) - \bar{\Gamma}_2 q_2(0), y^{**}(0)\rangle \\ &+ \mathbb{E}\int_0^T \Big[- \langle x_2, (A+C)y^{**}\rangle + \langle x_2, R_0^{-1}p^{**}\rangle - \langle x_2, C\delta y_i\rangle + \langle x_2, R_0^{-1}\delta p_i\rangle + \langle \alpha_2, y^{**}\rangle \\ &+ \sum_{k=1}^N \langle \beta_2^k, z_k^{**}\rangle\Big] dt + \mathbb{E}\langle q_2(0), -Gy^{**}(0) + \bar{\Gamma}_2\delta y_i(0) + \bar{\Gamma}_2 y^{**}(0)\rangle + \mathbb{E}\int_0^T \Big[\langle q_2, (A+C)^\top p^{**}\rangle \\ &+ \langle q_2, (\bar{\Gamma}_1 - Q)y^{**}\rangle + \langle q_2, C^\top\delta p_i\rangle + \langle q_2, \bar{\Gamma}_1\delta y_i\rangle + \langle \gamma_2, p^{**}\rangle\Big] dt + \sum_{l=1}^9 \varepsilon_l \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\int_0^T \Big(\langle R_1\bar{u}_i, \delta u_i\rangle + \langle Q\bar{y}_i, \delta y_i\rangle + \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, \delta y_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle Q\bar{y}_j, y_j^*\rangle \\
&\quad + \langle(-Q\Gamma_1 - \Gamma_1^\top Q + \Gamma_1^\top Q\Gamma_1)\hat{y}, y^{**}\rangle + \langle H\bar{z}_i, \delta z_i\rangle + \frac{1}{N}\sum_{j\neq i}\langle H\bar{z}_j, z_j^*\rangle - \langle R_0^{-1}\hat{p}, \delta p_i\rangle \\
&\quad - \langle R_0^{-1}\hat{p}, p^{**}\rangle\Big)dt + \langle G\bar{y}_i(0), \delta y_i(0)\rangle - \langle\bar{\Gamma}_2\hat{y}(0), \delta y_i(0)\rangle + \langle\bar{\Gamma}_2 q_2(0), \delta y_i(0)\rangle\Big] \\
&\quad - \frac{1}{N}\sum_{j\neq i}\mathbb{E}\int_0^T\Big[\langle -A^\top x_1^j + \alpha_1^j, y_j^*\rangle + \langle\beta_1^j, z_j^*\rangle - \langle C^\top x_1^j, y^{**}\rangle + \langle R_0^{-1}x_1^j, p^{**}\rangle \\
&\quad - \langle C^\top x_1^j, \delta y_i\rangle + \langle R_0^{-1}x_1^j, \delta p_i\rangle\Big]dt + \mathbb{E}\int_0^T\Big[-\langle x_2, (A+C)y^{**}\rangle + \langle x_2, R_0^{-1}p^{**}\rangle - \langle x_2, C\delta y_i\rangle \\
&\quad + \langle x_2, R_0^{-1}\delta p_i\rangle + \langle\alpha_2, y^{**}\rangle + \sum_{k=1}^N\langle\beta_2^k, z_k^{**}\rangle\Big]dt + \mathbb{E}\int_0^T\Big[\langle q_2, (A+C)^\top p^{**}\rangle + \langle q_2, (\bar{\Gamma}_1 - Q)y^{**}\rangle \\
&\quad + \langle q_2, C^\top\delta p_i\rangle + \langle q_2, \bar{\Gamma}_1\delta y_i\rangle + \langle\gamma_2, p^{**}\rangle\Big]dt + \sum_{l=1}^9\varepsilon_l.
\end{aligned}
$$

Let

$$
\begin{cases}
\alpha_1^j = Q\bar{y}_j + A^\top x_1^j, \\
\alpha_2 = \bar{\Gamma}_1\hat{y} - C^\top\mathbb{E}x_1^1 + (A+C)^\top x_2 - (\bar{\Gamma}_1 - Q)^\top q_2, \\
\beta_1^j = H\bar{z}_j, \\
\beta_2^k = 0, \\
\gamma_2 = R_0^{-1}\hat{p} + R_0^{-1}\mathbb{E}x_1^1 - R_0^{-1}x_2 - (A+C)q_2,
\end{cases}
\tag{13}
$$

then

$$
\begin{aligned}
\delta\mathcal{J}_{soc}^{(N)} =& \mathbb{E}\Big[\int_0^T\Big(\langle R_1\bar{u}_i, \delta u_i\rangle + \langle Q\bar{y}_i, \delta y_i\rangle + \langle -\bar{\Gamma}_1\hat{y} + C^\top\mathbb{E}x_1^1 - C^\top x_2 + \bar{\Gamma}_1 q_2, \delta y_i\rangle + \langle H\bar{z}_i, \delta z_i\rangle \\
&+ \langle -R_0^{-1}\hat{p} - R_0^{-1}\mathbb{E}x_1^1 + R_0^{-1}x_2 + Cq_2, \delta p_i\rangle\Big)dt + \langle G\bar{y}_i(0) - \bar{\Gamma}_2\hat{y}(0) + \bar{\Gamma}_2 q_2(0), \delta y_i(0)\rangle\Big] + \sum_{l=1}^{11}\varepsilon_l,
\end{aligned}
$$

where

$$
\begin{cases}
\varepsilon_{10} = \mathbb{E}\displaystyle\int_0^T\left(\left\langle C^\top\left(\mathbb{E}x_1^1 - \frac{1}{N}\sum_{j\neq i}x_1^j\right), y^{**}\right\rangle + \left\langle R_0^{-1}\left(\frac{1}{N}\sum_{j\neq i}x_1^j - \mathbb{E}x_1^1\right), p^{**}\right\rangle\right)dt, \\
\varepsilon_{11} = \mathbb{E}\displaystyle\int_0^T\left(\left\langle C^\top\left(\frac{1}{N}\sum_{j\neq i}x_1^j - \mathbb{E}x_1^1\right), \delta y_i\right\rangle + \left\langle R_0^{-1}\left(\mathbb{E}x_1^1 - \frac{1}{N}\sum_{j\neq i}x_1^j\right), \delta p_i\right\rangle\right)dt.
\end{cases}
$$

Note that x_1^j, $j \neq i$ are exchangeable. Hence in the above notations we use x_1^1 when we consider the expectation. Moreover, following (13), the adjoint equation (12) becomes

$$
\begin{cases}
dx_1^j = (Q\bar{y}_j + A^\top x_1^j)dt + H\bar{z}_j dW_j(t), & x_1^j(0) = G\bar{y}_j(0), \\
dx_2 = \Big[\bar{\Gamma}_1\hat{y} - C^\top\hat{x}_1 + (A+C)^\top x_2 - (\bar{\Gamma}_1 - Q)^\top q_2\Big]dt, & x_2(0) = \bar{\Gamma}_2\hat{y}(0) + Gq_2(0) - \bar{\Gamma}_2 q_2(0), \\
dq_2 = \Big[R_0^{-1}\hat{p} + R_0^{-1}\hat{x}_1 - R_0^{-1}x_2 - (A+C)q_2\Big]dt, & q_2(T) = 0,
\end{cases}
\tag{14}
$$

where $\hat{y}, \hat{p}, \hat{x}_1, x_2, q_2$ will be determined by the consistency condition in Section 5. Consequently, we introduce the variation of the decentralized auxiliary cost functional J_i as follows:

$$
\begin{aligned}
\delta J_i = \mathbb{E}\Big[\int_0^T&\Big(\langle R_1\bar{u}_i, \delta u_i\rangle + \langle Q\bar{y}_i, \delta y_i\rangle + \langle -\bar{\Gamma}_1\hat{y} + C^\top\hat{x}_1 - C^\top x_2 + \bar{\Gamma}_1 q_2, \delta y_i\rangle + \langle H\bar{z}_i, \delta z_i\rangle \\
&+ \langle -R_0^{-1}\hat{p} - R_0^{-1}\hat{x}_1 + R_0^{-1}x_2 + Cq_2, \delta p_i\rangle\Big)dt + \langle G\bar{y}_i(0) - \bar{\Gamma}_2\hat{y}(0) + \bar{\Gamma}_2 q_2(0), \delta y_i(0)\rangle\Big].
\end{aligned}
\tag{15}
$$

Note that in (15), the variation only depends on δu_i and δx_i. Therefore we can construct the corresponding auxiliary control problem, from which decentralized strategies can be derived. Furthermore, in the above procedures we only ignore the error term $\varepsilon_l, l = 1, \cdots, 11$. The asymptotic optimality can be guaranteed as long as we can obtain the exact estimations of the error terms, which are provided in Section 6.

4.2 Decentralized strategy

Motivated by (15), introduce the following auxiliary problem.

Problem 3. Minimize $J_i(u_i)$ over $u_i \in \mathcal{U}_i^d$ where

$$\begin{cases} dy_i = -\Big(Ay_i + Bu_i + C\hat{y} - R_0^{-1}\hat{p}\Big)dt + z_i dW_i(t), & y_i(T) = \xi, \\ dp_i = \Big(A^\top p_i + C^\top \hat{p} - Qy_i + \bar{\Gamma}_1 \hat{y}\Big)dt - Hz_i dW_i(t), & p_i(0) = -Gy_i(0) + \bar{\Gamma}_2 \hat{y}(0), \end{cases}$$

and

$$J_i(u_i) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(\langle R_1 u_i, u_i\rangle + \langle Qy_i, y_i\rangle + 2\langle \Theta_1, y_i\rangle + \langle Hz_i, z_i\rangle + 2\langle \Theta_2, p_i\rangle\Big)dt + \langle Gy_i(0), y_i(0)\rangle + 2\langle \Theta_3, y_i(0)\rangle\Big]$$

with $\Theta_1 = -\bar{\Gamma}_1\hat{y} + C^\top \hat{x}_1 - C^\top x_2 + \bar{\Gamma}_1 q_2, \Theta_2 = -R_0^{-1}\hat{p} - R_0^{-1}\hat{x}_1 + R_0^{-1}x_2 + Cq_2, \Theta_3 = -\bar{\Gamma}_2\hat{y}(0) + \bar{\Gamma}_2 q_2(0)$. The above problem is a stochastic optimal control problem with the state being FBSDE. Similar to Section 3, we will apply variation analysis to find the optimal control.

