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Two-stage linear quadratic stochastic optimal control problem under model uncertainty

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Abstract This study focuses on a two-stage linear quadratic stochastic optimal control problem under model uncertainty (TSLQU), where a system's coefficients are uncertain and the state switches at a certain time. The cost function of the TSLQU problem is designed to be robust to model uncertainty, that is, to be insensitive to unmatched coefficients. The TSLQU problem can characterize the common requirement of the robust optimization of a two-stage stochastic system. To obtain a robust optimal control, using the convex variational method and convergence analysis, a necessary optimality condition relying on an optimal parameter is designed, which is also sufficient. Then, through the feedback form of the candidate robust optimal control and continuity analysis, the existence of a robust optimal control and the corresponding optimal parameter is obtained. Through explicit representation of the optimal cost vector, the characterization of the optimal parameter is given. Additionally, a numerical simulation is presented to verify the theoretical results.

Keywords two-stage state, robust stochastic optimal control, necessary condition, convex variation, optimal parameter

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1 Introduction

Optimal control theory has received great attention since the mid-20th century, especially after Kalman, Pontriyagin, and Bellman announced the linear quadratic (LQ) optimal control theory, the maximum principle, and the dynamic programming principle, respectively, in the first International Federation of Automatic Control meeting in 1960. Compared with deterministic systems, stochastic systems with random noises can depict commonly existing phenomena where random factors affect systems in nature and production. In financial and industrial engineering, stochastic LQ theory is especially widely applied for its application convenience and effectiveness (see [1–4]). For a comprehensive study of stochastic optimal control theory and stochastic LQ theory, readers may refer to Yong and Zhou [5] and the references therein.

In the field of stochastic LQ theory, known models are inevitable for the analysis of controlled systems and the application of the maximum principle or the dynamic programming principle to derive optimal controls. Nonetheless, in reality, considering numerous factors such as measurement errors, lack of a priori experiences, and uncertainty of outer environments, system coefficients are not precisely known to decision-makers and thus are characteristically uncertain. This phenomenon is called model uncertainty, under which system coefficients are unmatched by the designed optimal control, resulting in nonoptimality. To deal with model uncertainty, a min–max approach was proposed by Poznyak et al. [6]. Precisely, the optimization objective was to first maximize all possible cost functions for a given control and then minimize the aforementioned maximum value. Compared with the simple minimization problem, this optimization approach improves the robustness of a candidate robust optimal control. Meanwhile, there widely exists a special kind of stochastic system in reality, where a stochastic system's state stops evolving at a certain time and then switches to a different state dynamic (see Huang et al. [7]). For instance, scientific research typically switches from a long stage of preparation, exploration, and accumulation to a stage of vast production and finding of new scientific results.

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The analysis above motivates the study of the two-stage linear quadratic stochastic optimal control problem under model uncertainty (TSLQU), where a linear system state switches at a certain time and the coefficients of the cost function and state are uncertain, characterizing a prevalent concern of the robust optimization of a two-stage stochastic system under model uncertainty. Using the convex variational technique and convergence analysis, a necessary and sufficient optimality condition depending on an optimal parameter is obtained. Based on the optimality condition along with two auxiliary stochastic Riccati equations and two linear differential equations, the feedback form of a candidate robust optimal control is designed. Combining the results above with a series of continuity analyses, the existence of robust optimal control and the corresponding optimal parameter is obtained. Moreover, using four Lyapunov equations and four linear equations, a further and more delicate characterization of the optimal parameter is given. Finally, a numerical example is shown to demonstrate the effectiveness of our designed robust optimal control.

Three features distinguish our study from previous literature.

• Previous studies of the stochastic optimal control problem under model uncertainty (cf. [8–13] for robust stochastic optimal control problems with the system states respectively being stochastic differential equations (SDEs), forward-backward SDEs, mean-field SDEs, regime-switching SDEs, quadratic-growth forward-backward SDEs, and partially observable states) only concerned single-stage system, which was unrealistic and less effective in dealing with model uncertainty, whereas the abrupt jump of a system's state, which lacked useful handling tools, was not investigated. Specifically, in [8–13], single-stage variational and adjoint equations were applied to obtain robust stochastic maximum principles for single-stage problems, which became invalid when impulses and dimensional changes of the state occurred in the system state equation. Meanwhile, Huang et al. [7] dealt with a mixed LQ problem from a two-stage project management, where an optimal pair of a stopping time and an admissible control was sought, the models were known to the decision maker, and optimality conditions were discussed, which lacked robustness for practical application. Thus, herein, we come up with a necessary and sufficient condition for robust optimal controls via an adjoint system consisting of two backward SDEs (BSDEs) coupled with a boundary condition, convex variational method, and convergence analysis, which are substantially innovative and technical compared with those in previous studies [7–13].

• More importantly, compared with previous studies [8–12], where only the existence of an optimal parameter was discussed, we demonstrate a more delicate characterization of an optimal parameter, which was obtained through the explicit computation of the optimal cost vector using Lyapunov equations and linear differential equations.

• Through the aforementioned explicit computation of the optimal cost vector, we can illustrate the effectiveness of our designed robust optimal control through numerical simulation, which is also novel compared with those methods in previous studies [7–12].

The organization of this paper is as follows. Section 2 is devoted to the formulation of the TSLQU problem and some preliminaries. Section 3 is dedicated to solving the TSLQU problem. Section 4 is committed to a numerical simulation to illustrate the validity of our theoretical results. Section 5 concludes the whole paper and points out some topics in future study.

2 Problem formulation

We consider a model in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ be the $\mathbb{P}$-null set in $\mathcal{F}$ and $\mathcal{F}_s := \sigma\{W_t, t \leqslant s\} \vee \mathcal{N}$, where $\{W_t, t \in [0,T]\}$ is a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\sigma\{\xi\}$ denotes the σ-algebra generated by the random variable ξ. To simplify these notations, some time variables are omitted throughout the paper.

We introduce herein some notations. For $k \in \mathbb{N}^+$, $\mathbb{S}^k$ denotes the set of symmetric matrices of the k-th order. For an $m \times n$ matrix M, the norm of M is $|M| = \sqrt{\mathrm{Tr}(MM^\top)}$. For $l \geqslant 1$, $T \geqslant \tau \geqslant 0$, and a Euclidian space $\mathbb{D}$ with the norm $|\cdot|$, $C^1(0,T;\mathbb{D})$ represents the set of all $\mathbb{D}$-valued differentiable functions on $[0,T]$, $\mathcal{H}^{2l}([\tau,T];\mathbb{D})$ consists of all $\mathcal{F}_r$-adapted processes $Z \in \mathbb{D}$ s.t. $E[\int_\tau^T |Z|^{2l}\mathrm{d}r] < +\infty$, $\mathcal{S}^{2l}([\tau,T];\mathbb{D})$ comprises continuous $\mathcal{F}_r$-adapted processes $Y \in \mathbb{D}$ s.t. $E[\sup_{r\in[\tau,T]} |Y|^{2l}] < +\infty$, and $L^\infty([\tau,T];\mathbb{D})$ is made of essentially bounded deterministic functions valued in $\mathbb{D}$. An $\mathbb{S}^k$-valued function $Q(t)$ on $[0,T]$ is said to be uniformly positive definite if there exists $\varepsilon > 0$, s.t. $Q(t) \geqslant \varepsilon \mathbb{I}_{k\times k}$, uniformly on $[0,T]$.

