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Measure-based uncertainty with Dempster-Shafer structure

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Abstract Fuzzy measure, also referred to as a monotonic measure, proves to be a powerful tool for describing the value of the uncertain variable. Its effectiveness stems from the accommodation of many uncertainties, including probabilistic, possibilistic, and epistemic, due to their relaxed constraints. However, measure-based uncertainty has not been widely developed due to difficulties in agreeing with other theories in modeling and handling uncertainty. In this paper, we develop the construction and processing approaches for measure-based uncertainty utilizing the Dempster-Shafer structure. In the realm of uncertainty modeling, the t canonical decomposition operates on fuzzy measures and offers an interpretable construction approach, which boasts more convenient interfaces for both knowledge and data. In the realm of uncertainty handling, the conjunctive and disjunctive combination rules of Dempster-Shafer granules are extended to handle measure-based uncertainty, thereby achieving information processing consistency across various uncertainty frameworks.

Keywords fuzzy measure, information fusion, uncertainty representation, Dempster-Shafer theory, evidential reasoning

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1 Introduction

Fuzzy measure [1], also called the monotonic measure, is an uncertainty representation that imposes monotonicity constraints on generalized measure theory [2]. It can be seen as an information granule to describe the restriction of an uncertain variable via assigning anticipations to the elements of a power set [3]. In measure-based uncertainty, monotonicity means that for propositions with entailment relations, the specific proposition's anticipation cannot be larger than the general proposition's. This constraint is without loss of generality and in line with the agent's attitude toward uncertain variables, intuitively. A fuzzy measure provides a unified information distribution across probability, possibility, and their imprecise versions [4]. When the measure satisfies the additivity, the anticipation of the proposition equals the probability. When the measure satisfies maxitivity and minitivity, the anticipation of the proposition equals possibility or necessity. Furthermore, a Dempster-Shafer granule's belief function and plausibility function are both fuzzy measures and correspond to pessimistic and optimistic scenarios, respectively [5]. Due to its powerful ability to model uncertainty, fuzzy measure is widely applied in multi-source information fusion [6, 7], system reliability modeling [8, 9], and decision making in uncertain environments [10]. Despite the presence of mathematically equivalent representations with other information granules, the processing consistency between fuzzy measures and their respective frameworks remains unsatisfactory. This discrepancy arises from the distinct generation and processing logic inherent in various uncertainty theories, such as additive-based probabilistic uncertainty and max-min-based possibilistic uncertainty. Therefore, how to establish a universal framework for modeling and handling measure-based uncertainty remains an open issue.

In terms of handling uncertainty, the fusion of fuzzy measures is commonly similar to the fuzzy set. Given that the aggregation operator inherently satisfies monotonicity, aggregating arbitrary fuzzy measures yields a result that remains a fuzzy measure [11]. However, ignoring the specific meaning of the

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uncertainty represented by the measure usually produces counter-intuitive results. For instance, consider two belief functions from independent and reliable sources. When employing the conjunctive aggregation operator (product operator) to fuse them, the conjunction implies an "anding" operation, suggesting a more specific result [11]. However, in Dempster-Shafer theory, the product of belief functions is equivalent to the disjunctive combination for normalized granules, resulting in a more general outcome [12]. It is indeed disconcerting that employing the same information within a different framework yields entirely contrasting outcomes. In addition, a crucial concern is the effective combination of bodies of evidence characterized by different types of uncertainty, especially for probability and possibility [13]. Yager [11] employed the product t-norm and probability t-conorm to fuse possibilistic and probabilistic uncertainty in conjunctive and disjunctive cases, respectively. However, the specific implications of the generated measure lack justification. Therefore, an additional objective of this paper is to introduce novel combination rules tailored to measure-based uncertainties. The proposed approaches should reach the consistency of information processing under different uncertainty theory frameworks.

Dempster-Shafer theory serves as an effective tool for reasoning in uncertain environments and has established a mature theoretical framework in modeling and handling uncertainty [14,15]. In Dempster-Shafer theory, the most prevalent information distribution is a mass function that allocates normalized belief masses to the power set of the frame, commonly known as a basic probability assignment (BPA) [16]. In this paper, information distributions with identical BPA's information content are collectively termed Dempster-Shafer granules. In the realm of uncertainty modeling, Dempster-Shafer theory, as a non-additive extension of probability theory, adeptly models unreliable testimonies and empirical ignorance in a more rational manner. Moreover, it exhibits domain overlap with various uncertainty theories regarding representations, encompassing possibility theory [17], order-2 information granules [18], and imprecise probability theory [19,20]. In the realm of uncertainty handling, Dempster's rule of combination effectively fuses BPAs from independent and reliable sources [21, 22]. This not only aligns with the principles of the Bayesian rule but also models the conflict relationships between Dempster-Shafer granules [23]. Leveraging the advantages of modeling and handling uncertainty, Dempster-Shafer theory finds extensive applications in areas such as unreliable information fusion [24–26], expert decision-making [27, 28], state assessment [29], and machine learning within uncertain environments [30–32]. Consequently, our goal is to derive insights for treating fuzzy measures from the Dempster-Shafer theory.

In this paper, we characterize a fuzzy measure through its assurance and opportunity measures, considering them as pseudo belief and plausibility functions, respectively. The inverse transformation of these functions can generate a pseudo BPA, which still adheres to the normalization requirement but does not ensure that the belief masses fall within the interval $[0, 1]$. Several effective evidential information measures are introduced to quantify uncertainty and similarity in measure-based uncertainty. Then, the t function derived from t canonical decomposition [33] provides a novel information representation for fuzzy measures, which decomposes the fuzzy measure as two components: propensity and attitude. Subsequently, a general construction approach for fuzzy measures with an interpretable process is proposed. In terms of information processing, combination rules for fuzzy measures are developed within the Dempster-Shafer structure, aiming to resolve counter-intuitive results observed in previous fuzzy measure fusion techniques.

The structure of this paper is organized as follows. The basic notions of uncertainty theories are introduced in Section 2. Section 3 represents fuzzy measure using the Dempster-Shafer structure and provides some novel information metrics. In Section 4, we employ the t function to interpret measure-based uncertainty and offer an interpretable construction approach. Section 5 discusses the conjunctive and disjunctive rules in measure-based uncertainty from the perspective of Dempster-Shafer theory. Section 6 summarizes the whole paper and discusses the further research directions.

2 Uncertainty theories

2.1 Dempster-Shafer theory

Consider an uncertain variable X with values located in the frame $\Omega = \{\omega_1, \ldots, \omega_n\}$. A mapping m on $2^\Omega \to [0, 1]$ is employed to model the restriction of X, satisfying $\sum_{F_i \in 2^\Omega} m(F_i) = 1$. This mapping is referred to as a BPA, also known as a mass function. For a proposition F_i that satisfies $m(F_i) > 0$, it is denoted by a focal set. When $m(\emptyset) > 0$, m is an unnormalized mass function, implying that the value of X may lie outside the frame Ω. In an n-element frame, its subsets can be efficiently represented using n-ary binary code, offering a convenient order for programming m as a vector, $\boldsymbol{m} = [m(\emptyset), m(\{\omega_1\}), m(\{\omega_2\}), m(\{\omega_1\omega_2\}), \ldots, m(\Omega)]^{\mathrm{T}}$. Based on the set-relation, there are some identical information content representations, belief (Bel) function, plausibility (Pl) function, implicability (b) function, commonality (q) function, and their inverse transformations are defined as follows [34]:

$$\begin{aligned}
&\mathrm{Bel}(F_i) = \sum_{\emptyset \neq F_j \subseteq F_i} m(F_j), \quad m(F_i) = \sum_{F_j \subseteq F_i} (-1)^{|F_i|-|F_j|}\mathrm{Bel}(F_j), \quad m(\emptyset) = 1 - \mathrm{Bel}(\Omega),\\
&\mathrm{Pl}(F_i) = \sum_{F_i \cap F_j \neq \emptyset} m(F_j), \quad m(F_i) = \sum_{F_j \subseteq F_i} (-1)^{|F_i|-|F_j|+1}\mathrm{Pl}(\overline{F_j}), \quad m(\emptyset) = 1 - \mathrm{Pl}(\Omega),\\
&b(F_i) = \sum_{F_j \subseteq F_i} m(F_j), \quad m(F_i) = \sum_{F_i \subseteq F_j} (-1)^{|F_i|-|F_j|} b(F_j),\\
&q(F_i) = \sum_{F_i \subseteq F_j} m(F_j), \quad m(F_i) = \sum_{F_i \subseteq F_j} (-1)^{|F_j|-|F_i|} q(F_j).
\end{aligned} \tag{1}$$