Proposition 1 (Decentralized optimality condition)**.** Under Assumptions 1–4, $\bar{u}_i$ is the optimal control of Problem 2 if and only if $\bar{u}_i$ satisfies

$$\bar{u}_i = R_1^{-1}B^\top X_i,$$

where the adjoint variables (X_i, ρ_i) are governed by

$$\begin{cases} dX_i = (A^\top X_i - Q\bar{y}_i - \Theta_1 + Q\rho_i)dt - H\bar{z}_i dW_i(t), & X_i(0) = G\rho_i(0) - G\bar{y}_i(0) - \Theta_3, \\ d\rho_i = -(A\rho_i + \Theta_2)dt, & \rho_i(T) = 0. \end{cases}$$

Proof. Let $(\bar{y}_i, \bar{z}_i, \bar{p}_i)$ and $(y_i^\theta, z_i^\theta, p_i^\theta)$ denote the states corresponding to $\bar{u}_i$ and $\bar{u}_i + \theta u_i$, respectively, where $u_i \in \mathcal{U}_i^d$ and $\theta \in \mathbb{R}$. Let $Y_i = \frac{y_i^\theta - \bar{y}_i}{\theta}$, $Z_i = \frac{z_i^\theta - \bar{z}_i}{\theta}$, $P_i = \frac{p_i^\theta - \bar{p}_i}{\theta}$, then

$$\begin{cases} dY_i = -\Big(AY_i + Bu_i\Big)dt + Z_i dW_i(t), & Y_i(T) = 0, \\ dP_i = \Big(A^\top P_i - QY_i\Big)dt - HZ_i dW_i(t), & P_i(0) = -GY_i(0). \end{cases}$$

Therefore,

$$\begin{aligned} & J(\bar{u}_i + \theta u_i) - J(\bar{u}_i) \\ =& \frac{1}{2}\Big[\mathbb{E}\int_0^T \Big(\langle R_1 u_i, u_i\rangle + \langle QY_i, Y_i\rangle + \langle HZ_i, Z_i\rangle\Big)dt + \langle GY_i(0), Y_i(0)\rangle\Big]\theta^2 \\ & + \Big[\mathbb{E}\int_0^T \Big(\langle R_1\bar{u}_i, u_i\rangle + \langle QY_i, \bar{y}_i\rangle + \langle \Theta_1, Y_i\rangle + \langle HZ_i, \bar{z}_i\rangle + \langle \Theta_2, P_i\rangle\Big)dt + \langle GY_i(0), \bar{y}_i(0)\rangle + \langle \Theta_3, Y_i(0)\rangle\Big]\theta. \end{aligned}$$

Introducing the following adjoint equation:

$$\begin{cases} dX_i = \alpha_i dt + \beta_i dW_i(t), & X_i(0) = G\rho_i(0) - G\bar{y}_i(0) - \Theta_3, \\ d\rho_i = \gamma_i dt, & \rho_i(T) = 0, \end{cases}$$

and applying Itô's formula to $\langle X_i, Y_i\rangle$ and $\langle P_i, \rho_i\rangle$, we have

$$\begin{aligned} & J(\bar{u}_i + \theta u_i) - J(\bar{u}_i) \\ =& \frac{1}{2}\Big[\mathbb{E}\int_0^T \Big(\langle R_1 u_i, u_i\rangle + \langle QY_i, Y_i\rangle + \langle HZ_i, Z_i\rangle\Big)dt + \langle GY_i(0), Y_i(0)\rangle\Big]\theta^2 + \Big[\mathbb{E}\int_0^T \Big(\langle R_1\bar{u}_i - B^\top X_i, u_i\rangle \\ & + \langle Y_i, \alpha_i + Q\bar{y}_i - A^\top X_i + \Theta_1 - Q\rho_i\rangle + \langle Z_i, \beta_i + H\bar{z}_i\rangle + \langle A\rho_i + \gamma_i + \Theta_2, P_i\rangle\Big)dt\Big]\theta. \end{aligned}$$

Then $\bar{u}_i$ is the optimal control if and only if

$$\mathbb{E}\int_0^T \Big(\langle R_1 u_i, u_i\rangle + \langle QY_i, Y_i\rangle + \langle HZ_i, Z_i\rangle\Big)dt + \langle GY_i(0), Y_i(0)\rangle \geqslant 0$$

and

$$\mathbb{E}\int_0^T \Big(\langle R_1\bar{u}_i - B^\top X_i, u_i\rangle + \langle Y_i, \alpha_i + Q\bar{y}_i - A^\top X_i + \Theta_1 - Q\rho_i\rangle + \langle Z_i, \beta_i + H\bar{z}_i\rangle + \langle A\rho_i + \gamma_i + \Theta_2, P_i\rangle\Big)dt = 0.$$

Therefore, $R_1\bar{u}_i - B^\top X_i = 0, \alpha_i + Q\bar{y}_i - A^\top X_i + \Theta_1 - Q\rho_i = 0, \beta_i + H\bar{z}_i = 0, A\rho_i + \gamma_i + \Theta_2 = 0$. Then the adjoint equation is

$$\begin{cases} dX_i = \Big(A^\top X_i - Q\bar{y}_i - \Theta_1 + Q\rho_i\Big)dt - H\bar{z}_i dW_i(t), & X_i(0) = G\rho_i(0) - G\bar{y}_i(0) - \Theta_3, \\ d\rho_i = -\Big(A\rho_i + \Theta_2\Big)dt, & \rho_i(T) = 0. \end{cases}$$

For the optimal control $\bar{u}_i = R_1^{-1}B^\top X_i$, the related Hamiltonian system becomes

$$\begin{cases} d\bar{y}_i = -\Big(A\bar{y}_i + BR_1^{-1}B^\top X_i + C\hat{y} - R_0^{-1}\hat{p}\Big)dt + \bar{z}_i dW_i(t), & \bar{y}_i(T) = \xi, \\ d\bar{p}_i = \Big(A^\top \bar{p}_i - Q\bar{y}_i + C^\top\hat{p} + \bar{\Gamma}_1\hat{y}\Big)dt - H\bar{z}_i dW_i(t), & \bar{p}_i(0) = -G\bar{y}_i(0) + \bar{\Gamma}_2\hat{y}(0), \\ d\rho_i = -\Big(A\rho_i + \Theta_2\Big)dt, & \rho_i(T) = 0, \\ dX_i = \Big(A^\top X_i + Q\rho_i - Q\bar{y}_i - \Theta_1\Big)dt - H\bar{z}_i dW_i(t), & X_i(0) = G\rho_i(0) - G\bar{y}_i(0) - \Theta_3. \end{cases} \tag{16}$$

5 Consistency condition

Note that the optimal decentralized strategy for the auxiliary control problem involves some undetermined terms. In this section, we will characterize these terms, particularly the frozen state-average limit $\hat{y}$, using some consistency matching scheme or fixed-point principle. Under the standard assumption in the mean-field framework that all agents are independent and identically distributed (i.i.d.), and governed by the same dynamics and cost structures, the agents are statistically symmetric and exchangeable, and therefore the empirical average over the population converges to the expected value of any single agent as the population size tends to infinity. By combining the consistency condition and law of large number, we can derive the limiting consistency condition system through identifying $\hat{y} = \mathbb{E}\bar{y}_i$ and integrating (16) with (14) [50]. Given that all agents are statistically identical in the distribution sense, we will use a generic Brownian motion $W(\cdot)$ to represent the consistency condition system in the following analysis.

Proposition 2. The parameters in problem 3 can be determined by

$$(\hat{y}, \hat{p}, \hat{x}_1, x_2, q_2) = \Big(\mathbb{E}\check{y}, \mathbb{E}\check{p}, \mathbb{E}\check{x}_1, \check{x}_2, \check{q}_2\Big),$$

where $(\check{y}, \check{z}, \check{p}, \check{X}, \check{\rho}, \check{q}_2, \check{x}_1, \check{x}_2) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times d}) \times L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is the solution of the following mean-field FBSDE:

$$\begin{cases} d\check{y} = -\Big(A\check{y} + BR_1^{-1}B^\top\check{X} + C\mathbb{E}\check{y} - R_0^{-1}\mathbb{E}\check{p}\Big)dt + \check{z}dW(t), \quad \check{y}(T) = \xi, \\ d\check{p} = \Big(A^\top\check{p} - Q\check{y} + C^\top\mathbb{E}\check{p} + \bar{\Gamma}_1\mathbb{E}\check{y}\Big)dt - H\check{z}dW(t), \quad \check{p}(0) = -G\check{y}(0) + \bar{\Gamma}_2\mathbb{E}\check{y}(0), \\ d\check{\rho} = -\Big(A\check{\rho} - R_0^{-1}\mathbb{E}\check{p} - R_0^{-1}\mathbb{E}\check{x}_1 + R_0^{-1}\check{x}_2 + C\check{q}_2\Big)dt, \quad \check{\rho}(T) = 0, \\ d\check{X} = \Big(A^\top\check{X} + Q\check{\rho} - Q\check{y} + \bar{\Gamma}_1\mathbb{E}\check{y} - C^\top\mathbb{E}\check{x}_1 + C^\top\check{x}_2 - \bar{\Gamma}_1\check{q}_2\Big)dt - H\check{z}dW(t), \\ d\check{x}_1 = (Q\check{y} + A^\top\check{x}_1)dt + H\check{z}dW(t), \quad \check{x}_1(0) = G\check{y}(0), \\ d\check{x}_2 = \Big[\bar{\Gamma}_1\mathbb{E}\check{y} - C^\top\mathbb{E}\check{x}_1 + (A+C)^\top\check{x}_2 - (\bar{\Gamma}_1 - Q)^\top\check{q}_2\Big]dt, \\ d\check{q}_2 = \Big[R_0^{-1}\mathbb{E}\check{p} + R_0^{-1}\mathbb{E}\check{x}_1 - R_0^{-1}\check{x}_2 - (A+C)\check{q}_2\Big]dt, \quad \check{q}_2(T) = 0, \\ \check{X}(0) = G\check{\rho}(0) - G\check{y}(0) + \bar{\Gamma}_2\mathbb{E}\check{y}(0) - \bar{\Gamma}_2\check{q}_2(0), \quad \check{x}_2(0) = \bar{\Gamma}_2\mathbb{E}\check{y}(0) + G\check{q}_2(0) - \bar{\Gamma}_2\check{q}_2(0). \end{cases} \tag{17}$$