The coefficients of our model satisfy the following assumption.

Assumption 1. For $\theta \in \{1,2\}$, $k,m,n \in \mathbb{N}^+$, and $T \geqslant \tau \geqslant 0$,

(i) $x_\theta \in \mathbb{R}^n, K_\theta \in \mathbb{R}^{m\times n}$.

(ii) $\tilde{A}_\theta, \tilde{C}_\theta \in L^\infty([0,\tau];\mathbb{R}^{n\times n}), \tilde{B}_\theta, \tilde{D}_\theta \in L^\infty([0,\tau];\mathbb{R}^{n\times k}), \tilde{b}_\theta, \tilde{\sigma}_\theta \in L^\infty([0,\tau];\mathbb{R}^n)$.

(iii) $\tilde{a}_\theta \in L^\infty([0,\tau];\mathbb{S}^n)$ is nonnegative definite, $\tilde{c}_\theta \in L^\infty([0,\tau];\mathbb{S}^k)$ is uniformly positive definite, and $\tilde{G}_\theta \in \mathbb{S}^n$ is nonnegative definite.

(iv) $A_\theta, C_\theta \in L^\infty([\tau,T];\mathbb{R}^{m\times m}), B_\theta, D_\theta \in L^\infty([\tau,T];\mathbb{R}^{m\times k}), b_\theta, \sigma_\theta \in L^\infty([\tau,T];\mathbb{R}^m)$.

(v) $a_\theta \in L^\infty([\tau,T];\mathbb{S}^m)$ is nonnegative definite, $c_\theta \in L^\infty([\tau,T];\mathbb{S}^k)$ is uniformly positive definite, and $G_\theta \in \mathbb{S}^m$ is nonnegative definite.

For $\theta \in \{1,2\}$ and $T \geqslant \tau \geqslant 0$, consider the following two-stage and nonhomogeneous linear SDE as a state equation:

$$\begin{cases} \mathrm{d}\tilde{X}_\theta = (\tilde{A}_\theta \tilde{X}_\theta + \tilde{B}_\theta v + \tilde{b}_\theta)\mathrm{d}t + (\tilde{C}_\theta \tilde{X}_\theta + \tilde{D}_\theta v + \tilde{\sigma}_\theta)\mathrm{d}W, \quad t \in [0,\tau), \\ \mathrm{d}X_\theta = (A_\theta X_\theta + B_\theta v + b_\theta)\,\mathrm{d}t + (C_\theta X_\theta + D_\theta v + \sigma_\theta)\,\mathrm{d}W, \quad t \in [\tau, T], \\ \tilde{X}_\theta(0) = x_\theta, \quad X_\theta(\tau) = K_\theta \tilde{X}_\theta(\tau). \end{cases} \tag{1}$$

The robust cost function is defined as

$$J[v] = \sup_{\lambda\in[0,1]} \{\lambda Y_1[v] + (1-\lambda)Y_2[v]\} = Y_1[v] \vee Y_2[v], \tag{2}$$

where

$$\begin{aligned} Y_\theta[v] = \frac{1}{2}E\bigg\{ \int_0^\tau \Big[\Big\langle \tilde{a}_\theta \tilde{X}_\theta, \tilde{X}_\theta \Big\rangle + \langle \tilde{c}_\theta v, v\rangle\Big]\,\mathrm{d}t + \Big\langle \tilde{G}_\theta \tilde{X}_\theta(\tau), \tilde{X}_\theta(\tau)\Big\rangle \\ + \int_\tau^T [\langle a_\theta X_\theta, X_\theta\rangle + \langle c_\theta v, v\rangle]\,\mathrm{d}t + \langle G_\theta X_\theta(T), X_\theta(T)\rangle \bigg\}. \end{aligned} \tag{3}$$

The set of admissible controls is $\mathcal{H}^2([0,T];\mathbb{R}^k)$, denoted by $\mathcal{U}$.

For any $v \in \mathcal{U}$, under Assumption 1, by the classical theories of SDE (cf. [5]), Eq. (1) is uniquely solvable and the cost function (2) is finite.

TSLQU problem. Seek for a robust optimal control $\bar{v} \in \mathcal{U}$, s.t.

$$J[\bar{v}] = \inf_{v\in\mathcal{U}} J[v].$$

Remark 1. (i) In the TSLQU problem, there exist two uncertain system states $(X_1, \tilde{X}_1)$ and $(X_2, \tilde{X}_2)$. For each state $(X_\theta, \tilde{X}_\theta)$, the decision maker aims to minimize the cost functional $Y_\theta[v]$, where $\theta = 1, 2$. As decision makers cannot know whether the true system state is $(X_1, \tilde{X}_1)$ or $(X_2, \tilde{X}_2)$ because of model uncertainty, they have to utilize a robust methodology and therefore minimize the largest loss in the worst case scenario (i.e., minimize $Y_1[v] \vee Y_2[v]$).

(ii) On the one hand, notice that $\int_0^T a_1 \mathrm{d}s \vee \int_0^T a_2 \mathrm{d}s \leqslant \int_0^T a_1 \vee a_2 \mathrm{d}s$, where a_1 and a_2 are integrable functions on $[0,T]$. Furthermore, normally, the left side is strictly smaller than the right side. Hence, we cannot simply transform the cost function (2) into a quadratic function, indicating the necessity of this study. On the other hand, the TSLQU problem can also be seen as a degenerate zero-sum game problem where only one player's control enters the system state, which cannot be solved by classical game theories.

3 Main results

This section is devoted to solving the TSLQU problem. The solution procedure is as follows.

(i) Using the convex variational method, seek for the necessary and sufficient optimality condition satisfied by the robust optimal control $\bar{v}$.

(ii) Using the decoupling method, obtain the feedback form of $\bar{v}$.

(iii) Analyze the continuity dependency of the feedback state and cost function w.r.t. the parameter λ and obtain the existence and characterization of the optimal parameter $\bar{\lambda}$.

We illustrate each step separately in Subsections 3.1–3.3.