These transformations can also be implemented via the matrix calculus efficiently:

$$\begin{aligned}
&\mathbf{Bel} = \boldsymbol{m2Bel} \cdot \boldsymbol{m}, \ m2\mathrm{Bel}(F_i, F_j) = \begin{cases} 1, & \emptyset \neq F_j \subseteq F_i,\\ 0, & \text{others,} \end{cases}\\
&\boldsymbol{b} = \boldsymbol{m2b} \cdot \boldsymbol{m}, \ m2b(F_i, F_j) = \begin{cases} 1, & F_j \subseteq F_i,\\ 0, & \text{others,} \end{cases}\\
&\mathbf{Pl} = \boldsymbol{m2Pl} \cdot \boldsymbol{m}, \ m2\mathrm{Pl}(F_i, F_j) = \begin{cases} 1, & F_j \cap F_i \neq \emptyset, F_i = F_j = \emptyset,\\ 0, & \text{others,} \end{cases}\\
&\boldsymbol{q} = \boldsymbol{m2q} \cdot \boldsymbol{m}, \ m2q(F_i, F_j) = \begin{cases} 1, & F_i \subseteq F_j,\\ 0, & \text{others.} \end{cases}
\end{aligned} \tag{2}$$

The belief interval $\mathrm{BI} = [\mathrm{Bel}, \mathrm{Pl}]$ serves as an effective tool for quantifying the uncertainty of Dempster-Shafer granules, representing the lower and upper beliefs (and probability) of the proposition. The aforementioned measures not only capture uncertainty from comprehensive and intuitive perspectives but also facilitate convenient operations in information processing. The most popular combination rules, namely the conjunctive combination rule (CCR) and the disjunctive combination rule (DCR), are defined as follows [35]:

$$\begin{aligned}
&m_1 \textcircled{\cap} m_2(F_i) = \sum_{F_j \cap F_k = F_i} m_1(F_j) m_2(F_k), \quad q_1 \textcircled{\cap} q_2(F_i) = q_1(F_i) q_2(F_i),\\
&m_1 \textcircled{\cup} m_2(F_i) = \sum_{F_j \cup F_k = F_i} m_1(F_j) m_2(F_k), \quad b_1 \textcircled{\cup} b_2(F_i) = b_1(F_i) b_2(F_i).
\end{aligned} \tag{3}$$

In addition, canonical decomposition can also generate identical information representation. Based on the Smets' canonical decomposition, the diffidence function σ and its dual v are defined as

$$\sigma(F_i) = \frac{\prod_{F_i \subseteq F_j; |F_j|-|F_i| \in 2\mathbb{N}-1} q(F_j)}{\prod_{F_i \subseteq F_k; |F_k|-|F_i| \in 2\mathbb{N}} q(F_k)}, \quad v(F_i) = \overline{\sigma}(\overline{F_i}), \tag{4}$$

where $\overline{\sigma}$ is the σ function of the negation of m. Based on the σ and v functions, the cautious and bold rules are extended via minimum t-norm [35]. Pichon proposed the t-canonical decomposition based on the representation multivariate Bernoulli distribution (MBD), for a mass function m, its t function is defined as [33]

$$t(F_i) = \begin{cases} \mathrm{Pl}(F_i), & |F_i| = 1,\\ \bigotimes_{j=1}^{n} \begin{bmatrix} 1 & 1\\ -\mathrm{Pl}(\{\omega_j\}) & 1 - \mathrm{Pl}(\{\omega_j\}) \end{bmatrix} \boldsymbol{m}, & \text{others.} \end{cases} \tag{5}$$

Within the Dempster-Shafer theory framework, the t function decomposes the mass function into two components: propensity and dependency, corresponding to the marginal probability and variance in the MBD.

There are co-dominated information representations of the Dempster-Shafer granules with both probability and possibility theories. When the beliefs are concentrated on singletons, it is called a Bayesian mass function and is equivalent to a probability distribution. When the focal sets of a mass function are nested, i.e., satisfying inclusion relationships, it is called a consonant mass function and is equivalent to a possibility distribution. For general mass functions, pignistic probability transformation (PPT) can generate probability distribution BetP_m, and contour functions can generate possibility distributions Poss_m, which are denoted as [36]

$$\mathrm{BetP}_m(\omega) = \sum_{\omega \in F_i} \frac{m(F_i)}{|F_i|}, \quad \mathrm{Poss}_m(\omega) = \mathrm{Pl}(\{\omega\}) = \sum_{\omega \in F_i} m(F_i). \tag{6}$$

2.2 Fuzzy measure theory

Similar to Dempster-Shafer theory to discuss the value of variable X, a fuzzy measure μ is defined as the restriction of X through a mapping $2^\Omega \to [0,1]$. If μ satisfies the property: $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and $\mu(F_i) \geqslant \mu(F_j)$ for $F_j \subseteq F_i$, it will be a fuzzy measure, also called a monotonic measure [37]. $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$ state that the value of X must be within Ω. In terms of uncertainty modeling, for a fuzzy measure μ over an n-element frame Ω, its identical information representation, the dual measure $\hat{\mu} = 1 - \mu(\overline{F_i})$, also plays an important role in modeling the uncertainty of X. $\mu(F_i)$ denotes the anticipation of the value of X located in F_i, and $\hat{\mu}(F_i)$ denotes the anticipation that the value of X is not in the subset $\Omega \setminus F_i$. It is worth noting that if $\mu(\emptyset) \neq 0$, its dual measure is $\hat{\mu} = \mu(\Omega) - \mu(\overline{F_i})$. When μ adheres to additional constraints, it signifies specific uncertainties. For more detailed presentations, please refer to [37]. Assurance and opportunity are dual measures to represent the lower and upper anticipation of measure-based uncertainty. For a fuzzy measure μ, its assurance λ and opportunity ψ are defined as [38]

$$\lambda(F_i) = \mu(F_i) \wedge \hat{\mu}(F_i), \quad \psi(F_i) = \mu(F_i) \vee \hat{\mu}(F_i). \tag{7}$$

In terms of uncertainty handling, since executing aggregation operators and triangular norms on fuzzy measures results in fuzzy measures, the fuzzy measure shares the same fusion framework as fuzzy sets. For a detailed presentation, please refer to [11]. In terms of the connections with singletons' contributions, the Shapley index is the effective choice. For a fuzzy measure μ under an n-element frame Ω, its marginal contribution is denoted as [37]

$$S(\omega) = \sum_{j=0}^{n-1} \frac{(n-j-1)!j!}{n!} \sum_{F_i \subseteq \Omega \setminus \{\omega\}; |F_i| = j} \left(\mu(F_i \cup \{\omega\}) - \mu(F_i)\right). \tag{8}$$

Yager [4] had demonstrated that for fuzzy measures generated from the belief structure m, their Shapley indices are identical. Since the Bel function is one of these measures [5], they are equal to BetP_m in (6). In terms of information measures, the entropy H and the attitudinal character AC are used together to describe the information content of a fuzzy measure, which are defined as follows [4]:

$$\begin{aligned} H(\mu) &= -\sum_{\omega \in \Omega} S(\omega) \log S(\omega), \\ \mathrm{AC}(\mu) &= \frac{1}{n-1} \sum_{k=0}^{n-1} \frac{(n-k-1)(n-k-1)!k!}{n!} \sum_{|F_i| = k} \sum_{\omega \notin F_i} \left(\mu(F_i \cup \{\omega\}) - \mu(F_i)\right). \end{aligned} \tag{9}$$

3 Representing measure-based uncertainty as Dempster-Shafer granule

3.1 Motivation

As mentioned in Section 1, although fuzzy measures have powerful uncertainty modeling capabilities, they may produce counter-intuitive results when updating information due to the lack of a specific theoretical

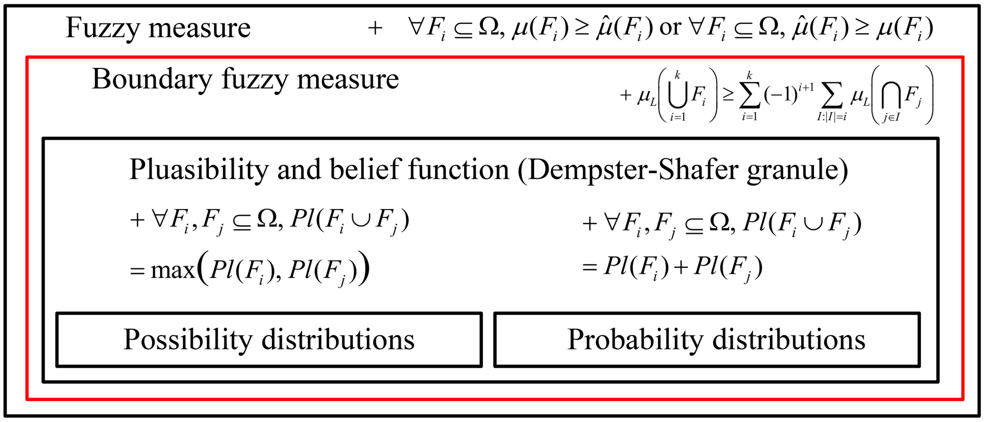

Figure 1 (Color online) Relationships between boundary fuzzy measures and other uncertainty representation frameworks.

basis. Therefore, we aim to find an information representation whose range is narrower than that of fuzzy measures but broader than the Dempster-Shafer granule, and whose uncertainty can be managed using the methods of Dempster-Shafer theory.