In this subsection, we will use the Riccati decoupling method or four-step method to study the well-posedness of FBSDE. To begin with, we give some new formulation of (17). Define $\mathbb{X}=(\check{p}^\top,\check{X}^\top,\check{x}_1^\top,\check{x}_2^\top)^\top$, $\mathbb{Y}=(\check{y}^\top,\check{\rho}^\top,0,\check{q}_2^\top)^\top$, $\mathbb{Z}=(\check{z}^\top,0,0,0)^\top$; then the mean-field FBSDE (17) take the following form:

$$\begin{cases} d\mathbb{Y}=-\Big[\mathbb{A}_1\mathbb{X}+\bar{\mathbb{A}}_1\mathbb{E}[\mathbb{X}]+\mathbb{B}_1\mathbb{Y}+\bar{\mathbb{B}}_1\mathbb{E}[\mathbb{Y}]\Big]dt+\mathbb{Z}dW(t), & \mathbb{X}(0)=\mathbb{H}_1\mathbb{Y}(0)+\mathbb{H}_2\mathbb{E}\mathbb{Y}(0),\\ d\mathbb{X}=\Big[\mathbb{A}_2\mathbb{X}+\bar{\mathbb{A}}_2\mathbb{E}[\mathbb{X}]+\mathbb{B}_2\mathbb{Y}+\bar{\mathbb{B}}_2\mathbb{E}[\mathbb{Y}]\Big]dt+\mathbb{C}_3\mathbb{Z}dW(t), & \mathbb{Y}(T)=(\xi^\top,0,0,0)^\top, \end{cases} \tag{18}$$

where

$$\mathbb{A}_1=\begin{pmatrix}0 & BR_1^{-1}B^\top & 0 & 0\\ 0 & 0 & 0 & R_0^{-1}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & R_0^{-1}\end{pmatrix},\bar{\mathbb{A}}_1=\begin{pmatrix}-R_0^{-1} & 0 & 0 & 0\\ -R_0^{-1} & 0 & -R_0^{-1} & 0\\ 0 & 0 & 0 & 0\\ -R_0^{-1} & 0 & -R_0^{-1} & 0\end{pmatrix},\mathbb{B}_1=\begin{pmatrix}A & 0 & 0 & 0\\ 0 & A & 0 & C\\ 0 & 0 & 0 & 0\\ 0 & 0 & C & A+C\end{pmatrix},\bar{\mathbb{B}}_1=\begin{pmatrix}C & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\mathbb{A}_2=\begin{pmatrix}A^\top & 0 & 0 & 0\\ 0 & A^\top & 0 & C^\top\\ 0 & 0 & A^\top & 0\\ 0 & 0 & 0 & (A+C)^\top\end{pmatrix},\bar{\mathbb{A}}_2=\begin{pmatrix}C^\top & 0 & 0 & 0\\ 0 & 0 & -C^\top & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -C^\top & 0\end{pmatrix},\mathbb{B}_2=\begin{pmatrix}-Q & 0 & 0 & 0\\ -Q & Q & 0 & -\bar{\Gamma}_1\\ Q & 0 & 0 & 0\\ 0 & 0 & 0 & -(\bar{\Gamma}_1-Q)^\top\end{pmatrix},$$

$$\bar{\mathbb{B}}_2=\begin{pmatrix}\bar{\Gamma}_1 & 0 & 0 & 0\\ \bar{\Gamma}_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \bar{\Gamma}_1 & 0 & 0 & 0\end{pmatrix},\mathbb{C}_3=\begin{pmatrix}-H & 0 & 0 & 0\\ -H & 0 & 0 & 0\\ H & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\mathbb{H}_1=\begin{pmatrix}-G & 0 & 0 & 0\\ -G & G & 0 & -\bar{\Gamma}_2\\ G & 0 & 0 & 0\\ 0 & 0 & 0 & G-\bar{\Gamma}_2\end{pmatrix},\mathbb{H}_2=\begin{pmatrix}\bar{\Gamma}_2 & 0 & 0 & 0\\ \bar{\Gamma}_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \bar{\Gamma}_2 & 0 & 0 & 0\end{pmatrix},$$

and 0 denotes the zero vector or zero matrix with suitable dimensions.

Proposition 3. Let

$$\mathcal{A}_1=\begin{pmatrix}\mathbb{A}_1+\bar{\mathbb{A}}_1 & 0\\ 0 & \mathbb{A}_1\end{pmatrix},\quad \mathcal{B}_1=\begin{pmatrix}\mathbb{B}_1+\bar{\mathbb{B}}_1 & 0\\ 0 & \mathbb{B}_1\end{pmatrix},\quad \mathcal{H}=\begin{pmatrix}\mathbb{H}_1+\bar{\mathbb{H}}_1 & 0\\ 0 & 0\end{pmatrix},$$
$$\mathcal{A}_2=\begin{pmatrix}\mathbb{A}_2+\bar{\mathbb{A}}_2 & 0\\ 0 & \mathbb{A}_2\end{pmatrix},\quad \mathcal{B}_2=\begin{pmatrix}\mathbb{B}_2+\bar{\mathbb{B}}_2 & 0\\ 0 & \mathbb{B}_2\end{pmatrix},\quad \mathcal{C}_3=\begin{pmatrix}0 & 0\\ 0 & \mathbb{C}_3\end{pmatrix}.$$

Supposing the Ricatti equation

$$\begin{cases}\dot{\Phi}+\Phi(\mathcal{A}_2+\mathcal{H}\mathcal{A}_1)+(\mathcal{B}_1+\mathcal{A}_1\mathcal{H})\Phi+\Phi[\mathcal{H}\mathcal{B}_1+(\mathcal{A}_2+\mathcal{H}\mathcal{A}_1)\mathcal{H}+\mathcal{B}_2]\Phi+\mathcal{A}_1=0,\\ \Phi(T)=0\end{cases} \tag{19}$$

admits a unique solution $\Phi(\cdot)$ over $[0,T]$ such that $I-\Phi(\mathcal{C}_3-\mathcal{H})$ is invertible, then the CC system (18) admits a solution.

Proof. Taking the expectation of (18), we can get

$$\begin{cases} d\mathbb{E}[\mathbb{Y}]=-\left[\left(\mathbb{A}_1+\bar{\mathbb{A}}_1\right)\mathbb{E}[\mathbb{X}]+\mathbb{B}_1\mathbb{E}[\mathbb{Y}]\right]dt, & \mathbb{E}[X(0)]=\left(\mathbb{H}_1+\mathbb{H}_2\right)\mathbb{E}[\mathbb{Y}(0)],\\ d\mathbb{E}[\mathbb{X}]=\left[\left(\mathbb{A}_2+\bar{\mathbb{A}}_2\right)\mathbb{E}[\mathbb{X}]+\left(\mathbb{B}_2+\bar{\mathbb{B}}_2\right)\mathbb{E}[\mathbb{Y}]\right]dt, & \mathbb{E}[\mathbb{Y}(T)]=\Big(\mathbb{E}[\xi]^\top\ 0\ 0\ 0\Big)^\top. \end{cases} \tag{20}$$

From (18) and (20), it follows that

$$\begin{cases} d\left(\mathbb{Y}-\mathbb{E}[\mathbb{Y}]\right)=-\left[\mathbb{A}_1\left(\mathbb{X}-\mathbb{E}[\mathbb{X}]\right)+\mathbb{B}_1\left(\mathbb{Y}-\mathbb{E}[\mathbb{Y}]\right)\right]dt+\mathbb{Z}dW(t),\\ d\left(\mathbb{X}-\mathbb{E}[\mathbb{X}]\right)=\left[\mathbb{A}_2\left(\mathbb{X}-\mathbb{E}[\mathbb{X}]\right)+\mathbb{B}_2\left(\mathbb{Y}-\mathbb{E}[\mathbb{Y}]\right)\right]dt+\mathbb{C}_3\mathbb{Z}dW(t),\\ \mathbb{X}(0)-\mathbb{E}[\mathbb{X}](0)=0,\quad (\mathbb{Y}-\mathbb{E}[\mathbb{Y}])(T)=\Big(\xi^\top-\mathbb{E}[\xi]^\top 0,0,0\Big)^\top. \end{cases}$$

Let

$$\mathcal{Y}=\begin{pmatrix}\mathbb{E}[\mathbb{Y}]\\ \mathbb{Y}-\mathbb{E}[\mathbb{Y}]\end{pmatrix},\quad \mathcal{X}=\begin{pmatrix}\mathbb{E}[\mathbb{X}]\\ \mathbb{X}-\mathbb{E}[\mathbb{X}]\end{pmatrix}.$$

Then mean-field FBSDE (18) is equivalent to the following FBSDE:

$$\begin{cases} d\mathcal{Y} = -\left[\mathcal{A}_1\mathcal{X} + \mathcal{B}_1\mathcal{Y}\right]dt + \mathcal{Z}dW(t), & \mathcal{X}(0) = \mathcal{H}\mathcal{Y}(0), \\ d\mathcal{X} = \left[\mathcal{A}_2\mathcal{X} + \mathcal{B}_2\mathcal{Y}\right]dt + \mathcal{C}_3\mathcal{Z}dW(t), & \mathcal{Y}(T) = \Xi, \end{cases} \tag{21}$$

where

$$\mathcal{Z} = \begin{pmatrix} 0 \\ \mathbb{Z} \end{pmatrix}, \quad \Xi = \Big(\mathbb{E}\xi^\top, 0, 0, 0, \xi^\top - \mathbb{E}\xi^\top, 0, 0, 0\Big)^\top.$$