3.1 Seeking for the robust optimal control $\bar{v}$

Suppose $\bar{v} \in \mathcal{U}$ is a robust optimal control of the TSLQU problem. Let $(\bar{\tilde{X}}_\theta, \bar{X}_\theta)$ denote the solution of (1) under $\bar{v}$. Introduce the following adjoint system coupled by a terminal condition:

$$\begin{cases} \mathrm{d}p_\theta = -\left[a_\theta \bar{X}_\theta + A_\theta^\top p_\theta + C_\theta^\top q_\theta\right] \mathrm{d}t + q_\theta \mathrm{d}W, \quad t \in [\tau, T], \\ \mathrm{d}\tilde{p}_\theta = -\left[\tilde{a}_\theta \bar{\tilde{X}}_\theta + \tilde{A}_\theta^\top \tilde{p}_\theta + \tilde{C}_\theta^\top \tilde{q}_\theta\right] \mathrm{d}t + \tilde{q}_\theta \mathrm{d}W, \quad t \in [0, \tau), \\ p_\theta(T) = G_\theta \bar{X}_\theta(T), \quad \tilde{p}_\theta(\tau) = \tilde{G}_\theta \bar{\tilde{X}}_\theta(\tau) + K_\theta^\top p_\theta(\tau). \end{cases} \tag{4}$$

System (4) is made of two linear BSDEs coupled by a terminal condition. For $\bar{v} \in \mathcal{U}$, Eq. (4) is uniquely solved by $(p_\theta, q_\theta) \in \mathcal{S}^2([0,T];\mathbb{R}^m) \times \mathcal{H}^2([0,T];\mathbb{R}^m)$ and $(\tilde{p}_\theta, \tilde{q}_\theta) \in \mathcal{S}^2([0,T];\mathbb{R}^n) \times \mathcal{H}^2([0,T];\mathbb{R}^n)$(cf. [5]).

Remark 2. Here, we emphasize that, significantly different from a one-state stochastic optimal control problem, where there is only one adjoint equation in the optimality condition, two adjoint equations coupled by a novel terminal condition $\tilde{p}_\theta(\tau) = \tilde{G}_\theta \bar{\tilde{X}}_\theta(\tau) + K_\theta^\top p_\theta(\tau)$ are needed to deal with addition jump terms in the cost function and the system state of the TSLQU problem.

We first give the necessary and sufficient optimality condition of the TSLQU problem.

Theorem 1. Suppose Assumption 1 holds. Then, $\bar{v} \in \mathcal{U}$ is a robust optimal control if and only if there exist $\bar{v} \in \mathcal{U}$ and $\bar{\lambda} \in [0,1]$ satisfying

$$\begin{cases} \bar{Y}_1 \vee \bar{Y}_2 = \bar{\lambda}\bar{Y}_1 + (1-\bar{\lambda})\bar{Y}_2, & \text{(5a)} \\ \bar{\lambda}(\tilde{B}_1^\top \tilde{p}_1 + \tilde{D}_1^\top \tilde{q}_1 + \tilde{c}_1 \bar{v}) + (1-\bar{\lambda})(\tilde{B}_2^\top \tilde{p}_2 + \tilde{D}_2^\top \tilde{q}_2 + \tilde{c}_2 \bar{v}) = 0, \quad t \in [0, \tau), & \text{(5b)} \\ \bar{\lambda}(B_1^\top p_1 + D_1^\top q_1 + c_1 \bar{v}) + (1-\bar{\lambda})\left(B_2^\top p_2 + D_2^\top q_2 + c_2 \bar{v}\right) = 0, \quad t \in [\tau, T], & \text{(5c)} \end{cases}$$

where $(p_\theta, q_\theta, \tilde{p}_\theta, \tilde{q}_\theta)$ is a solution to the adjoint equation (4) and $\bar{Y}_\theta$ is the value of (3) corresponding to $\bar{v}$, $\theta = 1, 2$.

Proof. Part I. Necessity of (5).

Step 1. Transformation of the necessary condition.

Suppose $\bar{v} \in \mathcal{U}$ is a robust optimal control. Let $\bar{\mathcal{Q}}$ denote the set of $\bar{\lambda}$ satisfying $\bar{Y}_1 \vee \bar{Y}_2 = \bar{\lambda}\bar{Y}_1 + (1-\bar{\lambda})\bar{Y}_2$. If $\bar{Y}_1 \geqslant \bar{Y}_2$, then, $\bar{\lambda} = 1 \in \bar{\mathcal{Q}}$; else, $\bar{\lambda} = 0 \in \bar{\mathcal{Q}}$. Therefore, $\bar{\mathcal{Q}}$ is not empty.

Then, the necessity condition is equivalent to that there exists a $\bar{\lambda} \in \bar{\mathcal{Q}}$ s.t. for any $v \in \mathcal{U}$,

$$\begin{aligned} &E\left[\int_\tau^T \left\langle \bar{\lambda}\left(B_1^\top p_1 + D_1^\top q_1 + c_1 \bar{v}\right) + (1-\bar{\lambda})\left(B_2^\top p_2 + D_2^\top q_2 + c_2 \bar{v}\right), v - \bar{v}\right\rangle \mathrm{d}t\right] \\ &+ E\left[\int_0^\tau \left\langle \bar{\lambda}\left(\tilde{B}_1^\top \tilde{p}_1 + \tilde{D}_1^\top \tilde{q}_1 + \tilde{c}_1 \bar{v}\right) + (1-\bar{\lambda})\left(\tilde{B}_2^\top \tilde{p}_2 + \tilde{D}_2^\top \tilde{q}_2 + \tilde{c}_2 \bar{v}\right), v - \bar{v}\right\rangle \mathrm{d}t\right] \geqslant 0. \end{aligned} \tag{6}$$

For any admissible control v, letting $\alpha_\theta = X_\theta - \bar{X}_\theta$, $\tilde{\alpha}_\theta = \tilde{X}_\theta - \bar{\tilde{X}}_\theta$, where $(\tilde{X}_\theta, X_\theta)$ is the solution of (1) corresponding to v, we have

$$\begin{cases} \mathrm{d}\alpha_\theta = [A_\theta \alpha_\theta + B_\theta(v - \bar{v})] \mathrm{d}s + [C_\theta \alpha_\theta + D_\theta(v - \bar{v})] \mathrm{d}W, \\ \mathrm{d}\tilde{\alpha}_\theta = \left[\tilde{A}_\theta \tilde{\alpha}_\theta + \tilde{B}_\theta(v - \bar{v})\right] \mathrm{d}s + \left[\tilde{C}_\theta \tilde{\alpha}_\theta + \tilde{D}_\theta(v - \bar{v})\right] \mathrm{d}W, \\ \tilde{\alpha}_\theta(0) = 0, \quad \alpha_\theta(\tau) = K_\theta \tilde{\alpha}_\theta(\tau). \end{cases} \tag{7}$$