3.2 Boundary fuzzy measure

In order to establish the entailment relationship in measure-based uncertainty, Yager [38] introduced the assurance and opportunity to represent the lower and upper anticipations. For a proposition F_i, assurance is the pessimistic result between μ and $\hat{\mu}$, and opportunity is the optimistic result between μ and $\hat{\mu}$. In the case of the probabilistic uncertainty, μ satisfies $\lambda = \mu = \psi$, which indicates that the anticipation of μ is certain. In the case of the possibilistic uncertainty, μ satisfies $\hat{\mu} = \lambda < \psi = \mu$, i.e., $\forall\ F_i \subseteq \Omega$ $\mu(F_i) \geqslant \hat{\mu}(F_i)$. In the case of the Dempster-Shafer structure, the Bel function and Pl function equal assurance and opportunity, respectively. Unlike the reversible transformations of belief and plausibility functions, the fuzzy measure cannot be inversely generated given λ and ψ. Thus, for fuzzy measures having the same λ and ψ, they are indistinguishable in the entailment relationship. Therefore, from the perspective of the entailment relationship, assurance and opportunity provide the distinguishable boundary of the measure-based uncertainty. In this paper, we continue this idea and formalize this boundary into a new uncertainty representation, called the boundary fuzzy measure.

Definition 1 (Boundary fuzzy measure)**.** Under an n-element frame Ω, its boundary fuzzy measure $\mathcal{BFM} = (\mu_{\rm L}, \mu_{\rm U})$ are mapping $2^{\Omega} \to [0, 1]$, where $\mu_{\rm L}$ is the lower fuzzy measure, and $\mu_{\rm U}$ is the upper fuzzy measure. They satisfy the following properties: (1) $\mu_{\rm L,U}(\emptyset) = 0$; (2) $\mu_{\rm L,U}(\Omega) = 1$; (3) If $F_i \subseteq F_j$, $\mu_{\rm L,U}(F_i) \leqslant \mu_{\rm L,U}(F_j)$; (4) $\forall F_i \subseteq \Omega$, $\mu_{\rm U}(F_i) \geqslant \mu_{\rm L}(F_i)$; (5) $\mu_{\rm L}(F_i) = 1 - \hat{\mu}_{\rm U}(\overline{F_i})$.

In cases where a fuzzy measure and its dual exhibit a complete dominance relationship, it is denoted that the general measure is $\mu_{\rm U}$ and the specific measure is $\mu_{\rm L}$. As a measure-based uncertainty with additional constraints on fuzzy measures, the relationship between boundary fuzzy measure and other uncertainty theories is shown in Figure 1.

Proposition 1 (Fuzzy measure to boundary fuzzy measure)**.** Given a fuzzy measure μ under an n-element frame Ω, its assurance λ and opportunity ψ compose a boundary fuzzy measure $\mathcal{BFM} = (\lambda, \psi)$.

Proof. Please refer to Section 1 in the supplementary material.

To assess the competency of the boundary fuzzy measure in modeling measure-based uncertainty, we discuss the distinction between the boundary fuzzy measure and the fuzzy measure from the information measurement perspective.

Proposition 2 (Entropy)**.** Given a fuzzy measure μ and its boundary measure (λ, ψ), the Shapley indexes and corresponding entropies satisfy $S_\mu = S_{\hat{\mu}} = S_\lambda = S_\psi$ and $H(\mu) = H(\hat{\mu}) = H(\lambda) = H(\psi)$.

Proof. Please refer to Section 1 in the supplementary material.

Proposition 3 (Attitudinal character)**.** Given a fuzzy measure μ and its boundary measure (λ, ψ), their attitudinal characters satisfy, $\mathrm{AC}(\psi) \geqslant \mathrm{AC}(\lambda)$ and $\mathrm{AC}(\mu), \mathrm{AC}(\hat{\mu}) \in [\mathrm{AC}(\psi), \mathrm{AC}(\lambda)]$.

Proof. Please refer to Section 1 in the supplementary material.

As demonstrated in Propositions 2 and 3, the Shapley indices of μ and $\mathcal{BFM}$ are identical, with $\mathcal{BFM}$ providing a broader attitudinal range where μ and $\hat{\mu}$ are located. From the perspective of information transformation, multiple fuzzy measures can point to the same boundary fuzzy measure, yet they share the same anticipation interval for each proposition. In addition, the opportunity and assurance can reflect the upper and lower bounds more intuitively. Instead of fuzzy measures, the boundary fuzzy measure is employed in this paper to model measure-based uncertainty.

3.3 Representing boundary fuzzy measure on Dempster-Shafer structure

In Dempster-Shafer theory and possibility theory, fuzzy measures consistently appear in pairs (belief and plausibility functions, possibility and necessity distributions), utilizing the range of the interval to depict the uncertainty of the corresponding subset's anticipation. Eq. (2) provides some reversible paths between the fuzzy measure and BPA, in this paper, these paths are used to represent the boundary fuzzy measure using Dempster-Shafer structure.

Definition 2 (Pseudo Dempster-Shafer granules)**.** Given a boundary fuzzy measure $\mathcal{BFM} = (\mu_{\rm L}, \mu_{\rm U})$ on n-element frame Ω, its pseudo Dempster-Shafer granule can be generated using the inverse path from BPA to Bel and Pl functions. When the $\mu_{\rm L}$ is seen as the pseudo belief function pBel, or $\mu_{\rm U}$ is seen as the pseudo plausibility function pPl, the pseudo BPA (PBPA) pm is denoted as

$$\begin{aligned} pm(F_i) &= \sum_{F_j \subseteq F_i} (-1)^{|F_i|-|F_j|} \mu_L(F_j), \quad pm(\emptyset) = 1 - \mu_{\rm L}(\Omega), \quad \boldsymbol{pm} = \boldsymbol{m2\mathrm{Bel}}^{-1} \cdot \boldsymbol{\mu}_{\rm L}, \\ pm(F_i) &= \sum_{F_j \subseteq F_i} (-1)^{|F_i|-|F_j|+1} \mu_U(\overline{F_j}), \quad pm(\emptyset) = 1 - \mu_{\rm U}(\Omega), \quad \boldsymbol{pm} = \boldsymbol{m2\mathrm{Pl}}^{-1} \cdot \boldsymbol{\mu}_{\rm U}. \end{aligned} \tag{10}$$

As depicted in Figure 1, the constraints of the boundary fuzzy measure are smoother than those of belief and plausibility functions.

Example 1. Given a fuzzy measure $\mu = \{0, 0.1, 0.2, 0.8, 0.15, 0.9, 0.95, 1\}$, it can be written as the boundary fuzzy measure $\mathcal{BFM} = \{\mu_{\rm L}, \mu_{\rm U}\}$, where $\mu_{\rm L} = p\mathrm{Bel} = \{0, 0.05, 0.1, 0.8, 0.15, 0.8, 0.9, 1\}$ and $\mu_{\rm U} = p\mathrm{Pl} = \{0, 0.1, 0.2, 0.85, 0.2, 0.9, 0.95, 1\}$. According to (10), (2), (4), and (5), the pseudo Dempster-Shafer granules are

$$\begin{aligned} &pm = \{0, 0.05, 0.1, 0.65, 0.15, 0.6, 0.65, -1.2\},\ pq = \{1, 0.1, 0.2, -0.55, 0.2, -0.6, -0.55, -1.2\}, \\ &p\sigma = \{0.026, -2.75, -1.26, 2.18, -1.375, 2, 2.18\},\ pt = \{1, 0.1, 0.2, -0.57, 0.2, -0.62, -0.59, -0.907\}. \end{aligned}$$

Considering the meanings of the pseudo Dempster-Shafer granules in Example 1, the pseudo Bel and Pl functions maintain their original interpretations, representing the lower and upper bounds of propositions' beliefs. However, regarding the mass function pm, the commonality function pq, and the diffidence function $p\sigma$, they all produce negative values and cannot be interpreted the meanings within the Dempster-Shafer structure. Therefore, these information representations should be used cautiously in information processing to prevent the generation of results that cannot be inverted to the fuzzy measure. Nevertheless, in the current representation, they can still reverse the boundary fuzzy measure; it is possible to quantify the uncertainty of the fuzzy measure by substituting them into the information measures in Dempster-Shafer theory.