Defining

$$\tilde{\mathcal{X}}(t) = \mathcal{X}(t) - \mathcal{H}\mathcal{Y}(t), \quad t \in [0, T],$$

we have

$$\begin{cases} d\tilde{\mathcal{X}} = \big[(\mathcal{A}_2 + \mathcal{H}\mathcal{A}_1)\tilde{\mathcal{X}} + (\mathcal{H}\mathcal{B}_1 + (\mathcal{A}_2 + \mathcal{H}\mathcal{A}_1)\mathcal{H} + \mathcal{B}_2)\mathcal{Y}\big]dt + (\mathcal{C}_3 - \mathcal{H})\mathcal{Z}dW(t), & \tilde{\mathcal{X}}(0) = 0, \\ d\mathcal{Y} = -\big[(\mathcal{B}_1 + \mathcal{A}_1\mathcal{H})\mathcal{Y} + \mathcal{A}_1\tilde{\mathcal{X}}\big]dt + \mathcal{Z}dW(t), & \mathcal{Y}(T) = \Xi, \end{cases} \tag{22}$$

which is also a fully-coupled FBSDE while the initial condition is decoupled. We assume that $\tilde{\mathcal{X}}$ and $\mathcal{Y}$ are related by

$$\mathcal{Y}(t) = \phi(t)\tilde{\mathcal{X}}(t) + \Psi(t), \quad t \in [0, T], \quad \text{a.s},$$

where $\phi : [0, T] \to \mathbb{R}^{8n\times 8n}$ is a deterministic matrix-valued function with $\phi(T) = 0$ and $\Psi : [0, T] \times \Omega \to \mathbb{R}^{8n}$ satisfies the following BSDE:

$$\begin{cases} d\Psi(t) = a(t)dt + \Upsilon(t)dW(t), \\ \Psi(T) = \Xi \end{cases} \tag{23}$$

with the generator $a(\cdot)$ being undetermined. Applying Itô's formula and comparing the diffusions, we get

$$\mathcal{Z} = [I - \Phi(\mathcal{C}_3 - \mathcal{H})]^{-1}\Upsilon.$$

It then follows from the drifts that ϕ satisfies (19) and

$$\Upsilon + \Phi(\mathcal{C}_3 - \mathcal{H})\mathcal{Z} = \mathcal{Z}.$$

Then Eq. (23) has the following form:

$$\begin{cases} d\Psi = -\Big[\mathcal{B}_1 + \mathcal{A}_1\mathcal{H} + \Phi(\mathcal{H}\mathcal{B}_1 + (\mathcal{A}_2 + \mathcal{H}\mathcal{A}_1)\mathcal{H} + \mathcal{B}_2)\Big]\Psi dt + \Upsilon dW(t), \\ \Psi(T) = \Xi. \end{cases} \tag{24}$$

When Eq. (19) admits a unique solution $\Phi(\cdot)$ such that $I - \Phi(\mathcal{C}_3 - \mathcal{H})$ is invertible, BSDE (24) admits a unique adapted solution $(\Psi(\cdot), \Upsilon(\cdot))$. Then the equation of $\tilde{\mathcal{X}}$ becomes

$$\begin{cases} d\tilde{\mathcal{X}} = \Big\{\Big[\mathcal{A}_2 + \mathcal{H}\mathcal{A}_1 + \big(\mathcal{H}\mathcal{B}_1 + (\mathcal{A}_2 + \mathcal{H}\mathcal{A}_1)\mathcal{H} + \mathcal{B}_2\big)\Phi\Big]\tilde{\mathcal{X}} + \big(\mathcal{H}\mathcal{B}_1 + (\mathcal{A}_2 + \mathcal{H}\mathcal{A}_1)\mathcal{H} + \mathcal{B}_2\big)\Psi\Big\}dt \\ \qquad\quad + (\mathcal{C}_3 - \mathcal{H})(I - \Phi(\mathcal{C}_3 - \mathcal{H}))^{-1}\Upsilon dW(t), \\ \tilde{\mathcal{X}}(0) = 0, \end{cases}$$

which admits a unique solution $\tilde{\mathcal{X}}(\cdot)$. Furthermore, the second equation in (21) (BSDE) admits a unique solution $(\mathcal{Y}(\cdot), \mathcal{Z}(\cdot))$. Then the existence of $\mathcal{X}(\cdot)$ is obtained.

In Proposition 3, we obtain the solution of (17) through Riccati equation (19). Following [51, Theorem 5.3], we can give the existence and uniqueness of the solution to Riccati equation (17).

Proposition 4. For any $s \in [0, T]$, let $\Psi(\cdot, s)$ be the solutions of the following ordinary differential equation (ODE):

$$\begin{cases} \frac{d}{dt}\Psi_1(t, s) = \hat{A}_1(t)\Psi_1(t, s), \quad t \in [s, T], \\ \Psi_1(s, s) = I, \end{cases}$$

where

$$\hat{A}_1(\cdot) = \begin{pmatrix} \mathcal{A}_2(\cdot) + \mathcal{H}\mathcal{A}_1(\cdot) & \mathcal{H}\mathcal{B}_1(\cdot) + [\mathcal{A}_2(\cdot) + \mathcal{H}\mathcal{A}_1(\cdot)]\mathcal{H} + \mathcal{B}_2(\cdot) \\ -\mathcal{A}_1(\cdot) & -[\mathcal{B}_1(\cdot) + \mathcal{A}_1(\cdot)\mathcal{H}] \end{pmatrix}.$$

Supposing

$$\left[\begin{pmatrix} 0 & I \end{pmatrix} \Psi_1(T,t) \begin{pmatrix} 0 \\ I \end{pmatrix}^{-1} \right] \in L^1(0,T;\mathbb{R}^{8n\times 8n}),$$

then Ricatti equation (19) admits a unique solution $\Phi(\cdot)$, which is given by

$$\Phi(t) = -\left[\begin{pmatrix} 0 & I \end{pmatrix} \Psi_1(T,t) \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 & I \end{pmatrix} \Psi_1(T,t) \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad t \in [0,T]. \tag{25}$$

To further verify the existence of solutions for the Riccati equation under specific conditions, we consider a special case where the matrix $\hat{A}_1(\cdot)$ is assumed to be a constant matrix (denoted as Λ). Under this assumption, we obtained the following proposition, which explicitly demonstrates that, under the given conditions, the solution to (19) can be expressed as follows (see [52]).

Proposition 5. Let $\hat{A}_1(\cdot)$ be constant-valued matrices and $\hat{A}_1(t) \equiv \Lambda$. Suppose $\forall t \in [0,T]$, $\det \begin{pmatrix} 0 & I \end{pmatrix} e^{\Lambda t} \begin{pmatrix} 0 \\ I \end{pmatrix} > 0$ holds. Then Eq. (25) admits a unique solution $\Phi(\cdot)$, which has the following representation:

$$\Phi(t) = -\left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\Lambda(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix}^{-1} \begin{pmatrix} 0 & I \end{pmatrix} e^{\Lambda(T-t)} \begin{pmatrix} I \\ 0 \end{pmatrix} \right], \quad t \in [0,T].$$

6 Asymptotic optimal

In this section, we assume the following.

Assumption 5. $Q = \Gamma_1 = 0, (A+C)^\top(\bar{\Gamma}_2 - G) + (\bar{\Gamma}_2 - G)(A+C) - (\bar{\Gamma}_2 - G)R_0^{-1}(\bar{\Gamma}_2 - G) > 0$. Let $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_N)$ denote the set of decentralized strategies given by

$$\widetilde{u}_i(t) = R_1^{-1}B^\top X_i,$$

where

$$\begin{cases} dy_i = -\Big(Ay_i + BR_1^{-1}B^\top X_i + C\mathbb{E}\check{y} - R_0^{-1}\mathbb{E}\check{p}\Big)dt + z_i dW_i(t), \quad y_i(T) = \xi_i, \\ dp_i = \Big(A^\top p_i + C^\top \mathbb{E}\check{p}\Big)dt - Hz_i dW_i(t), \quad p_i(0) = -Gy_i(0) + \bar{\Gamma}_2\mathbb{E}\check{y}(0), \\ d\rho_i = -\Big(A\rho_i - R_0^{-1}\mathbb{E}\check{p} - R_0^{-1}\mathbb{E}\check{x}_1 + R_0^{-1}\check{x}_2 + C\check{q}_2\Big)dt, \quad \rho_i(T) = 0, \\ dX_i = \Big(A^\top X_i - C^\top\mathbb{E}\check{x}_1 + C^\top\check{x}_2\Big)dt - Hz_i dW_i(t), \\ X_i(0) = -Gy_i(0) + G\rho_i(0) + \bar{\Gamma}_2\mathbb{E}\check{y}(0) - \bar{\Gamma}_2\check{q}_2(0), \end{cases} \tag{26}$$

and the frozen terms $(\check{y}, \check{\rho}, \check{q}_2, \check{z}, \check{p}, \check{X}, \check{x}_1, \check{x}_2)$ is the solution of (17). Correspondingly, the realized decentralized state $(\widetilde{y}_1, \widetilde{p}_1, \cdots, \widetilde{y}_N, \widetilde{p}_N)$ satisfies

$$\begin{cases} d\widetilde{y}_i = -\Big(A\widetilde{y}_i + BR_1^{-1}B^\top X_i + C\widetilde{y}^{(N)} - R_0^{-1}\widetilde{p}^{(N)}\Big)dt + \widetilde{z}_i dW_i(t), \quad \widetilde{y}_i(T) = \xi_i, \\ d\widetilde{p}_i = \Big(A^\top\widetilde{p}_i + C^\top\widetilde{p}^{(N)}\Big)dt - H\widetilde{z}_i dW_i(t), \quad \widetilde{p}_i(0) = -G\widetilde{y}_i(0) + \bar{\Gamma}_2\widetilde{y}^{(N)}(0) \end{cases} \tag{27}$$

with $\widetilde{y}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widetilde{y}_i$ and $\widetilde{p}^{(N)} = \frac{1}{N}\sum_{i=1}^N \widetilde{p}_i$. Let us consider the case that the agent $\mathcal{A}_i$ uses an alternative strategy u_i while the other agents $\mathcal{A}_j$, $j \neq i$ apply the strategy $\widetilde{u}_{-i}$. The realized state with the i-th agent's perturbation is

$$\begin{cases} d\acute{y}_i = -\Big(A\acute{y}_i + Bu_i + C\acute{y}^{(N)} - R_0^{-1}\acute{p}^{(N)}\Big)dt + \acute{z}_i dW_i(t), \quad \acute{y}_i(T) = \xi_i, \\ d\acute{p}_i = \Big(A^\top \acute{p}_i + C^\top \acute{p}^{(N)}\Big)dt - H\acute{z}_i dW_i(t), \quad \acute{p}_i(0) = -G\acute{y}_i(0) + \bar{\Gamma}_2 \acute{y}^{(N)}(0), \end{cases}$$

and for $j \neq i$,

$$\begin{cases} d\acute{y}_j = -\Big(A\acute{y}_j + BR_1^{-1}B^\top X_j + C\acute{y}^{(N)} - R_0^{-1}\acute{p}^{(N)}\Big)dt + \acute{z}_j dW_j(t), \quad \acute{y}_j(T) = \xi_j, \\ d\acute{p}_j = \Big(A^\top \acute{p}_j + C^\top \acute{p}^{(N)}\Big)dt - H\acute{z}_j dW_j(t), \quad \acute{p}_j(0) = -G\hat{y}_j(0) + \bar{\Gamma}_2 \acute{y}^{(N)}(0) \end{cases}$$

with $\acute{y}^{(N)} = \frac{1}{N}\sum_{i=1}^N \acute{y}_i$ and $\acute{p}^{(N)} = \frac{1}{N}\sum_{i=1}^N \acute{p}_i$. Define

$$\delta u_j = u_j - \widetilde{u}_j, \delta y_j = \acute{y}_j - \widetilde{y}_j, \delta\mathcal{J}_j = J_j(u_i, \widetilde{u}_{-i}) - J_j(\widetilde{u}_i, \widetilde{u}_{-i}).$$