Let

$$\begin{aligned} m_\theta[v] =& E\left[\int_\tau^T \left[\left\langle a_\theta \bar{X}_\theta, \alpha_\theta\right\rangle + \langle c_\theta \bar{v}, v - \bar{v}\rangle\right] \mathrm{d}t + \left\langle G_\theta \bar{X}_\theta(T), \alpha_\theta(T)\right\rangle\right] \\ &+ E\left[\int_0^\tau \left[\left\langle \tilde{a}_\theta \bar{\tilde{X}}_\theta, \tilde{\alpha}_\theta\right\rangle + \langle \tilde{c}_\theta \bar{v}, v - \bar{v}\rangle\right] \mathrm{d}t + \left\langle \tilde{G}_\theta \bar{\tilde{X}}_\theta(\tau), \tilde{\alpha}_\theta(\tau)\right\rangle\right], \theta = 1, 2. \end{aligned}$$

Applying Itô's formula to $\langle p, \alpha\rangle$ and $\langle \tilde{p}, \tilde{\alpha}\rangle$, it can be obtained that the necessary condition (6) is equivalent to the condition where there exists a $\bar{\lambda} \in \bar{\mathcal{Q}}$ s.t. the following variational inequality:

$$\inf_{v \in \mathcal{U}} \{\bar{\lambda} m_1[v] + (1-\bar{\lambda}) m_2[v]\} \geqslant 0. \tag{8}$$

Step 2. Proof of the variational inequality (8)**.**
Set

$$\mathbb{J}_{\theta,1}[v] = \frac{1}{2}E\bigg\{ \int_0^\tau \Big[\Big\langle \tilde{a}_\theta \tilde{X}_\theta, \tilde{X}_\theta \Big\rangle + \langle \tilde{c}_\theta v, v\rangle\Big] \mathrm{d}t + \Big\langle \tilde{G}_\theta \tilde{X}_\theta(\tau), \tilde{X}_\theta(\tau)\Big\rangle \bigg\}, \tag{9}$$

$$\mathbb{J}_{\theta,2}[v] = \frac{1}{2}E\bigg\{ \int_\tau^T [\langle a_\theta X_\theta, X_\theta\rangle + \langle c_\theta v, v\rangle] \mathrm{d}t + \langle G_\theta X_\theta(T), X_\theta(T)\rangle \bigg\}. \tag{10}$$

Let $(\tilde{X}^l, X^l)$ denote the solution of (1) corresponding to $v^l = \bar{v}+l(v-\bar{v})$; then, it holds that $X^l_\theta = \bar{X}_\theta + l\alpha_\theta$, $\tilde{X}^l_\theta = \bar{\tilde{X}}_\theta + l\tilde{\alpha}_\theta$. Simple calculations lead to

$$\begin{aligned}\mathbb{J}_{\theta,2}[v^l] - \mathbb{J}_{\theta,2}[\bar{v}] = \frac{1}{2}E\Bigg\{ &\int_\tau^T \left[\left\langle a_\theta(2\bar{X}_\theta + l\alpha_\theta), l\alpha_\theta\right\rangle + \langle c_\theta(2\bar{v} + l(v-\bar{v})), l(v-\bar{v})\rangle\right] \mathrm{d}t \\ &+ \left\langle G_\theta(2\bar{X}_\theta(T) + l\alpha_\theta(T)), l\alpha_\theta(T)\right\rangle \Bigg\},\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbb{J}_{\theta,1}[v^l] - \mathbb{J}_{\theta,1}[\bar{v}] = \frac{1}{2}E\Bigg\{ &\int_0^\tau \left[\left\langle \tilde{a}_\theta(2\bar{\tilde{X}}_\theta + l\tilde{\alpha}_\theta), l\tilde{\alpha}_\theta\right\rangle + \langle \tilde{c}_\theta(2\bar{v} + l(v-\bar{v})), l(v-\bar{v})\rangle\right] \mathrm{d}t \\ &+ \left\langle \tilde{G}_\theta(2\bar{\tilde{X}}_\theta(\tau) + l\tilde{\alpha}_\theta(\tau)), l\tilde{\alpha}_\theta(\tau)\right\rangle \Bigg\}.\end{aligned} \tag{12}$$

For any $\lambda \in \bar{\mathcal{Q}}$, by the definition of $J[v]$, (11), and (12), it holds that

$$\begin{aligned}\liminf_{l\to 0}\frac{J[v^l] - J[\bar{v}]}{l} &\geqslant \liminf_{l\to 0}\frac{\lambda(Y_1[v^l] - \bar{Y}_1) + (1-\lambda)(Y_2[v^l] - \bar{Y}_2)}{l} \\ &= \lambda m_1[v] + (1-\lambda) m_2[v].\end{aligned}$$

Therefore,

$$\liminf_{l\to 0}\frac{J[v^l] - J[\bar{v}]}{l} \geqslant \sup_{\lambda\in\bar{\mathcal{Q}}}\{\lambda m_1[v] + (1-\lambda)m_2[v]\}. \tag{13}$$

We intend to prove that the reverse inequality of (13) is true. By the property of upper limits, there exists a sequence $\{l_n\}$ s.t. $l_n \to 0$, $n\to +\infty$, and $\limsup_{l\to 0}(J[v^l] - J[\bar{v}])/l = \lim_{n\to+\infty}(J[v^{l_n}] - J[\bar{v}])/l_n$. For any $n \in \mathbb{N}^+$, there exists an $\lambda_n \in [0,1]$ s.t.

$$J[v^{l_n}] = \lambda_n Y_1[v^{l_n}] + (1-\lambda_n) Y_2[v^{l_n}]. \tag{14}$$

Therefore, by (11), (12), and (14),

$$\begin{aligned}\frac{J[v^{l_n}] - J[\bar{v}]}{l_n} &\leqslant \frac{\lambda_n(Y_1[v^{l_n}] - \bar{Y}_1) + (1-\lambda_n)(Y_2[v^{l_n}] - \bar{Y}_2)}{l_n} \\ &\leqslant \lambda_n m_1[v] + (1-\lambda_n)m_2[v] + o(l_n).\end{aligned} \tag{15}$$