In t-canonical decomposition (Eq. (5)), the t function can be decomposed into two components. The first component comprises the singletons, which equal the contour function of m and represent the marginal probability in the MBD. The second component comprises the propositions with multiple elements, representing the dependence among its elements. In the pseudo t function, it retains the same meaning as in the Dempster-Shafer theory. Therefore, in subsequent information processing, we favor the Bel, Pl, and t functions as the information representations or processing mediums.

It is important to note that the primary aim of modeling the fuzzy measure within the Dempster-Shafer structure is not to directly manipulate the pseudo-information distribution using the rules of Dempster-Shafer theory. Instead, it is to leverage its well-developed semantics and comprehensive information handling methods, which are effective for guiding measure-based uncertainty processing and managing uncertainty under a unified framework.

3.4 Information measures of boundary fuzzy measure

We will offer some information measures for the fuzzy measure from the perspectives of uncertainty and distance measures, respectively, leveraging the aid of pseudo-Dempster-Shafer granules.

3.4.1 *Uncertainty measure*

For a fuzzy measure, the Shannon entropy of the Shapley index and the attitudinal characters can adequately describe its uncertainty from two perspectives. However, for the boundary fuzzy measure proposed in this paper, these two metrics may produce counter-intuitive results. For instance, consider two fuzzy measures $\mu_1 = \{0,0,0,0,0,0,0,1\}$ and $\mu_2 = \{0,1,1,1,1,1,1,1\}$, according to (9), the entropies of them both are $\log 3$, and $\mathrm{AC}(\mu_1) = 0$ and $\mathrm{AC}(\mu_2) = 1$. However, according to Proposition 1, their boundary fuzzy measures are identical, with $\mu_{\mathrm{L}} = \mu_1$ and $\mu_{\mathrm{U}} = \mu_2$, and their uncertainties should also be identical. To address this issue, we extend the total uncertainty measures in Dempster-Shafer theory to measure-based uncertainty and assess their validity through several numerical examples. The total uncertainty measure under the Dempster-Shafer structure is usually defined through the Bel function, the Pl function, and the BPA [39]. For a boundary fuzzy measure, their μ_{L} and μ_{U} have the same meaning as Bel and Pl functions, respectively, allowing them to be directly substituted into the total uncertainty measures. Since PBPA may include negative values, performing a logarithmic operation is not feasible. Moreover, the multiplication of two negative values results in a positive number, leading to counter-intuitive outcomes. Hence, for the total uncertainty measures proposed based on BPA, this paper employs a heuristic approach to improve their performance under measure-based uncertainty.

Definition 3 (Dempster-Shafer structure-based uncertainty measures)**.** Given a fuzzy measure μ under an n-element frame Ω, and its boundary fuzzy measure is $\mathcal{BFM} = \{\mu_{\mathrm{L}}, \mu_{\mathrm{U}}\}$. There are four Dempster-Shafer structure-based uncertainty measures of μ, which are extended from the distance-based uncertainty measure [40], Jirošuek and Sheony's (JS) entropy [41], SU measurement [42], and fractal-based belief (FB) entropy [43], respectively.

(1) The interval distance-based uncertainty measure TU_{d} of μ is defined as

$$TU_{\mathrm{d}}(\mu) = 1 - \frac{\sqrt{3}}{n}\sum_{\omega\in\Omega}\left(\frac{1}{3}\left[\mu_L^2(\{\omega\}) + \mu_U^2(\{\omega\}) + \mu_L(\{\omega\})\mu_U(\{\omega\}) - 2\mu_U(\{\omega\}) - \mu_L(\{\omega\}) + 1\right]\right)^{\frac{1}{2}};$$

(2) JS entropy E_{JS} of μ is defined as

$$E_{\mathrm{JS}}(\mu) = -\sum_{\omega\in\Omega}\frac{\mu_{\mathrm{U}}(\{\omega\})}{\sum_{\omega_j\in\Omega}\mu_{\mathrm{U}}(\{\omega_j\})}\log\frac{\mu_{\mathrm{U}}(\{\omega\})}{\sum_{\omega_j\in\Omega}\mu_{\mathrm{U}}(\{\omega_j\})} + \sum_{F_i\subseteq\Omega} pm(F_i)\log|F_i|;$$

(3) SU measurement SU of μ is defined as

$$SU(\mu) = -\sum_{\omega\in\Omega}\left[-\frac{\mu_{\mathrm{L}}(\{\omega\}) + \mu_{\mathrm{U}}(\{\omega\})}{2}\log\frac{\mu_{\mathrm{L}}(\{\omega\}) + \mu_{\mathrm{U}}(\{\omega\})}{2} + \frac{1}{2}(\mu_{\mathrm{U}}(\{\omega\}) - \mu_{\mathrm{L}}(\{\omega\}))\right];$$

(4) The FB entropy E_{FB} of μ is defined as

$$E_{\mathrm{FB}}(\mu) = -\sum_{F_i\in\{2^\Omega\setminus\emptyset\}} pm_{\mathrm{FB}}(F_i)\cdot\log \mathrm{abs}(pm_{\mathrm{FB}}(F_i)),\ pm_{\mathrm{FB}}(F_j) = \sum_{F_i\subseteq F_j}\frac{pm(F_i)}{2^{|F_j|}-1},$$

where $\mathrm{abs}(\cdot)$ is the absolute value.

In Definition 3, the Bel and Pl functions in total uncertainty measures of Dempster-Shafer theory are replaced by μ_{L} and μ_{U}, respectively. For the PBPA pm, when it represents the weights of information content, it remains consistent with m; however, when it represents the information content itself, it is replaced by the absolute value.

Example 2. Consider a cardinality-based fuzzy measure μ is determined by an orness measure. Under a 4-element frame, a 4-dimensional weight vector $[w(1), w(2), w(3), w(4)]$ can be determined by an orness measure Orn based on the maximum entropy OWA. The cardinality-based measure is denoted as $\mu(F_i) = \sum_{j\leqslant|F_i|} w(j)$. When Orn changes from 0 to 1, the variation of four uncertainty measures in Definition 3 are shown in Figure 2.

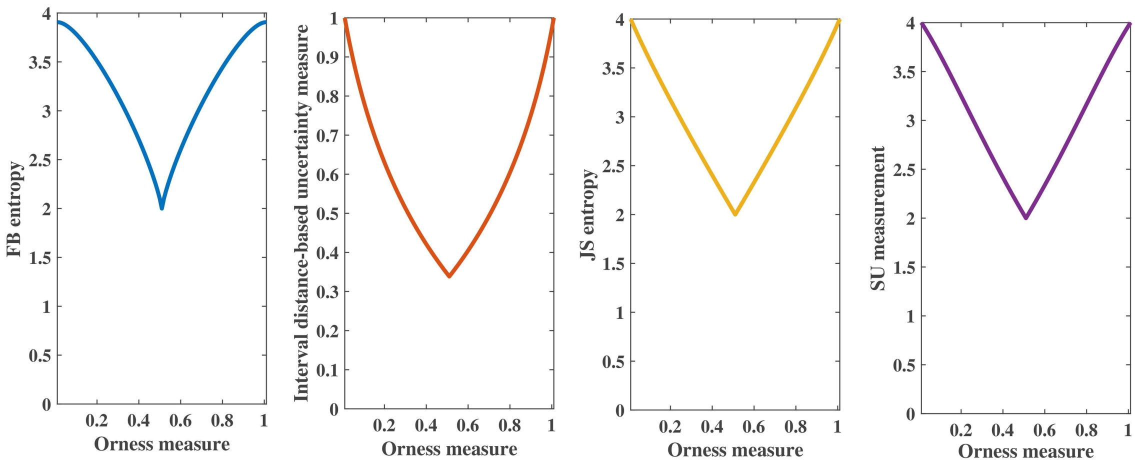

Figure 2 (Color online) Variations of four uncertainty measures in cardinality-based measures.

In Example 2, uncertainty measures inspired by the Dempster-Shafer theory reveal that variations in uncertainties are symmetrical, which resolves the above issue. When Orn $= 0.5$, i.e., when μ is a probabilistic measure, the uncertainty reaches its minimum value. When modeling uncertainty using the boundary fuzzy measure, for a proposition, an interval value is used to describe its anticipation. In contrast to the fuzzy measure, the interval value indicates the corresponding pessimistic and optimistic scenarios. In Example 2, when Orn changes from 0 to 0.5, the corresponding boundary fuzzy measures are the same as when Orn changes from 1 to 0.5. Hence, their uncertainties' variations are symmetry. From the perspective of total uncertainty measures [44], non-additive measures represent imprecise extensions of probabilistic measures. Consequently, the total uncertainty associated with non-additive measures tends to be larger than that of additive measures when they possess equal discord measures, i.e., equal entropy of Shapley indexes.