Similar to Subsection 4.1, we have

$$\begin{aligned} \delta\mathcal{J}_{soc}^{wo} = \mathbb{E}\Big[\int_0^T \Big(&\langle R_1\widetilde{u}_i, \delta u_i\rangle + \langle C^\top \mathbb{E}\check{x}_1 - C^\top \check{x}_2, \delta y_i\rangle + \langle -R_0^{-1}\hat{p} - R_0^{-1}\mathbb{E}\check{x}_1 + R_0^{-1}\check{x}_2 + C\check{q}_2, \delta p_i\rangle \\ &+ \langle H\widetilde{z}_i, \delta z_i\rangle\Big)dt + \langle G\widetilde{y}_i(0) - \bar{\Gamma}_2\mathbb{E}\check{y}(0) + \bar{\Gamma}_2\check{q}_2(0), \delta y_i(0)\rangle\Big] + \sum_{l=1}^8 \varepsilon_l, \end{aligned} \tag{28}$$

where

$$\begin{cases} \varepsilon_1 = E\int_0^T N\langle R_0^{-1}(\mathbb{E}\check{p} - \widetilde{p}^{(N)}), \delta p^{(N)}\rangle dt, \\ \varepsilon_2 = \mathbb{E}\langle (G\Gamma_2 + \Gamma_2^\top G - \Gamma_2^\top G\Gamma_2)(\mathbb{E}\check{y}(0) - \widetilde{y}^{(N)}(0)), N\delta y^{(N)}(0)\rangle, \\ \varepsilon_3 = \mathbb{E}\int_0^T \dfrac{1}{N}\sum_{j\neq i}\langle H\widetilde{z}_j, N\delta z_j - z_j^*\rangle dt, \\ \varepsilon_4 = \mathbb{E}\int_0^T \langle R_0^{-1}\mathbb{E}\check{p}, p^{**} - \delta p_{-i}\rangle dt, \\ \varepsilon_5 = \mathbb{E}\int_0^T \langle (G\Gamma_2 + \Gamma_2^\top G - \Gamma_2^\top G\Gamma_2)\mathbb{E}\check{y}(0), y^{**}(0) - \delta y_{-i}(0)\rangle, \\ \varepsilon_6 = \mathbb{E}\int_0^T \dfrac{1}{N}\sum_{j\neq i}\langle G\widetilde{y}_j(0), N\delta y_j(0) - y_j^*(0)\rangle, \\ \varepsilon_7 = \mathbb{E}\int_0^T \Big(C^\top(\mathbb{E}x_1^1 - \dfrac{1}{N}\sum_{j\neq i} x_1^j), y^{**}\rangle + \langle R_0^{-1}(\dfrac{1}{N}\sum_{j\neq i} x_1^j - \mathbb{E}x_1^1), p^{**}\rangle\Big)dt, \\ \varepsilon_8 = \mathbb{E}\int_0^T \Big(\langle C^\top(\dfrac{1}{N}\sum_{j\neq i} x_1^j - \mathbb{E}x_1^1), \delta y_i\rangle + \langle R_0^{-1}(\mathbb{E}x_1^1 - \dfrac{1}{N}\sum_{j\neq i} x_1^j), \delta p_i\rangle\Big)dt. \end{cases} \tag{29}$$

Let $a^* = \sup\limits_{0\leqslant s\leqslant T} \Lambda_{\max}(-\frac{1}{2}(A(s) + A(s)^\top))$, where $\Lambda_{\max}(M)$ is the largest eigenvalue of the matrix M. In order to verify the asymptotic optimality, first we need to give some estimations on the error terms in (29). In the proofs, K will denote a constant whose value may change from line to line. It follows from Proposition 3 that Eq. (17) has a solution. Therefore, there exists a constant K independent of N such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\Big[|\check{y}(t)|^2 + |\check{p}(t)|^2 + |\check{\rho}(t)|^2 + |\check{X}(t)|^2 + |\check{x}_1(t)|^2 + |\check{x}_2(t)|^2 + |\check{q}_2(t)|^2\Big] + \mathbb{E}\int_0^T |\check{z}(t)|^2 dt \leqslant K. \tag{30}$$

Lemma 1. Under Assumptions 1–5, let $a^* < 0$. There exists some $\delta > 0$ depending on a^* such that when $\|B\|, \|G\|, \|R_1^{-1}\|, \|H\| \in [0, \delta)$,

$$\sup_{1\leqslant i\leqslant N}\mathbb{E}\sup_{0\leqslant t\leqslant T}\Big[|\widetilde{y}_i(t)|^2 + |\widetilde{p}_i(t)|^2 + |\acute{y}_i(t)|^2 + |\acute{p}_i(t)|^2\Big] \leqslant K. \tag{31}$$

Proof. For simplicity, let

$$\Theta_1 = C\mathbb{E}\check{y} - R_0^{-1}\mathbb{E}\check{p},\ \Theta_2 = -C^\top\mathbb{E}\check{x}_1 + C^\top\check{x}_2,\ \Theta_3 = G\rho_i(0) + \bar{\Gamma}_2\mathbb{E}\check{y}(0) - \bar{\Gamma}_2\check{q}_2(0).$$

First, for some $\lambda \in \mathbb{R}$, applying Itô's formula to $e^{-\lambda t}|X_i(t)|^2$, we have

$$\mathbb{E}[e^{-\lambda t}|X_i(t)|^2] + \bar{\lambda}_1\mathbb{E}\int_0^t e^{-\lambda s}|X_i(s)|^2 ds \leqslant K + 2\|G\|^2\mathbb{E}|y_i(0)|^2 + \|H\|^2\mathbb{E}\int_0^t e^{-\lambda s}|z_i(s)|^2 ds,$$

where $\bar{\lambda}_1 = \lambda - 2a^* - \frac{1}{k_1}$ for some $k_1 > 0$. Similarly, applying Itô's formula to $e^{-\bar{\lambda}_1(t-s)-\lambda s}|X_i(s)|^2$,

$$\mathbb{E}\left[e^{-\lambda t}|X_i(t)|^2\right] \leqslant Ke^{-\bar{\lambda}_1 t} + 2\|G\|^2|y_i(0)|^2 e^{-\bar{\lambda}_1 t} + \|H\|^2\mathbb{E}\int_0^t e^{-\bar{\lambda}_1(t-s)-\lambda s}|z_i(s)|^2 ds.$$

Integrating on $[0,T]$ and noting $\frac{1-e^{-\bar{\lambda}_1(T-s)}}{\bar{\lambda}_1} \leqslant \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}$ for $s \in [0,T]$, we obtain

$$\mathbb{E}\int_0^T e^{-\lambda t}|X_i(t)|^2 dt \leqslant K\frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1} + \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}2\|G\|^2|y_i(0)|^2 + \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}\|H\|^2\mathbb{E}\int_0^T e^{-\lambda t}|z_i(t)|^2 dt.$$

Especially, if $\bar{\lambda}_1 > 0$, we have

$$\mathbb{E}\int_0^T e^{-\lambda t}|X_i(t)|^2 dt \leqslant \frac{K}{\bar{\lambda}_1} + \frac{2\|G\|^2}{\bar{\lambda}_1}|y_i(0)|^2 + \frac{\|H\|^2}{\bar{\lambda}_1}\mathbb{E}\int_0^T e^{-\lambda t}|z_i(t)|^2 dt. \tag{32}$$

Next, for some $\lambda \in \mathbb{R}$, applying Itô's formula to $e^{-\lambda t}|y_i(t)|^2$, we have

$$\mathbb{E}\left[e^{-\lambda t}|y_i(t)|^2\right] + \bar{\lambda}_2\mathbb{E}\int_t^T e^{-\lambda s}|y_i(s)|^2 ds + \mathbb{E}\int_t^T e^{-\lambda s}|z_i(s)|^2 ds \leqslant K + k_2\|BR_1^{-1}B^\top\|\mathbb{E}\int_t^T e^{-\lambda s}|X_i(s)|^2 ds, \tag{33}$$

where $\bar{\lambda}_2 = -\lambda - 2a^* - \frac{\|BR_1^{-1}B^\top\|}{k_2} - \frac{1}{k_3}$ for some $k_2, k_3 > 0$. Similarly, applying Itô's formula to $e^{-\bar{\lambda}_1(s-t)-\lambda t}|X_i(s)|^2$ and integrating on $[0,T]$, we have

$$\mathbb{E}\int_0^T e^{-\lambda s}|y_i(s)|^2 ds \leqslant \frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2}\Big[K + k_2\|BR_1^{-1}B^\top\|\mathbb{E}\int_t^T e^{-\lambda s}|X_i(s)|^2 ds\Big].$$

Especially, if $\bar{\lambda}_2 > 0$, letting $t = 0$ in (33), we have

$$|y_i(0)|^2 + \mathbb{E}\int_t^T e^{-\lambda s}|z_i(s)|^2 ds \leqslant K + k_2\|BR_1^{-1}B^\top\|\mathbb{E}\int_t^T e^{-\lambda s}|X_i(s)|^2 ds. \tag{34}$$