If $Y_1[v^{l_n}] \geqslant Y_2[v^{l_n}]$, let $\lambda_n = 1$. Else, let $\lambda_n = 0$. Then, there exists a subsequence $\{n_k\}$ satisfying (c1) or (c2), where (c1) $\lambda_{n_k} \equiv 1$, $\forall k\in\mathbb{N}^+$ and (c2) $\lambda_{n_k} \equiv 0$, $\forall k \in \mathbb{N}^+$. Because the analysis under (c1) and (c2) is symmetrical, we only consider (c1), where $\lambda_{n_k}\equiv 1$, $\forall k\in\mathbb{N}^+$. Then, it holds that $Y_1[v^{l_{n_k}}] \geqslant Y_2[v^{l_{n_k}}]$. By (11) and (12), we obtain that $Y_\theta[v^{l_{n_k}}] \to \bar{Y}_\theta$, $k\to+\infty$. Therefore, $\bar{Y}_1 \geqslant \bar{Y}_2$, which means that $1 \in \bar{\mathcal{Q}}$. Taking the limit on both sides of (15), we arrive at

$$\limsup_{l\to 0}\frac{J[v^l] - J[\bar{v}]}{l} = \lim_{k\to+\infty}\frac{J[v^{l_k}] - J[\bar{v}]}{l_k} \leqslant m_1[v]. \tag{16}$$

Combining (13) and (16), along with $1\in\bar{\mathcal{Q}}$ and the optimality of $\bar{v}$, we obtain that

$$\sup_{\lambda\in\bar{\mathcal{Q}}}\{\lambda m_1[v] + (1-\lambda)m_2[v]\} = \lim_{l\to 0}\frac{J[v^l] - J[\bar{v}]}{l} \geqslant 0$$

holds for any $v \in \mathcal{U}$. Thus,

$$\inf_{v\in\mathcal{U}}\sup_{\lambda\in\bar{\mathcal{Q}}}\{\lambda m_1[v]+(1-\lambda)m_2[v]\}\geqslant 0.$$

We plan to derive

$$\sup_{\lambda\in\bar{\mathcal{Q}}}\inf_{v\in\mathcal{U}}\{\lambda m_1[v]+(1-\lambda)m_2[v]\}\geqslant 0. \tag{17}$$

Therefore, when $\bar{Y}_1 \neq \bar{Y}_2$, then $\bar{\mathcal{Q}}$ is a singleton; clearly, Eq. (17), which is equivalent to (8), holds. Moreover, when $\bar{Y}_1 = \bar{Y}_2$, it holds that $\bar{\mathcal{Q}} = [0,1]$. Through (17), $\forall \varepsilon_n > 0$, there exists $\lambda_n \in [0,1]$ s.t., $\forall v \in \mathcal{U}$,

$$\lambda_n m_1[v]+(1-\lambda_n)m_2[v]\geqslant \inf_{v\in\mathcal{U}}\{\lambda_n m_1[v]+(1-\lambda_n)m_2[v]\}\geqslant -\varepsilon_n. \tag{18}$$

Then, choosing a subsequence, if needed, there exists a $\bar{\lambda} \in [0,1]$ independent of v s.t. $\lambda_n \to \bar{\lambda}$. Thus, for any $v \in \mathcal{U}$, taking the limit on both sides of (18), we obtain that there exists a $\bar{\lambda} \in \bar{\mathcal{Q}}$ s.t., for any $v \in \mathcal{U}$,

$$\bar{\lambda} m_1[v]+(1-\bar{\lambda})m_2[v]\geqslant 0, \tag{19}$$

which proves (8).

It only remains to prove (17) given that $\bar{Y}_1 = \bar{Y}_2$. In this case, $\bar{\mathcal{Q}} = [0,1]$. Let $g(\lambda, v) = \lambda m_1[v] + (1-\lambda)m_2[v]$. It can be easily obtained that $g(\lambda, v)$ is linear and hence convex w.r.t. v and linear w.r.t. λ. For $v' \in \mathcal{U}$, let $(\tilde{\alpha}^{v'}, \alpha^{v'})$ denote the solution of (7) under v'. By classical methods such as Gronwall's inequality and the Burkholder-Davis-Gundy inequality, we derive (hereafter, $\mathbb{C} > 0$ represents a constant independent of (v, v') and may vary from line to line)

$$E\left[\sup_{t\in[0,\tau]}\left|\tilde{\alpha}^v_\theta-\tilde{\alpha}^{v'}_\theta\right|^2\right]\leqslant \mathbb{C}E\left[\int_0^\tau |v-v'|^2\mathrm{d}r\right],$$

and

$$\begin{aligned}
&E\left[\sup_{t\in[\tau,T]}\left|\alpha^v_\theta-\alpha^{v'}_\theta\right|^2\right]\\
&\leqslant \mathbb{C}E\left[\int_\tau^T |v-v'|^2\mathrm{d}r+\left|K\left(\tilde{\alpha}^v_\theta(\tau)-\tilde{\alpha}^{v'}_\theta(\tau)\right)\right|^2\right]\\
&\leqslant \mathbb{C}E\left[\int_0^T |v-v'|^2\mathrm{d}r\right].
\end{aligned}$$

Therefore, by the definition of m_θ, we can deduce

$$|g(\lambda,v)-g(\lambda,v')|\leqslant \mathbb{C}E\left[\int_0^T |v-v'|^2\mathrm{d}r\right],$$

implying the continuity of $g(\lambda, v)$ w.r.t. v. As $g(\lambda, v)$ is also linear w.r.t. λ and convex w.r.t. v, by the mini-max theorem (cf. Theorem B.1.2 in [14]),

$$\sup_{\lambda\in[0,1]}\inf_{v\in\mathcal{U}}g(\lambda,v)=\inf_{v\in\mathcal{U}}\sup_{\lambda\in[0,1]}g(\lambda,v)\geqslant 0,$$

which is (17). The necessity of (5) is proven.

Part II. Sufficiency of (5).

Suppose that there exist $\bar{\lambda}$ and $\bar{v}$ satisfying (5). For any admissible control v, still let $\alpha_\theta = X_\theta - \bar{X}_\theta$, $\tilde{\alpha}_\theta = \tilde{X}_\theta - \bar{\tilde{X}}_\theta$ as in Part I. Applying Itô's formula to $\langle p_\theta, \alpha_\theta\rangle$, it holds that