Example 3. Consider two fuzzy measures under a 4-element frame

$$\mu_1 = \{0, 0.2, 0.3, 0.5, 0.1, 0.3, 0.3, 0.8, 0.2, 0.3, 0.5, 0.9, 0.6, 0.85, 0.8, 1\},$$
$$\mu_2 = \{0, 0.2, 0.3, 0.6, 0.1, 0.5, 0.5, 0.8, 0.2, 0.4, 0.6, 0.9, 0.7, 0.85, 0.8, 1\}.$$

Their Dempster-Shafer structure-based uncertainty measures are

$$E_{\mathrm{FB}}(\mu_1) = 2.0706, \quad TU_{\mathrm{d}}(\mu_1) = 0.2663, \quad E_{\mathrm{JS}}(\mu_1) = 2.2074, \quad SU(\mu_1) = 1.8202,$$
$$E_{\mathrm{FB}}(\mu_2) = 2.0362, \quad TU_{\mathrm{d}}(\mu_2) = 0.2663, \quad E_{\mathrm{JS}}(\mu_2) = 1.9934, \quad SU(\mu_2) = 1.8202.$$

For measures in Example 3, the anticipations of propositions which satisfy $|F_i| = 1$ or $|F_i| = 3$ are equal. According to the Proposition 1, their $\mu_{\mathrm{L}}(\{\omega\})$ or $\mu_{\mathrm{U}}(\{\omega\})$ are also equal, i.e., $\mu_{\mathrm{LU1}}(\{\omega\}) = \mu_{\mathrm{LU2}}(\{\omega\})$. Despite their obvious differences in $\mu(F_i)$ where $|F_i| = 2$, the interval distance-based uncertainty measure and SU measurement can not distinguish them. Therefore, the proposed uncertainty measures relying only on $\mathrm{Bel}(\{\omega\})$ and $\mathrm{Pl}(\{\omega\})$ cannot be fully adapted to measure-based uncertainty.

For the total uncertainty measure in Dempster-Shafer theory [45], one of its requirements is that the measurement can be divided into two components, the discord measure and the non-specificity measure. The discord measure represents the internal conflict arising from the support given to different elements within the frame, while the non-specificity measure represents the ignorance involved in assigning support to these elements. The Shannon entropy of BetP is considered the discord measure of total uncertainty in FB entropy [43]. Based on (8) and (6), their Shapley indices are $S_1 = \{0.2375, 0.2875, 0.1875, 0.2875\}$ and $S_2 = \{0.2375, 0.2875, 0.2042, 0.2708\}$, which match the outcomes of PPT in their corresponding PBPAs. Therefore, the discord measure in FB entropy indicates that they exhibit different internal conflicts. However, the discord measure in JS entropy [41] is determined by the contour function, i.e., μ_{U}. For measure-based uncertainty, the anticipation of singletons is not influenced by multi-element propositions, and therefore cannot distinguish the internal conflicts within the fuzzy measures.

Based on the aforementioned discussion, FB entropy and JS entropy provide reasonable measures of total uncertainty; however, only FB entropy is effective when analyzing the composition of total uncertainty.

3.4.2 *Distance measure*

The distance measure between Dempster-Shafer granules is inspired by the Mahalanobis distance, where the covariance matrix is replaced by a matrix representing the relationships among focal sets. Although the PBPA of a fuzzy measure may include negative belief masses, this does not affect the use of distance measures in Euclidean space. In this paper, we extend Jousselme et al.'s distance measure [46] and fractal-based distance measure [47] to measure-based uncertainty.

Definition 4 (Dempster-Shafer structure-based distance measures)**.** Given two fuzzy measures μ_1 and μ_2 under an n-element frame Ω, and its boundary fuzzy measures are $\mathcal{BFM}_i = \{\mu_{i\mathrm{L}}, \mu_{i\mathrm{U}}\}$ $(i = \{1, 2\})$.

(1) Jousselme et al.'s distance Dis_{J} of the μ_1 and μ_2 is denoted as

$$Dis_{\mathrm{J}}(\mu_1, \mu_2) = \sqrt{\frac{1}{2}(\boldsymbol{pm}_1 - \boldsymbol{pm}_2)^{\mathrm{T}}\boldsymbol{D}(\boldsymbol{pm}_1 - \boldsymbol{pm}_2)},$$

where $\boldsymbol{pm}$ is the vector form of PBPA, and $\boldsymbol{D}$ is a 2^n-dimensional matrix satisfying $D(F_i, F_j) = \frac{|F_i \cap F_j|}{|F_i \cup F_j|}$.

(2) Fractal-based distance Dis_{FB} of the μ_1 and μ_2 is denoted as

$$Dis_{\mathrm{FB}}(\mu_1, \mu_2) = \sqrt{\frac{1}{2}(\boldsymbol{pm}_{1\mathrm{FB}} - \boldsymbol{pm}_{2\mathrm{FB}})^{\mathrm{T}}(\boldsymbol{pm}_{1\mathrm{FB}} - \boldsymbol{pm}_{2\mathrm{FB}})}, \quad pm_{\mathrm{FB}}(F_i) = \sum_{F_i \subseteq F_j} \frac{pm(F_j)}{2^{|F_j|} - 1}.$$

Example 4. Consider three fuzzy measures under 3-element frame

$$\mu_1 = \{0, 0.1, 0.2, 0.4, 0.2, 0.9, 0.8, 1\}, \mu_2 = \{0, 0.1, 0.4, 0.4, 0.2, 0.9, 1, 1\}, \mu_3 = \{0, 0.1, 0, 0.4, 0.2, 0.7, 0.8, 1\}.$$

The Euclidean distance[1)], Jousselme et al.'s distance and fractal-based distance of (μ_1, μ_2) and (μ_1, μ_3) are

$$d(\mu_1, \mu_2) = 0.2, \ Dis_{\mathrm{J}}(\mu_1, \mu_2) = 0.1291, \ Dis_{\mathrm{FB}}(\mu_1, \mu_2) = 0.0943,$$
$$d(\mu_1, \mu_3) = 0.2, \ Dis_{\mathrm{J}}(\mu_1, \mu_3) = 0.0861, \ Dis_{\mathrm{FB}}(\mu_1, \mu_3) = 0.0471.$$

Based on the Definition 8, the Shapley indices of them are $S_1 = \{0.25, 0.25, 0.5\}$, $S_2 = \{0.15, 0.35, 0.5\}$, $S_3 = \{0.25, 0.25, 0.5\}$.

As shown in Example 4, the Euclidean distance may generate counter-intuitive results in representing the dissimilarity of fuzzy measures. In terms of the distance measures in Dempster-Shafer theory, the linear transformations, FB[2)] and $\boldsymbol{D}$, realize the interaction between focal elements and satisfy the strong structural properties [46].

3.5 Discussion

In this section, we consider μ_{U} and μ_{L} in $\mathcal{BFM}$ as the Pl and Bel functions under the Dempster-Shafer structure, respectively, and combine them with Proposition 1 to demonstrate that a fuzzy measure can correspond to a pseudo Dempster-Shafer granule. Through Example 1, we discuss the effectiveness of various evidential representations in modeling measure-based uncertainty. Additionally, by utilizing information measures in Dempster-Shafer theory, we gain a fresh perspective on quantifying the information content and distance of fuzzy measures. In the remaining sections, we delve into how to construct and handle boundary fuzzy measures via pseudo Dempster-Shafer granules.

4 Constructing measure-based uncertainty via canonical decomposition

The construction of fuzzy measures has been a prominent topic, especially during the era of expert systems in AI [48]. In the age of generative AI, experts have transitioned from specific individuals to intelligent agents, such as large language models [49]. As a result, the interpretable and generalizable construction of fuzzy measures has become an essential area of research. Previous approaches have been explored through the perspectives of transformation, optimization, and migration.

1) Euclidean distance: $d(\mu_1, \mu_2) = \sqrt{\frac{1}{2}\sum_{F_i \subseteq \Omega}(\mu_1(F_i) - \mu_2(F_i))^2}$.

2) Generation of $\boldsymbol{pm}_{\mathrm{FB}}$.

In the case of transformation, generating functions have been proposed to construct fuzzy measures; however, the corresponding parameters often lack physical meaning. For optimization, genetic algorithms are used to cluster data into fuzzy measures, but these methods lack appropriate information processing techniques. In terms of migration, fuzzy measures from other uncertainty theories [50], such as belief and possibility measures, are directly adopted, but these approaches cannot encompass the entire spectrum of fuzzy measures.