Since $a^* < 0$, we can choose sufficiently large k_1, k_2, k_3 such that

$$4a^* + \frac{1}{k_1} + \frac{\|BR_1^{-1}B^\top\|}{k_2} + \frac{1}{k_3} < 0.$$

Therefore, there exists $\lambda \in \mathbb{R}$ such $\bar{\lambda}_1, \bar{\lambda}_2 > 0$. Combining (32) and (34),

$$|y_i(0)|^2 + \mathbb{E}\int_t^T e^{-\lambda s}|z_i(s)|^2 ds \leqslant K + \frac{2k_2\|G\|^2\|BR_1^{-1}B^\top\|}{\bar{\lambda}_1}|y_i(0)|^2 + \frac{k_2\|H\|^2\|BR_1^{-1}B^\top\|}{\bar{\lambda}_1}\mathbb{E}\int_t^T e^{\lambda s}|z_i(s)|^2 ds.$$

Note that there exists $\delta > 0$ depending on a^* such that when $\|B\|, \|G\|, \|R_1^{-1}\|, \|H\| \in [0,\delta)$,

$$\frac{2k_2\|G\|^2\|BR_1^{-1}B^\top\|}{\bar{\lambda}_1} \vee \frac{k_2\|H\|^2\|BR_1^{-1}B^\top\|}{\bar{\lambda}_1} < 1.$$

Therefore,

$$|y_i(0)|^2 + \mathbb{E}\int_0^T e^{-\lambda s}|z_i(t)|^2 dt \leqslant K.$$

It then follows from Burkholder-Davis-Gundy inequality that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|X_i(t)|^2\leqslant K.$$

Consequently,

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|X^{(N)}(t)|^2\leqslant \frac{1}{N}\mathbb{E}\sup_{0\leqslant t\leqslant T}|X_i(t)|^2\leqslant K. \tag{35}$$

Finally, it follows from (27) that

$$\begin{cases} d\widetilde{y}^{(N)}=-\Big[(A+C)\widetilde{y}^{(N)}+BR_1^{-1}B^\top X^{(N)}-R_0^{-1}\widetilde{p}^{(N)}\Big]dt+\dfrac{1}{N}\displaystyle\sum_{j=1}^N\widetilde{z}_j dW_j(t),\\ d\widetilde{p}^{(N)}=(A+C)^\top\widetilde{p}^{(N)}dt-\dfrac{H}{N}\displaystyle\sum_{j=1}^N\widetilde{z}_j dW_j(t),\\ \widetilde{y}^{(N)}(T)=\dfrac{1}{N}\displaystyle\sum_{j=1}^N\xi_j,\quad \widetilde{p}^{(N)}(0)=(\bar{\Gamma}_2-G)\widetilde{y}^{(N)}(0). \end{cases} \tag{36}$$

Defining $\hat{p}^{(N)}=\widetilde{p}^{(N)}-(\bar{\Gamma}_2-G)\widetilde{y}^{(N)}$, then

$$\begin{aligned} d\hat{p}^{(N)}=&(A+C)^\top\widetilde{p}^{(N)}dt-\frac{H}{N}\sum_{j=1}^N\widetilde{z}_j dW_j(t)+(\bar{\Gamma}_2-G)\Big[(A+C)\widetilde{y}^{(N)}+BR_1^{-1}B^\top X^{(N)}-R_0^{-1}\widetilde{p}^{(N)}\Big]dt\\ &-(\bar{\Gamma}_2-G)\frac{1}{N}\sum_{j=1}^N\widetilde{z}_j dW_j(t)\\ =&\Big[(A+C)^\top\widetilde{p}^{(N)}+(\bar{\Gamma}_2-G)(A+C)\widetilde{y}^{(N)}+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}-(\bar{\Gamma}_2-G)R_0^{-1}\widetilde{p}^{(N)}\Big]dt\\ &-\sum_{j=1}^N\Big[\frac{H}{N}+(\bar{\Gamma}_2-G)\frac{1}{N}\Big]\widetilde{z}_j dW_j(t)\\ =&\Big[(A+C)^\top\hat{p}^{(N)}+(A+C)^\top(\bar{\Gamma}_2-G)\widetilde{y}^{(N)}+(\bar{\Gamma}_2-G)(A+C)\widetilde{y}^{(N)}+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}\\ &-(\bar{\Gamma}_2-G)R_0^{-1}\hat{p}^{(N)}-(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\widetilde{y}^{(N)}\Big]dt-\sum_{j=1}^N\frac{H+\bar{\Gamma}_2-G}{N}\widetilde{z}_j dW_j(t). \end{aligned}$$

Let $\widetilde{y}^{(N)}=\Phi\hat{p}^{(N)}+\phi$, where

$$d\phi=\dot{\phi}dt+\sum_{j=1}^N\vartheta_j dW_j,\quad \phi(T)=\frac{1}{N}\sum_{j=1}^N\xi_j,$$

then

$$\begin{aligned} d\widetilde{y}^{(N)}=&\dot{\Phi}\hat{p}^{(N)}dt+\Phi\Big[(A+C)^\top\hat{p}^{(N)}+(A+C)^\top(\bar{\Gamma}_2-G)\widetilde{y}^{(N)}+(\bar{\Gamma}_2-G)(A+C)\widetilde{y}^{(N)}\\ &+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}-(\bar{\Gamma}_2-G)R_0^{-1}\hat{p}^{(N)}-(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\widetilde{y}^{(N)}\Big]dt+\dot{\phi}dt\\ &+\sum_{j=1}^N\Big(\vartheta_j-\Phi\frac{H+\bar{\Gamma}_2-G}{N}\widetilde{z}_j\Big)dW_j(t)\\ =&\dot{\Phi}\hat{p}^{(N)}dt+\Phi\Big[(A+C)^\top\hat{p}^{(N)}+(A+C)^\top(\bar{\Gamma}_2-G)\Phi\hat{p}^{(N)}+(A+C)^\top(\bar{\Gamma}_2-G)\phi\\ &+(\bar{\Gamma}_2-G)(A+C)\Phi\hat{p}^{(N)}+(\bar{\Gamma}_2-G)(A+C)\phi+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}\\ &-(\bar{\Gamma}_2-G)R_0^{-1}\hat{p}^{(N)}-(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\Phi\hat{p}^{(N)}-(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\phi+\dot{\phi}\Big]dt \end{aligned}$$

$$+\sum_{j=1}^{N}\Big(\vartheta_j-\Phi\frac{H+\bar{\Gamma}_2-G}{N}\widetilde{z}_j\Big)dW_j(t).$$

Therefore,

$$\begin{cases}\dot{\Phi}\hat{p}^{(N)}+\Phi(A+C)^\top\hat{p}^{(N)}+\Phi(A+C)^\top(\bar{\Gamma}_2-G)\Phi\hat{p}^{(N)}+\Phi(A+C)^\top(\bar{\Gamma}_2-G)\phi\\+\Phi(\bar{\Gamma}_2-G)(A+C)\Phi\hat{p}^{(N)}+\Phi(\bar{\Gamma}_2-G)(A+C)\phi+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}\\-\Phi(\bar{\Gamma}_2-G)R_0^{-1}\hat{p}^{(N)}-\Phi(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\Phi\hat{p}^{(N)}-\Phi(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\phi+\dot{\phi}\\+\Big[(A+C)\Phi\hat{p}^{(N)}+(A+C)\phi+BR_1^{-1}B^\top X^{(N)}-R_0^{-1}\hat{p}^{(N)}-R_0^{-1}(\bar{\Gamma}_2-G)\Phi\hat{p}^{(N)}\\-R_0^{-1}(\bar{\Gamma}_2-G)\phi\Big]=0,\\\vartheta_j-\Phi\dfrac{H+\bar{\Gamma}_2-G}{N}\widetilde{z}_j-\dfrac{1}{N}\widetilde{z}_j=0\end{cases}$$

and

$$\begin{cases}\dot{\Phi}+\Phi(A+C)^\top+\Phi(A+C)^\top(\bar{\Gamma}_2-G)\Phi+\Phi(\bar{\Gamma}_2-G)(A+C)\Phi\\-\Phi(\bar{\Gamma}_2-G)R_0^{-1}-\Phi(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\Phi+(A+C)\Phi-R_0^{-1}-R_0^{-1}(\bar{\Gamma}_2-G)\Phi=0,\\\Phi(A+C)^\top(\bar{\Gamma}_2-G)\phi+\Phi(\bar{\Gamma}_2-G)(A+C)\phi+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}\\-\Phi(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\phi+\dot{\phi}+(A+C)\phi+BR_1^{-1}B^\top X^{(N)}-R_0^{-1}(\bar{\Gamma}_2-G)\phi=0,\\\vartheta_j=\Phi\dfrac{H+\bar{\Gamma}_2-G}{N}\widetilde{z}_j+\dfrac{1}{N}\widetilde{z}_j.\end{cases}$$

Therefore, the Riccati equation and BSDE take the following forms:

$$\begin{cases}\dot{\Phi}+\Phi\Big[(A+C)^\top-(\bar{\Gamma}_2-G)R_0^{-1}\Big]+\Big[(A+C)-R_0^{-1}(\bar{\Gamma}_2-G)\Big]\Phi\\+\Phi\Big[(A+C)^\top(\bar{\Gamma}_2-G)+(\bar{\Gamma}_2-G)(A+C)-(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\Big]\Phi-R_0^{-1}=0,\\\Phi(T)=0\end{cases}\tag{37}$$

and

$$\begin{cases}d\phi=-\Big[\Phi(A+C)^\top(\bar{\Gamma}_2-G)\phi+\Phi(\bar{\Gamma}_2-G)(A+C)\phi+(\bar{\Gamma}_2-G)BR_1^{-1}B^\top X^{(N)}\\-\Phi(\bar{\Gamma}_2-G)R_0^{-1}(\bar{\Gamma}_2-G)\phi+(A+C)\phi+BR_1^{-1}B^\top X^{(N)}-R_0^{-1}(\bar{\Gamma}_2-G)\phi\Big]dt+\displaystyle\sum_{j=1}^{N}\vartheta_jdW_j,\\\phi(T)=\dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\xi_j.\end{cases}\tag{38}$$

Under Assumption 4, Riccati equation (37) admits a unique solution. Then BSDE (38) admits a unique solution. Recalling (35), it follows from the standard estimation of BSDE and SDE that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{y}^{(N)}(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\hat{p}^{(N)}(t)|^2\leqslant K.$$

Consequently,

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{y}^{(N)}(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{p}^{(N)}(t)|^2\leqslant K.$$

Finally, it follows from the equation of $\widetilde{y}_i$ and $\widetilde{p}_i$ that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{y}_i(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{p}_i(t)|^2\leqslant K.$$

Similarly, we have

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\acute{y}_i(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\acute{p}_i(t)|^2\leqslant K.$$

In the following, we will assume the conditions in Lemma 1 hold.