$$\mathbb{J}_{\theta,2}[v]-\mathbb{J}_{\theta,2}[\bar{v}]$$

$$
\begin{aligned}
&\geqslant E\bigg\{ \left\langle G_\theta \bar{X}_\theta(T), \alpha_\theta(T) \right\rangle \\
&\quad + \mathbb{J}_{\theta,2}[v] - \mathbb{J}_{\theta,2}[\bar{v}] - \frac{1}{2}\left\langle G_\theta X_\theta(T), X_\theta(T) \right\rangle + \frac{1}{2}\left\langle G_\theta \bar{X}_\theta(T), \bar{X}_\theta(T) \right\rangle \bigg\} \\
&= E\bigg\{ \langle p_\theta(\tau), \alpha_\theta(\tau) \rangle + \int_\tau^T \Big\{ - \left\langle a_\theta \bar{X}_\theta, \alpha_\theta \right\rangle + \left\langle B_\theta^\top p_\theta, v - \bar{v} \right\rangle + \left\langle D_\theta^\top q_\theta, v - \bar{v} \right\rangle \\
&\quad + \frac{1}{2}\left[\langle a_\theta X_\theta, X_\theta \rangle - \left\langle a_\theta \bar{X}_\theta, \bar{X}_\theta \right\rangle + \langle c_\theta v, v \rangle - \langle c_\theta \bar{v}, \bar{v} \rangle \right] \Big\} \mathrm{d}s \bigg\} \\
&\geqslant E\left[\langle p_\theta(\tau), \alpha_\theta(\tau) \rangle + \int_\tau^T \left\langle B_\theta^\top p_\theta + D_\theta^\top q_\theta + c_\theta \bar{v}, v - \bar{v} \right\rangle \mathrm{d}s \right].
\end{aligned}
$$

Similarly, applying Itô's formula to $\langle \tilde{p}_\theta, \tilde{\alpha}_\theta \rangle$, it holds that

$$
\begin{aligned}
&\bar{\lambda} Y_1[v] + (1-\bar{\lambda}) Y_2[v] - \bar{\lambda} \bar{Y}_1 - (1-\bar{\lambda}) \bar{Y}_2 \\
&\quad = \bar{\lambda} \left\{ \mathbb{J}_{1,2}[v] + \mathbb{J}_{1,1}[v] - \mathbb{J}_{1,2}[\bar{v}] - \mathbb{J}_{1,1}[\bar{v}] \right\} + (1-\bar{\lambda}) \left\{ \mathbb{J}_{2,2}[v] + \mathbb{J}_{2,1}[v] - \mathbb{J}_{2,2}[\bar{v}] - \mathbb{J}_{2,1}[\bar{v}] \right\} \\
&\quad \geqslant \bar{\lambda} E\left[\langle p_1(\tau), \alpha_1(\tau) \rangle + \mathbb{J}_{1,1}[v] - \mathbb{J}_{1,1}[\bar{v}] + \int_\tau^T \left\langle B_1^\top p_1 + D_1^\top q_1 + c_1 \bar{v}, v - \bar{v} \right\rangle \mathrm{d}s \right] \\
&\qquad + (1-\bar{\lambda}) E\left[\langle p_2(\tau), \alpha_2(\tau) \rangle + \mathbb{J}_{2,1}[v] - \mathbb{J}_{2,1}[\bar{v}] + \int_\tau^T \left\langle B_2^\top p_2 + D_2^\top q_2 + c_2 \bar{v}, v - \bar{v} \right\rangle \mathrm{d}s \right] \\
&\quad \geqslant E\bigg[\int_0^\tau \left\langle \bar{\lambda}\left(\tilde{B}_1^\top \tilde{p}_1 + \tilde{D}_1^\top \tilde{q}_1 + \tilde{c}_1 \bar{v} \right) + (1-\bar{\lambda}) \left(\tilde{B}_2^\top \tilde{p}_2 + \tilde{D}_2^\top \tilde{q}_2 + \tilde{c}_2 \bar{v} \right), v - \bar{v} \right\rangle \mathrm{d}s \\
&\qquad + \int_\tau^T \left\langle \bar{\lambda}\left(B_1^\top p_1 + D_1^\top q_1 + c_1 \bar{v} \right) + (1-\bar{\lambda}) \left(B_2^\top p_2 + D_2^\top q_2 + c_2 \bar{v} \right), v - \bar{v} \right\rangle \mathrm{d}s \bigg] \\
&\quad = 0.
\end{aligned}
$$

Finally, we arrive at

$$
J[v] - J[\bar{v}] \geqslant \bar{\lambda} Y_1[v] + (1-\bar{\lambda}) Y_2[v] - \bar{\lambda} \bar{Y}_1 - (1-\bar{\lambda}) \bar{Y}_2 \geqslant 0.
$$

The proof is complete.

3.2 Designing the feedback form of the robust optimal control $\bar{v}$

Suppose that there exist $\bar{\lambda}$ and $\bar{v}$ satisfying (5). Set

$$
\mathcal{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},
$$

and

$$
K = \begin{bmatrix} K_1^\top & 0 \\ 0 & K_2^\top \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.
$$

Similarly, define $\tilde{\mathcal{A}}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{b}, \tilde{\sigma}$, and $\tilde{X}$. Then, the state equation (1) is reformulated as

$$
\begin{cases}
\mathrm{d}\tilde{X} = \left(\tilde{\mathcal{A}} \tilde{X} + \tilde{B} v + \tilde{b} \right) \mathrm{d}t + \left(\tilde{C} \tilde{X} + \tilde{D} v + \tilde{\sigma} \right) \mathrm{d}W, \quad t \in [0, \tau), \\
\mathrm{d}X = (\mathcal{A} X + B v + b)\, \mathrm{d}t + (C X + D v + \sigma)\, \mathrm{d}W, \quad t \in [\tau, T], \\
\tilde{X}(0) = x, \quad X(\tau) = K^\top \tilde{X}(\tau).
\end{cases} \tag{20}
$$

Set

$$
p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}.
$$

Similarly, set $\tilde{p}, \tilde{q}, \tilde{a}, \tilde{G}$, and

$$\tilde{\Lambda} = \begin{bmatrix} \lambda \mathbb{I}_{n\times n} & 0 \\ 0 & (1-\lambda)\mathbb{I}_{n\times n} \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda \mathbb{I}_{m\times m} & 0 \\ 0 & (1-\lambda)\mathbb{I}_{m\times m} \end{bmatrix}.$$

Then, the optimality conditions (5b) and (5c) are rewritten as

$$B^\top \Lambda p + D^\top \Lambda q + R^{\bar{\lambda}} \bar{v} = 0, \tilde{B}^\top \tilde{\Lambda} \tilde{p} + \tilde{D}^\top \tilde{\Lambda} \tilde{q} + \tilde{R}^{\bar{\lambda}} \bar{v} = 0, \tag{21}$$

where

$$\begin{cases} -\mathrm{d}p = \left(a\bar{X} + \mathcal{A}^\top p + C^\top q\right)\mathrm{d}t - q\mathrm{d}W, \\ p(T) = G\bar{X}(T), \end{cases}$$

$$\begin{cases} -\mathrm{d}\tilde{p} = \left(\tilde{a}\bar{\tilde{X}} + \tilde{\mathcal{A}}^\top \tilde{p} + \tilde{C}^\top \tilde{q}\right)\mathrm{d}t - \tilde{q}\mathrm{d}W, \\ \tilde{p}(\tau) = \tilde{G}\bar{\tilde{X}}(\tau) + Kp(\tau), \end{cases}$$

$$\tilde{R}^{\bar{\lambda}} = \bar{\lambda}\tilde{c}_1 + (1-\bar{\lambda})\tilde{c}_2, R^{\bar{\lambda}} = \bar{\lambda}c_1 + (1-\bar{\lambda})c_2, \tag{22}$$

and $(\bar{\tilde{X}}, \bar{X})$ is the solution of (20) under $\bar{v}$.