In this paper, we aim to develop a construction method that encompasses the entire spectrum of measure-based uncertainty from the perspective of transformation. By providing specific interpretations for the parameters in the generation process, a construction paradigm for measure-based uncertainty is established.

4.1 Canonical decomposition of fuzzy measure

In Example 1, the pseudo t-function in measure-based uncertainty retains its meaning within the Dempster-Shafer theory framework. Therefore, extending the t-canonical decomposition to fuzzy measures is a natural progression.

Definition 5 (t-canonical decomposition on measure-based uncertainty)**.** Given a fuzzy measure μ, and its boundary fuzzy measure $\mathcal{BFM}=\{\mu_{\rm L},\mu_{\rm U}\}$, PBPA pm, the t function of μ is defined as

$$t(F_i)=\begin{cases}\mu_{\rm U}(F_i), & |F_i|=1,\\ \bigotimes_{j=1}^{n}\begin{bmatrix}1 & 1\\ -\mu_{\rm U}(\{\omega_j\}) & 1-\mu_{\rm U}(\{\omega_j\})\end{bmatrix}\boldsymbol{pm}, & \text{others.}\end{cases}\tag{11}$$

If the t function is known, its corresponding $\mathcal{BFM}$ is defined as

$$\mu_{\rm U}(F_i)=\begin{cases}t(F_i), & |F_i|=1,\\ \boldsymbol{m2Pl}\bigotimes_{j=1}^{n}\begin{bmatrix}1-t(\{\omega_j\}) & -1\\ t(\{\omega_j\}) & 1\end{bmatrix}\boldsymbol{t}, & \text{others.}\end{cases}\tag{12}$$

According to (5) and [33], t-canonical decomposition is proposed based on the Teugels' representation of MBD [51]. BPAs generated by their t functions are numerically equivalent to MBDs, though they represent different uncertainties under their respective theoretical frameworks. In this paper, since PBPA may generate negative belief mass, using t function to model the measure-based uncertainty has exceeded the MBD's representation. We revisit the properties and meaning of the t function from the perspective of fuzzy measure.

Proposition 4. Given a boundary fuzzy measure $\mathcal{BFM}=\{\mu_{\rm L},\mu_{\rm U}\}$, and its t function can be calculated via (11). For a subset with multiple elements F_i, when $t(F_i)$ varies $\Delta t(F_i)$, the $\mu_{\rm U}(F_i)$ will vary $(-1)^{|F_i|-1}\Delta t(F_i)$ and $\forall F_j\not\supseteq F_i$ $\mu_{\rm U}$ will remain unchanged.

Proof. Please refer to Section 1 in the supplementary material.

Example 5. Given a fuzzy measure μ under a 4-element frame, where μ and its t function are

$$\begin{aligned}\mu&=\mu_{\rm U}\\ &=\{0,0.6034,0.5889,0.8440,0.4741,\mathbf{0.8388},0.8038,0.9496,\\ &\quad 0.6591,0.8939,0.8697,0.9727,0.7211,0.9349,0.9101,1.0000\},\\ t&=\{1,0.6034,0.5889,-0.0070,0.4741,\mathbf{-0.0474},-0.0200,0.0042,\\ &\quad 0.6591,-0.0291,-0.0098,0.0100,0.0996,0.0141,0.0128,-0.0090\}.\end{aligned}$$

When $t(\{\omega_1\omega_3\})$ varies from -0.0474 to -0.2086, according to Proposition 4, $\mu_{\rm U}(\{\omega_1\omega_3\})$ varies from 0.8388 to 1. The variations in subsets' anticipations and the trends of their entropies and attitudinal characters are depicted in Figure 3.

As shown in Example 5, altering $t(F_i)$ solely impacts $\mu_{\rm U}(F_j)$ with satisfying $F_i\subseteq F_j$, and their variation remains consistent when $F_i=F_j$. Regarding the attitudinal character, an increase in $\mu_{\rm U}(F_i)$ due to a change in $t(F_i)$ implies that the agent's anticipation improves to both propositions F_i and the propositions F_j with satisfying $F_i\subseteq F_j$. Thus, for the whole fuzzy measure, the agent holds a more optimistic attitude. Regarding the entropy, its trend is not monotonic, because increasing the

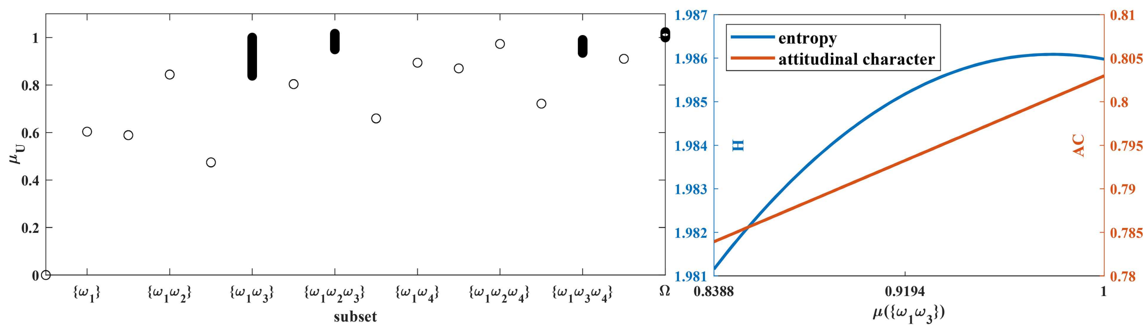

Figure 3 (Color online) Variations of subsets' anticipations and the trends of their entropies and attitudinal characters.

anticipations of different propositions has a different effect on the marginal contributions. Hence, we focus more on analyzing the relationship between the t function and measure-based uncertainty from the perspective of attitudinal character. Furthermore, as depicted in Figure 3, the initial anticipation of Ω is 1. With the increase in $\mu_{\mathrm{U}}(\{\omega_1\omega_3\})$, $\mu_{\mathrm{U}}(\Omega)$ surpasses 1. Consequently, directly operating on the t function for modeling measure-based uncertainty becomes untenable. In the subsequent discussion, we will introduce a modified form of the t function to address this issue.

4.2 Construction of fuzzy measure

Considering Proposition 4 and Example 5, altering $t(F_i)$ does not influence the values of $\mu_{\mathrm{U}}(F_j)$ where $F_i \nsubseteq F_j$. Based on the above, we propose a tr function. For n-element frames, tr function is derived from the t function and commences from propositions with smaller cardinalities, progressing through $n-1$ steps.

Definition 6 (tr function)**.** For a fuzzy measure μ under an n-element frame Ω, and its boundary fuzzy measure is $\mathcal{BFM} = \{\mu_{\mathrm{L}}, \mu_{\mathrm{U}}\}$, the tr function can be computed using Algorithm 1, while the inverse process is outlined in Algorithm 2. For the pseudo codes, please refer to Section 2 in the supplementary material.

Example 6. Continue to the Example 1, $t = pt = \{1, 0.1, 0.2, -0.57, 0.2, -0.62, -0.59, -0.907\}$, and its tr function is $tr = \{1, 0.1, 0.2, 0.8125, 0.2, 0.875, 0.9375, 1\}$. The specific generation process and its inverse process please refer to Section 3 in the supplementary material.

Example 6 extends the toy example to illustrate the relationship between the tr function, the t function, and the upper fuzzy measure. To showcase the advantages and significance of tr, specific properties are examined.

Proposition 5. When tr function satisfies $\forall F_i \in 2^{\Omega}$, $tr(F_i) \in [0, 1]$, and $\exists F_j \in \{2^{\Omega} \setminus \emptyset\}$, $tr(F_j) = 1$, Algorithm 2 generates a fuzzy measure.

Proof. Please refer to Section 1 in the supplementary material.

Proposition 6. When the tr function satisfies $\max_{\omega\in\Omega}(tr(\{\omega\})) = 1$ and $tr(F_i) = 0$ for $|F_i| > 1$, the corresponding fuzzy measure will be a possibility measure.

Proof. Please refer to Section 1 in the supplementary material.

Proposition 7. The fuzzy measure derived from the tr function may not necessarily serve as the upper fuzzy measure. In other words, two distinct tr functions may yield different fuzzy measures, yet they encapsulate the same boundary fuzzy measure.

Proof. Please refer to Section 1 in the supplementary material.

Proposition 8. There may be multiple trs pointing to the same fuzzy measure.

Proof. Please refer to Section 1 in the supplementary material.

According to Propositions 7 and 8, although the tr function and fuzzy measure can be transformed into each other, they do not exhibit a one-to-one correspondence. To simplify the subsequent discussion, the tr function generated by the boundary fuzzy measure through Algorithm 1 is referred to as the strict tr function.