Lemma 2. Under Assumptions 1–5, there exists a constant K independent of N such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\Big[|\delta p^{(N)}(t)|^2+\delta y^{(N)}(t)|^2\Big]\leqslant\frac{K}{N^2}\Big(1+\mathbb{E}\int_0^T|\delta u_i|^2ds\Big). \tag{39}$$

Proof. It follows from (8) and (9) that

$$\begin{cases} d\delta p^{(N)}=(A+C)^\top\delta p^{(N)}dt-\dfrac{1}{N}\displaystyle\sum_{j=1}^N H\delta z_j dW_j(t),\\ d\delta y^{(N)}=-\Big[(A+C)\delta y^{(N)}-R_0^{-1}\delta p^{(N)}+\dfrac{B}{N}\delta u_i\Big]dt+\dfrac{1}{N}\displaystyle\sum_{j=1}^N\delta z_j dW_j(t)+\dfrac{1}{N}\sum_{j=1}^N\sum_{k=1,k\neq j}^N\delta z_{jk}dW_k(t),\\ \delta p^{(N)}(0)=(\bar\Gamma_2-G)\delta y^{(N)}(0),\quad \delta y^{(N)}(T)=0. \end{cases}$$

Similar to (36), we can decompose the above FBSDE. Therefore, it follows from the standard estimation of BSDE and SDE that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\delta y^{(N)}(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\delta p^{(N)}(t)|^2\leqslant\frac{K}{N^2}\Big(1+\mathbb{E}\int_0^T|\delta u_i|^2ds\Big).$$

Remark 2. In Lemma 1, in order to decouple FBSDE (36), first we do some linear transformation such that the transformed FBSDE is not coupled at the initial time 0. Similarly, we can do this linear transformation, decouple the transformed FBSDE and then obtain the required estimations (39). Actually we can also decouple FBSDE directly without the transformation. For this, letting $\widetilde{y}^{(N)}=\widetilde\Phi\widetilde{p}^{(N)}+\widetilde\phi$, applying Itô's formula and comparing the coefficients, we have

$$\dot{\widetilde\Phi}+\widetilde\Phi(A+C)^\top+(A+C)\widetilde\Phi-R_0^{-1}=0,\ \widetilde\Phi(T)=0 \tag{40}$$

and

$$d\widetilde\phi=-\Big[(A+C)\widetilde\phi+\frac{B}{N}\delta u_i\Big]dt+\sum_{j=1}^N\theta_j dW_j,\ \widetilde\phi(T)=\frac{1}{N}\sum_{j=1}^N\xi_j. \tag{41}$$

Note that the above equation (40) admits a unique solution, so does (41). Therefore, if $I-(\bar\Gamma_2-G)\widetilde\Phi$ is invertible, from the initial condition $\delta p^{(N)}(0)=(\bar\Gamma_2-G)\delta y^{(N)}(0)$ we get that $\delta p^{(N)}(0)=(\bar\Gamma_2-G)[I-(\bar\Gamma_2-G)\widetilde\Phi]^{-1}\widetilde\phi(0)$. Then from the standard estimations of SDE and BSDE, we can also get (39).

Lemma 3. Under Assumptions 1–5, there exists a constant K independent of N such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{y}^{(N)}(t)-\mathbb{E}\check{y}(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{p}^{(N)}(t)-\mathbb{E}\check{p}(t)|^2\leqslant\frac{K}{N}.$$

Proof. First, similar to Lemma 1, we have $y_i(0)=\check{y}(0)$. Therefore, it follows from (26) and (17) that

$$\begin{cases} d(X^{(N)}-\mathbb{E}\check{X})=A^\top(X^{(N)}-\mathbb{E}\check{X})dt-\dfrac{1}{N}\displaystyle\sum_{j=1}^N Hz_j dW_j(t),\\ (X^{(N)}-\mathbb{E}\check{X})(0)=0. \end{cases} \tag{42}$$

It follows from the equation of $X^{(N)}(t)-\mathbb{E}\check{X}(t)$ that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|X^{(N)}(t)-\mathbb{E}\check{X}(t)|^2\leqslant\frac{K}{N}.$$

Note that

$$\begin{cases} d(\widetilde{y}^{(N)}-\mathbb{E}\check{y})=-\Big[(A+C)(\widetilde{y}^{(N)}-\mathbb{E}\check{y})+BR_1^{-1}B^\top(X^{(N)}-\mathbb{E}\check{X})-R_0^{-1}(\widetilde{p}^{(N)}-\mathbb{E}\check{p})\Big]dt+\dfrac{1}{N}\displaystyle\sum_{j=1}^N z_j dW_j(t),\\ d(\widetilde{p}^{(N)}-\mathbb{E}\check{p})=(A+C)^\top(\widetilde{p}^{(N)}-\mathbb{E}\check{p})dt-H\dfrac{1}{N}\displaystyle\sum_{j=1}^N z_j dW_j(t),\\ (\widetilde{y}^{(N)}-\mathbb{E}\check{y})(T)=\dfrac{1}{N}\displaystyle\sum_{j=1}^N\xi_j-\mathbb{E}\xi_1,\quad(\widetilde{p}^{(N)}-\mathbb{E}\check{p})(0)=(\bar\Gamma_2-G)(\widetilde{y}^{(N)}-\mathbb{E}\check{y})(0). \end{cases}$$

Similar to (36), we have

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{y}^{(N)}(t)-\mathbb{E}\check{y}(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\widetilde{p}^{(N)}(t)-\mathbb{E}\check{p}(t)|^2\leqslant\frac{K}{N}.$$

By Lemma 3, it is easy to see that

$$\sup_{1\leqslant j\leqslant N}\mathbb{E}\sup_{0\leqslant t\leqslant T}|y_j(t)-\widetilde{y}_j(t)|^2+\sup_{1\leqslant j\leqslant N}\mathbb{E}\sup_{0\leqslant t\leqslant T}|p_j(t)-\widetilde{p}_j(t)|^2\leqslant\frac{K}{N}. \tag{43}$$

Lemma 4. Under Assumptions 1–5, there exists a constant K independent of N such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\delta y_{-i}(t)-y^{**}(t)|^2+\mathbb{E}\sup_{0\leqslant t\leqslant T}|\delta p_{-i}(t)-p^{**}(t)|^2\leqslant\frac{K}{N^2}\Big(1+\mathbb{E}\int_0^T|\delta u_i|^2ds\Big). \tag{44}$$

Proof. It follows from (9) and (11) that

$$\begin{cases} d(\delta y_{-i}-y^{**})=-\Big[(A+C)(\delta y_{-i}-y^{**})+R_0^{-1}(\delta p_{-i}-p^{**})-C\delta y^{(N)}-R_0^{-1}\delta p^{(N)}\Big]dt\\ \qquad\qquad +\displaystyle\sum_{j=1,j\neq i}^N\delta z_jdW_j(t)+\sum_{j=1,j\neq i}^N\sum_{k=1,k\neq j}^N\delta z_{jk}dW_k(t)-\sum_{j=1}^N z_j^{**}dW_j(t),\\ d(\delta p_{-i}-p^{**})=\Big[(A+C)^\top(\delta p_{-i}-p^{**})-C^\top\delta p^{(N)}-\bar{\Gamma}_1\delta y^{(N)}\Big]dt-\displaystyle\sum_{j=1,j\neq i}H\delta z_jdW_j(t),\\ (\delta y_{-i}-y^{**})(T)=0,\quad(\delta p_{-i}-p^{**})(0)=(\bar{\Gamma}_2-G)(\delta y_{-i}-y^{**})-\bar{\Gamma}_2\delta y^{(N)}(0).\end{cases}$$

Similar to (36), we can get (44).

Lemma 5. Under Assumptions 1–5, there exists a constant K independent of N such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|N\delta y_j(t)-y_j^*(t)|^2+\mathbb{E}\int_0^T|N\delta z_j(t)-z_j^*(t)|^2dt\leqslant\frac{K}{N^2}\Big(1+\mathbb{E}\int_0^T|\delta u_i|^2ds\Big),\ j\neq i.$$

Proof. Note that

$$\begin{cases} d(N\delta y_j-y_j^*)=-\Big[A(N\delta y_j-y_j^*)+C(\delta y_{-i}-y^{**})-R_0^{-1}(\delta p_{-i}-p^{**})\Big]dt+(N\delta z_j-z_j^*)dW_j(t)\\ \qquad\qquad +\displaystyle\sum_{k=1,k\neq j}^N(N\delta z_{jk}-z_{jk}^*)dW_k(t),\\ (N\delta y_j-y_j^*)(T)=0.\end{cases}$$

Applying Itô's formula to $|N\delta y_j-y_j^*|^2$, we have

$$\begin{aligned}&\mathbb{E}|N\delta y_j(t)-y_j^*(t)|^2+\mathbb{E}\int_t^T|N\delta z_j-z_j^*|^2ds\\ &\leqslant(2\|A\|+\|C\|+\|R_0^{-1}\|)\mathbb{E}\int_t^T|N\delta y_j-y_j^*|^2ds+\mathbb{E}\int_t^T|\delta y_{-i}-y^{**}|^2ds+\mathbb{E}\int_t^T|\delta p_{-i}-p^{**}|^2ds.\end{aligned}$$

Therefore, the result follows from Gronwall inequality with (44).