Now, we want to give a feedback form of $\bar{v}$. Set $\Lambda p = P\bar{X} + \phi$ with $P \in C^1(0,T;\mathbb{S}^{2m})$ and $\phi \in C^1(0,T;\mathbb{R}^{2m})$. Applying Itô's formula to $P\bar{X}$, it holds that

$$\left[\dot{P}\bar{X} + P\left(\mathcal{A}\bar{X} + B\bar{v} + b\right) + \dot{\phi}\right]\mathrm{d}t + P\left(C\bar{X} + D\bar{v} + \sigma\right)\mathrm{d}W \tag{23}$$

$$\begin{aligned} &= \mathrm{d}\Lambda p \\ &= -\Lambda\left(a\bar{X} + \mathcal{A}p + Cq\right)\mathrm{d}t + \Lambda q\mathrm{d}W. \end{aligned} \tag{24}$$

Comparing (24) and (23), we have $\Lambda q = P\left(C\bar{X} + D\bar{v} + \sigma\right)$, and

$$\left(\dot{P} + P\mathcal{A} + \mathcal{A}P + a\Lambda\right)\bar{X} + PB\bar{v} + CP\left(C\bar{X} + D\bar{v} + \sigma\right) + Pb + \dot{\phi} + \mathcal{A}\phi = 0. \tag{25}$$

Inserting Λp and Λq into (21) results in

$$\bar{v} = -\left(R^{\bar{\lambda}} + D^\top PD\right)^{-1}\left[\left(B^\top P + D^\top PC\right)\bar{X} + B^\top\phi + D^\top P\sigma\right], t \in [\tau, T],$$

where

$$\begin{cases} \dot{P} + P\mathcal{A} + \mathcal{A}^\top P + a\Lambda + C^\top PC \\ \quad - \left(B^\top P + D^\top PC\right)^\top \left(R^{\bar{\lambda}} + D^\top PD\right)^{-1}\left(B^\top P + D^\top PC\right) = 0, \\ P(T) = \Lambda G, \end{cases} \tag{26}$$

$$\begin{cases} \dot{\phi} + \left[\mathcal{A} - B\left(R^{\bar{\lambda}} + D^\top PD\right)^{-1}\left(B^\top P + D^\top PC\right)\right]^\top \phi \\ \quad + \left[C - D\left(R^{\bar{\lambda}} + D^\top PD\right)^{-1}\left(B^\top P + D^\top PC\right)\right]^\top P\sigma + Pb = 0, \\ \phi(T) = 0. \end{cases} \tag{27}$$

Similarly, we have

$$\bar{v} = -\left(\tilde{R}^{\bar{\lambda}} + \tilde{D}^\top \tilde{P}\tilde{D}\right)^{-1}\left[\left(\tilde{B}^\top \tilde{P} + \tilde{D}^\top \tilde{P}\tilde{C}\right)\bar{\tilde{X}} + \tilde{B}^\top\tilde{\phi} + \tilde{D}^\top \tilde{P}\tilde{\sigma}\right], t \in [0, \tau),$$

where

$$\begin{cases} \dot{\tilde{P}} + \tilde{P}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^\top \tilde{P} + \tilde{a}\Lambda + \tilde{C}^\top \tilde{P}\tilde{C} \\ \quad - \left(\tilde{B}^\top \tilde{P} + \tilde{D}^\top \tilde{P}\tilde{C}\right)^\top \left(\tilde{R}^{\bar{\lambda}} + \tilde{D}^\top \tilde{P}\tilde{D}\right)^{-1}\left(\tilde{B}^\top \tilde{P} + \tilde{D}^\top \tilde{P}\tilde{C}\right) = 0, \\ \tilde{P}(\tau) = \tilde{\Lambda}\tilde{G} + KP(\tau)K^\top, \end{cases} \tag{28}$$

$$\begin{cases} \dot{\tilde{\phi}} + \left[\tilde{\mathcal{A}} - \tilde{B}\left(\tilde{R}^{\bar{\lambda}} + \tilde{D}^\top \tilde{P}\tilde{D}\right)^{-1}\left(\tilde{B}^\top \tilde{P} + \tilde{D}^\top \tilde{P}\tilde{C}\right)\right]^\top \tilde{\phi} \\ + \left[\tilde{C} - \tilde{D}\left(\tilde{R}^{\bar{\lambda}} + \tilde{D}^\top \tilde{P}\tilde{D}\right)^{-1}\left(\tilde{B}^\top \tilde{P} + \tilde{D}^\top \tilde{P}\tilde{C}\right)\right]^\top \tilde{P}\tilde{\sigma} + \tilde{P}\tilde{b} = 0, \\ \tilde{\phi}(\tau) = K\phi(\tau). \end{cases} \tag{29}$$

Eqs. (26) and (28) are stochastic Riccati equations. Under any $\bar{\lambda} \in [0, 1]$, Eqs. (26) and (28) admit unique nonnegative definite solutions (cf. Theorem 7.2, Chapter 6 in Yong and Zhou [5]). Therefore, the linear differential equations (27) and (29) are uniquely solvable.

The analysis above is summarized as the following lemma.

Lemma 1. Suppose Assumption 1 holds. Let $\bar{v} \in \bar{\mathcal{U}}$, $\bar{\lambda} \in [0, 1]$ satisfy (5). Then, $\bar{v}$ is a robust optimal control of the TSLQU problem, which admits the following feedback form:

$$\bar{v} = \begin{cases} -\left(\tilde{R}^{\bar{\lambda}} + \tilde{D}^\top \tilde{P}\tilde{D}\right)^{-1}\left[\left(\tilde{B}^\top \tilde{P} + \tilde{D}^\top \tilde{P}\tilde{C}\right)\bar{\tilde{X}} + \tilde{B}^\top \tilde{\phi} + \tilde{D}^\top \tilde{P}\tilde{\sigma}\right], t \in [0, \tau), \\ -\left(R^{\bar{\lambda}} + D^\top P D\right)^{-1}\left[\left(B^\top P + D^\top P C\right)\bar{X} + B^\top \phi + D^\top P\sigma\right], t \in [\tau, T], \end{cases} \tag{30}$$

where $P, \phi, \tilde{P}$, and $\tilde{\phi}$ solve the differential equations (26)–(29), respectively, $R^{\bar{\lambda}}$ and $\tilde{R}^{\bar{\lambda}}$ are given by (22), and $(\bar{\tilde{X}}, \bar{X})$ solves the two-stage SDE (20) under $\bar{v}$.