A probabilistic or fuzzy model can restrict the support degree for different elements by assigning weights, which collectively form an information distribution that represents the values of uncertain variables. However, assigning values directly to propositions with multiple elements is challenging due to

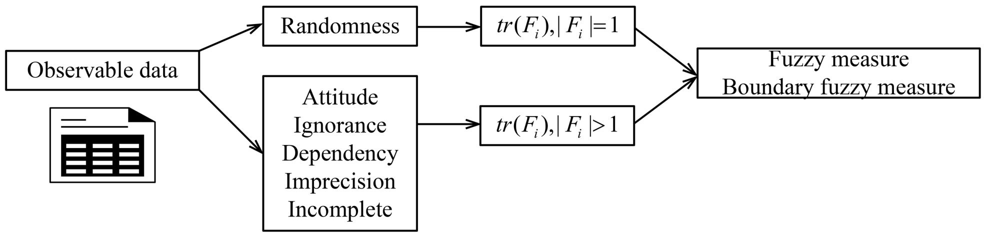

Figure 4 Construction of measure-based uncertainty via tr function in data and knowledge mixed environment.

set relations and the constraints of normalized numerical values. Dempster-Shafer structures require the normalization of belief masses, while fuzzy measures require monotonicity. In this paper, the tr function is proposed to address the above issues. It serves not only as a tool for directly assigning values to non-empty propositions to construct boundary fuzzy measures, but also provides an interpretable calculation process.

As shown in Figure 4, the tr function is divided into two components: the values of singletons represent the discord among elements, while the values of multi-element subsets represent the attitude information among the singletons. In the case of singletons, the propensity can be viewed as a precise yet uncertain information granule that expresses the agent's preference for whether each element is the value of the uncertain variable. In the case of multi-element subsets, they can be interpreted as a representation of the attitude toward the elements within the proposition. When $tr(F_i) = 0$, it signifies a cautious attitude toward the elements within the proposition. Consequently, the corresponding anticipation is determined by the maximum value among its subsets. Conversely, when $tr(F_i) = 1$, it indicates a bold attitude toward the elements of the proposition, resulting in the anticipation of the proposition being assigned a maximum value of 1, regardless of the anticipations of its subsets.

4.3 Discussion

In this section, the t function is extended from the Dempster-Shafer structure to measure-based uncertainty. t function serves as an identical information content representation for the boundary fuzzy measure. In contrast to the Dempster-Shafer granule, the t function holds a more transparent interpretation within the framework of measure-based uncertainty. Furthermore, we introduce a novel representation of the t function, referred to as the tr function, enabling agents to assign weights to all non-empty propositions of the power set with a more relaxed constraint.

5 Handling measure-based uncertainty on Dempster-Shafer structure

5.1 Overview and motivation

In the previous research, the fusion of fuzzy measures has typically been tackled by employing well-established aggregation operators, t-norm and t-conorm, to directly manipulate fuzzy measures [11, 13]. Although this ensures that the generated result is also a fuzzy measure, it cannot be consistent with the processing logic of other uncertainty theories, i.e., it will produce counter-intuitive results.

Example 7. Given two fuzzy measures $\mu_1 = \{0, 0.5, 0.1, 0.6, 0.4, 0.9, 0.5, 1\}$ and $\mu_2 = \{0, 0.8, 0.7, 0.8, 1, 1, 1, 1\}$, it is evident that μ_1 is probabilistic measure and μ_2 is possibilistic measure. According to the conjunctive rule in [11], its fused result and Shapley index are

$$\mu_1 \textcircled{\cap}_{\mathrm{Y}} \mu_2 = \boldsymbol{\mu_1} \circ \boldsymbol{\mu_2} = \mu_{\mathrm{Y}} = \{0, 0.4, 0.07, 0.48, 0.4, 0.9, 0.5, 1\}, \quad S_{\mu_{\mathrm{Y}}} = \{0.4517, 0.0867, 0.4617\}.$$

Under the Dempster-Shafer theory framework, they can be written as a Bayesian mass function m_1 and a consonant mass function m_2, and their fusion result via CCR (Eq. (3)) is

$$m_1 = \{0, 0.5, 0.1, 0, 0.4, 0, 0, 0\}, \quad m_2 = \{0, 0, 0, 0, 0.2, 0.1, 0, 0.7\},$$
$$m_{1 \textcircled{\cap} 2} = \{0.13, 0.4, 0.07, 0, 0.4, 0, 0, 0\}, \quad \mathrm{BetP}_m = \{0.4598, 0.0805, 0.4597\}.$$

Example 7 shows that the conjunction of fuzzy measures under distinct frameworks yields varying outcomes, and this challenge remains an unresolved issue. Since Dempster-Shafer theory has developed a mature high-level information fusion paradigm [35, 52] and can model both probabilistic and possibilistic measures. In this paper, the combination of fuzzy measures is regarded as an extension of handling Dempster-Shafer granules. The proposed method not only maintains semantic consistency with other uncertainty theories but also leverages its powerful modeling capabilities to accommodate more diverse information.

5.2 Approximation for boundary fuzzy measure to belief function

Example 8. Continue to Example 1, suppose $\mu_1 = \mu$ and $pm_1 = pm$, and given another fuzzy measure $\mu_2 = \{0, 0.2, 0.4, 0.7, 0.1, 0.85, 0.9, 1\}$. According to (10), the pseudo BPA of μ_2 is $pm_2 = \{0, 0.1, 0.15, 0.45, 0.1, 0.4, .55, -0.75\}$. If the conjunction of μ_1 and μ_2 is implemented via executing CCR on pm_1 and pm_2 directly, the Pl function of the normalized fusion result will be $\{0, 0.035, 0.139, -0.113, 0.104, -0.226, 0.052, 1\}$.

As shown in Example 8, it is evident that the methods for handling Dempster-Shafer granules cannot be directly extended to measure-based uncertainty, as the negative values produce counter-intuitive results in the multiplications. Hence, a reversible bridge should be established first between the fuzzy measure and the viable belief function. Similar to the construction, we also utilize the t function to resolve this issue.

Definition 7 (Belief approximation). Consider a fuzzy measure μ and its boundary fuzzy measure $\mathcal{BFM} = \{\mu_{\mathrm{L}}, \mu_{\mathrm{U}}\}$, it can be updated via the t function with a parameter α

$$t_\mu^\alpha(F_i)\begin{cases} t_\mu(F_i), & |F_i| = 1, \\ \alpha \times t_\mu(F_i), & |F_i| > 1, \end{cases} \quad \mu_{\mathrm{U}}^\alpha = \begin{cases} t_\mu^\alpha(F_i), & |F_i| = 1, \\ \boldsymbol{m2\mathrm{Pl}} \bigotimes_{j=1}^n \begin{bmatrix} 1 - t_\mu^\alpha(\{\omega_j\}) & -1 \\ t_\mu^\alpha(\{\omega_j\}) & 1 \end{bmatrix} \boldsymbol{t}_\mu^\alpha, & \text{others}, \end{cases}$$

where $\alpha \in [0, 1]$ and t_μ is the t function of the μ generated based on (11). If the parameter α satisfies (13), the pm^α will be the belief approximation of the μ, which is denoted as $m_\mu[\alpha]$,

$$\begin{aligned} &\alpha_0 \leqslant \alpha, \quad \forall F_i \in 2^\Omega, \quad pm^{\alpha_0}(F_i) \geqslant 0, \\ &1 \geqslant \alpha_0 > \alpha, \quad \exists F_i \in 2^\Omega, \quad pm^{\alpha_0}(F_i) < 0. \end{aligned} \tag{13}$$

Proposition 9. For an arbitrary fuzzy measure μ, there must be existing an α satisfying the requirement of $m_\mu[\alpha]$.

Proof. Please refer to Section 1 in the supplementary material.

Based on the Definition 7, a fuzzy measure μ can be written as a BPA $m_\mu[\alpha]$. Revisiting the foundation of the t-canonical decomposition [33], the Teugels' MBD representation [51] can be used to interpret the meaning of α. Since the t function of a multi-element subset can be viewed as the covariance among its elements, α can be interpreted as the compromise degree for the correlation among singletons, aimed at achieving a viable Dempster-Shafer granule.

5.3 Combination rules of fuzzy measures

We have implemented the approximation of the boundary fuzzy measure to a belief function using the t function, and this approximation is reversible, provided that the parameter α is known. Since this approach resolves the issue presented in Example 8, the combination rule for fuzzy measures on the Dempster-Shafer structure can now be developed, as outlined in Subsection 5.1.