Lemma 6. Under Assumption 1–5, there exists a constant K independent of N such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\frac{1}{N}\sum_{j\neq i}x_1^j-\mathbb{E}x_1^1|^2\leqslant\frac{K}{N}.$$

Proof. First, note that

$$\begin{cases} d\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}\rho_j-\rho_1\right)=-\left[A\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}\rho_j-\rho_1\right)+\dfrac{-R_0^{-1}\mathbb{E}\check{p}-R_0^{-1}\mathbb{E}\check{x}_1+R_0^{-1}\check{x}_2+C\check{q}_2}{N}\right]dt,\\ \dfrac{1}{N}\displaystyle\sum_{j\neq i}\rho_j(T)-\rho_1(T)=0.\end{cases}$$

We have

$$\sup_{0\leqslant t\leqslant T}\left|\frac{1}{N}\sum_{j\neq i}\rho_j(t)-\rho_1(t)\right|\leqslant\frac{K}{N}. \tag{45}$$

Therefore,

$$\begin{cases} d\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}y_j-Ey_1\right)=-\left[A\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}y_j-Ey_1\right)+BR_1^{-1}B^\top\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}X_j-\mathbb{E}X_1\right)-\dfrac{C\mathbb{E}\check{y}-R_0^{-1}\mathbb{E}\check{p}}{N}\right]dt \\ \qquad\qquad +\dfrac{1}{N}\displaystyle\sum_{j\neq i}z_j dW_j(t), \\ d\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}X_j-\mathbb{E}X_1\right)=\left[A^\top\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}X_j-\mathbb{E}X_1\right)-\dfrac{\bar{\Gamma}_1\mathbb{E}\check{y}-C^\top\mathbb{E}\check{x}_1+C^\top\check{x}_2-\bar{\Gamma}_1\check{q}_2}{N}\right]dt \\ \qquad\qquad -\dfrac{1}{N}\displaystyle\sum_{j\neq i}Hz_j dW_j(t), \\ \left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}y_j-\mathbb{E}y_1\right)(T)=\dfrac{1}{N}\displaystyle\sum_{j\neq i}\xi_j-\mathbb{E}\xi, \\ \left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}X_j-\mathbb{E}X_1\right)(0)=G\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}\rho_j(0)-\mathbb{E}\rho_1(0)\right)-\dfrac{Gy_1(0)+\bar{\Gamma}_2\mathbb{E}\check{y}(0)-\bar{\Gamma}_2\mathbb{E}\check{y}_1(0)-\bar{\Gamma}_2\check{q}_2(0)}{N}. \end{cases}$$

By (45), it follows from the equation of $\frac{1}{N}\sum_{j\neq i}X_j-\mathbb{E}X_1$ that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{N}\sum_{j\neq i}X_j(t)-\mathbb{E}X_1(t)\right|\leqslant\frac{K}{N}.$$

Then, from the equation of $\frac{1}{N}\sum_{j\neq i}y_j(t)-\mathbb{E}y_1(t)$,

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{N}\sum_{j\neq i}y_j(t)-\mathbb{E}y_1(t)\right|\leqslant\frac{K}{N}.$$

Therefore, combining (43) and the above inequality,

$$\begin{aligned}\mathbb{E}\left|\frac{1}{N}\sum_{j\neq i}\widetilde{y}_j-\mathbb{E}\widetilde{y}_1\right|^2 &\leqslant 3\mathbb{E}\left|\frac{1}{N}\sum_{j\neq i}(\widetilde{y}_j-y_j)\right|^2+3\mathbb{E}\left|\frac{1}{N}\sum_{j\neq i}y_j-\mathbb{E}y_1\right|^2+3|\mathbb{E}y_1-\mathbb{E}\widetilde{y}_1|^2 \\ &\leqslant 3\frac{N-1}{N^2}\sum_{j\neq i}\mathbb{E}|(\widetilde{y}_j-y_j)|^2+3\mathbb{E}\left|\frac{1}{N}\sum_{j\neq i}y_j-\mathbb{E}y_1\right|^2+3|\mathbb{E}y_1-\mathbb{E}\widetilde{y}_1|^2\leqslant\frac{K}{N}.\end{aligned}$$

Finally, note that

$$\begin{cases} d\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}x_1^j-\mathbb{E}x_1^1\right)=A^\top\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}x_1^j-\mathbb{E}x_1^1\right)dt+\dfrac{1}{N}\displaystyle\sum_{j\neq i}H\widetilde{z}_j dW_j(t), \\ \left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}x_1^j-\mathbb{E}x_1^1\right)(0)=G\left(\dfrac{1}{N}\displaystyle\sum_{j\neq i}\widetilde{y}_j-\mathbb{E}\widetilde{y}_1\right)(0). \end{cases}$$

Therefore, by the standard estimation of SDE,

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{N}\sum_{j\neq i}x_1^j-\mathbb{E}x_1^1\right|^2\leqslant\frac{K}{N}.$$

Since the representation of the social cost is very complicated, we will rewrite the social cost with an abstract operator. Note that the main purpose of the new representation is just to simplify the proof of the asymptotic optimality. We will only give the following formulation without the explicit form of the operators:

$$2J_{soc}^{wo}(u(\cdot)) = \langle \mathcal{M}_2 u(\cdot), u(\cdot)\rangle + 2\langle \mathcal{M}_1, u(\cdot)\rangle + \mathcal{M}_0, \tag{46}$$

where $\mathcal{M}_2$ is an L^2 bounded self-adjoint linear operator, $\mathcal{M}_1$ is an L^2 bounded operator and $\mathcal{M}_0 \in \mathbb{R}$. Please refer to [24, 53] for more information. The main result of this section is the following asymptotic optimality result.

Theorem 1. Let Assumptions 1–5 hold. Then $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_N)$ is a $\left(\frac{1}{\sqrt{N}}\right)$-optimal strategy for the agents.

Proof. In order to prove asymptotic optimality, it suffices to consider the perturbations $u_i \in \mathcal{U}_i^c$ such that $\mathcal{J}_{soc}^{wo}(u_1, \cdots, u_N) \leqslant \mathcal{J}_{soc}^{wo}(\widetilde{u}_1, \cdots, \widetilde{u}_N)$. It is easy to check that

$$\mathcal{J}_{soc}^{wo}(\widetilde{u}_1, \cdots, \widetilde{u}_N) \leqslant KN,$$

where K is a constant independent of N. Therefore, in the following we only consider the perturbations $u_i \in \mathcal{U}_i^c$ satisfying

$$\sum_{i=1}^{N} \mathbb{E}\int_0^T |u_i|^2 dt \leqslant KN. \tag{47}$$

Now consider a perturbation $u = \widetilde{u} + (\delta u_1, \cdots, \delta u_N) := \widetilde{u} + \delta u$, where $\delta u_i = u_i - \widetilde{u}_i$. Therefore, by Lemma 2 and (47), there exists a constant K independent of N such that

$$\sup_{1\leqslant j\leqslant N, j\neq i}\left[\mathbb{E}\sup_{0\leqslant t\leqslant T}|\delta y_j(t)|^2 + \mathbb{E}\sup_{0\leqslant t\leqslant T}|\delta p_j(t)|^2\right] \leqslant \frac{K}{N}.$$

Furthermore, by (46), we have

$$\begin{aligned}
2\mathcal{J}_{soc}^{wo}(\widetilde{u} + \delta u) &= \langle M_2(\widetilde{u} + \delta u), \widetilde{u} + \delta u\rangle + 2\langle M_1, \widetilde{u} + \delta u\rangle + M_0 \\
&= 2\mathcal{J}_{soc}^{wo}(\widetilde{u}) + 2\sum_{i=1}^{N}\langle M_2\widetilde{u} + M_1, \delta u_i\rangle + \langle M_2\delta u, \delta u\rangle,
\end{aligned}$$

where $\langle M_2\widetilde{u} + M_1, \cdot\rangle$ is the Fréchet differential of $\mathcal{J}_{soc}^{wo}$ with $\widetilde{u}$. Moreover, by (28) we know that

$$\begin{aligned}
\langle M_2\widetilde{u} + M_1, \delta u_i\rangle =& \mathbb{E}\Bigg[\int_0^T \Big(\langle R_1\widetilde{u}_i, \delta u_i\rangle + \langle C^\top \mathbb{E}\check{x}_1 - C^\top \check{x}_2, \delta y_i\rangle + \langle H\widetilde{z}_i, \delta z_i\rangle \\
&+ \langle -R_0^{-1}\hat{p} + R_0^{-1}\mathbb{E}\check{x}_1 - R_0^{-1}\check{x}_2 + C^\top \check{q}_2, \delta p_i\rangle\Big)dt \\
&+ \langle G\widetilde{y}_i(0) - \bar{\Gamma}_2\mathbb{E}\check{y}(0) + \bar{\Gamma}_2\check{q}_2(0), \delta y_i(0)\rangle\Bigg] + \sum_{l=1}^{8}\varepsilon_l.
\end{aligned} \tag{48}$$

Applying the Cauchy-Schwarz inequality, we obtain that

$$\mathcal{J}_{soc}^{wo}(\widetilde{u} + \delta u) - \mathcal{J}_{soc}^{wo}(\widetilde{u}) \geqslant -\sqrt{\sum_{i=1}^{N}|M_2\widetilde{u} + M_1|^2\sum_{i=1}^{N}|\delta u_i|^2} + \frac{1}{2}\langle M_2\delta u, \delta u\rangle \geqslant -|M_2\widetilde{u} + M_1|O(N).$$

7 Conclusion

This study investigates robust control in large-population systems through LQ mean-field teams under drift uncertainty, where agent dynamics is modeled by BSDEs. Unlike forward SDE frameworks, our BSDE-based formulation yields a coupled forward-backward Hamiltonian system with distinctive features. The BSDE approach incorporates terminal constraints and adapted processes $(y(\cdot), z(\cdot))$, making it suitable for problems with predefined terminal targets. To address drift uncertainty, we introduce a negative quadratic penalty on the adversarial disturbance f in the cost function and derive the worst-case scenario. By applying the person-by-person optimality principle in a forward-backward Hamiltonian system, we derive a robust decentralized strategy and establish the well-posedness of the consistency condition system via Riccati decoupling. The distinct formulation of the adjoint equations leads to an alternative convergence analysis for establishing asymptotic optimality. Theoretical contributions include an extension of mean-field team theory to backward stochastic frameworks and a general robust control structure for cooperative large population systems.

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