Definition 8 (Combination rules of fuzzy measures). Consider two fuzzy measures μ_1 and μ_2, and their boundary fuzzy measures are $\mathcal{BFM}_i = \{\mu_{\mathrm{L}i}, \mu_{\mathrm{U}i}\}$, $i = \{1, 2\}$. Suppose their belief approximations are $m_{\mu_1}[\alpha_1]$ and $m_{\mu_2}[\alpha_2]$, the combination $\mu_1 \odot \mu_2$ is defined as

$$\mu_{1\odot 2} = pm_1^\alpha \odot pm_2^\alpha \ominus \alpha, \tag{14}$$

where $\alpha = \min(\alpha_1, \alpha_2)$ and $\ominus\alpha$ means the inverse process of the belief approximation. The specific flowchart of the combination is shown in Figure 5.

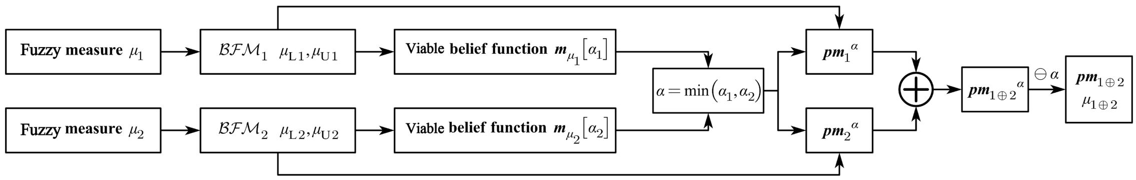

Figure 5 Combination of fuzzy measures on Dempster-Shafer structure.

Example 9. Continue to Example 8, when the $\odot$ is $\textcircled{\cap}$, the conjunction of μ_1 and μ_2 is

$$\mu_{L1\textcircled{\cap}2} = \{0, 0.085, 0.18, 0.244, 0.153, 0.224, 0.284, 1\},$$
$$\mu_{U1\textcircled{\cap}2} = \{0, 0.716, 0.776, 0.847, 0.756, 0.82, 0.915, 1\}.$$

The specific calculation process, please refer to Section 3 in the supplementary material.

In contrast to Example 8, the outcome in Example 9 remains a boundary fuzzy measure. Therefore, the proposed method not only enables the combination of fuzzy measures on the Dempster-Shafer structure, but also preserves the broader uncertainty representation for measure-based uncertainty.

Remark 1. For the inverse process of the compromise degree, i.e., the retraction of α, the generated pseudo Pl or pseudo Bel functions may take negative values, indicating that the retraction is excessive. To ensure that the fusion result remains a fuzzy measure, the approximation method given in Definition 7 can be applied to reduce the retraction until no negative values are present.

5.4 Property analysis

To justify the proposed combination rule, its properties from the perspective of information fusion are discussed.

Proposition 10 (Dempster-Shafer granules' consistency). Consider two fuzzy measures μ_1 and μ_2, if they can be written as the Bel function or Pl function, the proposed combination rule will degrade to the corresponding rule in the Dempster-Shafer theory.

Proof. Please refer to Section 1 in the supplementary material.

According to the Proposition 10, the contour intuitive result in Example 7 has been resolved, and the fused result is consistent with the Dempster-Shafer theory. Therefore, the proposed method addresses the core motivation of this paper, i.e., how to handle measure-based uncertainty while being consistent with other uncertainty theories.

Proposition 11 (Quasi-associativity). Although an operator satisfies associativity, its corresponding rule does not satisfy associativity in the context of measure-based uncertainty. However, the proposed method can be extended to a multi-source information fusion method, thereby satisfying quasi-associativity.

Proof. Please refer to Section 1 in the supplementary material.

Proposition 12 (Idempotency). When the operator satisfies idempotency, i.e., $m \odot m = m$, the proposed combination rule also satisfies the idempotency in the context of the measure-based uncertainty.

Proof. Please refer to Section 1 in the supplementary material.

Proposition 13 (Neutrality). For the conjunction and disjunction, the proposed combination rule has the same neutral element with Dempster-Shafer structure, i.e., $\mu \textcircled{\cap} \mu_\Omega = \mu$, and $\mu \textcircled{\cup} \mu_\emptyset = \mu$, where

$$\mu_{\Omega,L}(F_i) = \begin{cases} 1, & F_i = \Omega, \\ 0, & F_i \in \{2^\Omega \setminus \Omega\}, \end{cases} \quad \mu_{\Omega,U}(F_i) = \begin{cases} 1, & F_i \in \{2^\Omega \setminus \emptyset\}, \\ 0, & F_i = \emptyset, \end{cases}$$
$$\mu_{\emptyset,L}(F_i) = \mu_{\emptyset,U}(F_i) = \begin{cases} 1, & F_i = \emptyset, \\ 0, & F_i \in \{2^\Omega \setminus \emptyset\}. \end{cases}$$

Proof. Please refer to the Section 1 in supplementary material.

The entailment relationship is an important notion in the uncertainty handling [38]. Since the proposed method manage uncertainty within the Dempster-Shafer structure, we discuss the contour inclusion (Definition 6 in [53]) in this paper.

Table 1 Comparison of the properties of combination rules between Dempster-Shafer granules and measure-based uncertainty.

Properties	Dempster-Shafe granule				Measure-based uncertainty			
	$\textcircled{\cap}$	$\textcircled{\cup}$	$\textcircled{\wedge}$	$\textcircled{\vee}$	$\textcircled{\cap}$	$\textcircled{\cup}$	$\textcircled{\wedge}$	$\textcircled{\vee}$
Associativity	Y	Y	Y	Y	quasi	quasi	quasi	quasi
Idempotency	N	N	Y	Y	N	N	Y	Y
Neutrality	m_Ω	$m_\emptyset$	N	N	μ_Ω	$\mu_\emptyset$	N	N
Informative monotonicity	Y	Y	Y	Y	Y	Y	–	–

Proposition 14 (Informative monotonicity)**.** If the operator satisfies the contour inclusion, the proposed combination rule will also satisfy the contour inclusion, i.e., $\mu_1 \textcircled{\cap} \mu_2 \sqsubseteq_{\text{Poss}} \mu_i$, and $\mu_i \sqsubseteq_{\text{Poss}} \mu_1 \textcircled{\cup} \mu_2$, $i = \{1, 2\}$.

Proof. Please refer to Section 1 in the supplementary material.

Table 1 shows a comparison of the properties of the same rules in the Dempster-Shafer theory framework and measure-based uncertainty. As seen, the proposed method effectively transfers the advantages of the Dempster-Shafer structure in handling uncertainty to measure-based uncertainty.

5.5 Data-driven modeling and handling measure-based uncertainty

Based on the proposed methods for modeling and handling measure-based uncertainty, a data-driven example is given in this paper. The aim is to show the robustness of the measure-based uncertainty framework for multi-source information fusion tasks in a labeling uncertainty environment. Due to space constraints, please refer to Section 4 in the supplementary material. It is worth noting that the process of granulation of data to a fuzzy measure is a topic that deserves deep discussion, and this paper only gives one of the feasible paths.

6 Conclusion

In this paper, we revisit the potential relationship between measure-based uncertainty and other uncertainty theories and give a novel information representation-boundary fuzzy measure. The boundary fuzzy measure can clearly quantify both the upper and lower anticipations of a proposition, offering a more intuitive information representation than a fuzzy measure. We demonstrate not only that the fuzzy measure and its boundary version share the same Shapley indices, but also that the boundary version covers a larger attitude interval. Utilizing the upper and lower bound representations of anticipations, we model the boundary fuzzy measure within a Dempster-Shafer structure. In the realm of information measures, the uncertainty and distance measures within the Dempster-Shafer theory framework are introduced into measure-based uncertainty, with their validation discussed through numerical examples. In the realm of uncertainty modeling, the t canonical decomposition within the Dempster-Shafer theory framework is introduced to offer a novel information representation for measure-based uncertainty. This proposed representation aims to provide a more convenient information interface for constructing fuzzy measures, enhancing interpretability. In the realm of uncertainty handling, the combination rules of Dempster-Shafer granules are extended to the measure-based uncertainty via the belief approximation. Through the comparison and analysis of properties, the proposed method effectively resolves the previous issues, and the operators in the Dempster-Shafer theory facilitate a reasonable migration.

As a pioneering research effort, this paper introduces a novel approach to modeling and handling fuzzy measures within the Dempster-Shafer structure. By incorporating the generalized representation of uncertain information through fuzzy measures into established uncertainty frameworks, we provide a new method for uncertainty processing. However, adapting this generalized uncertainty modeling approach to a broader range of application scenarios, as well as developing more rational reasoning logics, remains an open challenge that is not fully addressed in this paper. In future research, we aim to further explore the advantages of information representation within measure-based uncertainty, with the goal of developing a comprehensive and effective set of generalized information reasoning models.

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Supporting information Proofs, algorithms, and examples. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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