Remark 3. There are two main differences between (30) and the feedback optimal controls of classical stochastic LQ problems (cf. [5]). First, we can see that unlike in stochastic LQ problems without model uncertainty, the robust optimal control (30) of the TSLQU problem usually relies on both coefficients under $\theta = 1$ and ones under $\theta = 2$, which means that because of model uncertainty, the robust optimal control relies on the information of two uncertain states. Secondly, because of the impulse term in the state equation, the robust optimal control (30) also switches at time τ, which is substantially different compared with those of classical stochastic LQ problems.

3.3 Seeking for the optimal parameter $\bar{\lambda}$

According to the optimality condition (5a), $\bar{\lambda}$ must satisfy $\bar{Y}_1 \vee \bar{Y}_2 = \bar{\lambda}\bar{Y}_1 + (1 - \bar{\lambda})\bar{Y}_2$. Nonetheless, because of Lemma 1, $\bar{v}, \bar{\tilde{X}}, \bar{X}, \bar{Y}_1$, and $\bar{Y}_2$ all depend on the parameter $\bar{\lambda}$. Consequently, for a given $\bar{\lambda}$, Eq. (5a) might not hold, which cannot guarantee the existence of the optimal parameter $\bar{\lambda}$. For $\lambda \in [0, 1]$ and a random variable ξ, define ξ^λ to emphasize the dependency of ξ on λ. Through a series of analyses, we can perform the continuity of $\bar{Y}_1^\lambda = Y_1[\bar{v}^\lambda]$ and $\bar{Y}_2^\lambda = Y_2[\bar{v}^\lambda]$ w.r.t. λ, which further leads to the existence of $\bar{\lambda}$ satisfying (5a). Furthermore, via the explicit presentation of $\bar{Y}_1^\lambda$ and $\bar{Y}_2^\lambda$ under λ, we give the characterization of $\bar{\lambda}$.

Lemma 2. $\bar{Y}_1^\lambda$ and $\bar{Y}_2^\lambda$ are continuous w.r.t. $\lambda \in [0, 1]$.

Proof. Hereafter, $\mathbb{C}$ represents a constant independent of λ. Let $\Delta\xi = \xi^\lambda - \xi^{\lambda'}$ for $\lambda, \lambda' \in [0, 1]$.

(i) Estimate $|\Delta\bar{\tilde{X}}|$ on $[0, \tau]$. Given Theorem 4.5 in Hu and Wang [9], the following continuity dependency of the solution of (26) w.r.t. λ holds

$$|\Delta P(t)| \leqslant \mathbb{C}|\lambda - \lambda'|, t \in [\tau, T]. \tag{31}$$

By Lemma 3.1 in Kohlmann and Tang [15], the uniform estimate of $|P^\lambda(t)|$ holds

$$|P^\lambda(t)| \leqslant \mathbb{C}, t \in [\tau, T]. \tag{32}$$

Rewrite (27) as

$$\begin{cases} \dot{\phi^\lambda} + \rho^\lambda \phi^\lambda + \varrho^\lambda = 0, \\ \phi(T) = 0. \end{cases}$$

Then, Eq. (32) and the uniformly positive definiteness of c_1 and c_2 result in

$$|\rho^\lambda(t)| \vee |\varrho^\lambda(t)| \leqslant \mathbb{C}, t \in [\tau, T]. \tag{33}$$

By Gronwall's inequality and (33), we arrive at

$$|\phi^{\lambda}(t)| \leqslant \mathbb{C}, t \in [\tau, T]. \tag{34}$$

Using the estimates of $|\Delta P(t)|$ (31), it can be obtained that

$$|\Delta\rho(t)| \vee |\Delta\varrho(t)| \leqslant \mathbb{C}|\lambda - \lambda'|, t \in [\tau, T]. \tag{35}$$

Organize the equation satisfied by $\Delta\phi$ as

$$\begin{cases} \Delta\dot{\phi} + \Delta\rho\phi^{\lambda} + \rho^{\lambda'}\Delta\phi + \Delta\varrho = 0, \\ \Delta\phi(T) = 0. \end{cases}$$

By Gronwall's inequality, Eqs. (33), (34), and (35) result in the estimates of $|\Delta\phi|$, and

$$|\Delta\phi(t)| \leqslant \mathbb{C}|\lambda - \lambda'|, t \in [\tau, T].$$

Substituting $\bar{v}$ into $\bar{\tilde{X}}$, the feedback state satisfied by $\bar{\tilde{X}}$ can be rewritten as

$$\mathrm{d}\bar{\tilde{X}}^{\lambda} = \left(\tilde{\mu}^{\lambda}\bar{\tilde{X}}^{\lambda} + \tilde{b}\right) + \left(\tilde{\nu}^{\lambda}\bar{\tilde{X}}^{\lambda} + \tilde{\sigma}\right)\mathrm{d}W, \bar{\tilde{X}}(0) = x.$$

Continue to estimate $\tilde{\mu}^{\lambda}$ and $\tilde{\nu}^{\lambda}$. By the estimates $|\Delta P(t)| \vee |\Delta\phi(t)| \leqslant \mathbb{C}|\lambda - \lambda'|$, $t \in [\tau, T]$, it holds that $|\Delta\tilde{P}(\tau)| \vee |\Delta\tilde{\phi}(\tau)| \leqslant \mathbb{C}|\lambda - \lambda'|$. By the procedures dealing with P and ϕ, we obtain $|\Delta\tilde{P}(t)| \vee |\Delta\tilde{\phi}(t)| \leqslant \mathbb{C}|\lambda - \lambda'|$, $t \in [0, \tau]$. Therefore, $|\Delta\tilde{\mu}(t)| \vee |\Delta\tilde{\nu}(t)| \leqslant \mathbb{C}|\lambda - \lambda'|$, $t \in [0, \tau]$.

Rearrange $\Delta\bar{\tilde{X}}$ as follows:

$$\mathrm{d}\Delta\bar{\tilde{X}} = \left(\Delta\tilde{\mu}\bar{\tilde{X}}^{\lambda} + \tilde{\mu}^{\lambda'}\Delta\bar{\tilde{X}}\right)\mathrm{d}s + \left(\Delta\tilde{\nu}\bar{\tilde{X}}^{\lambda} + \tilde{\nu}^{\lambda'}\Delta\bar{\tilde{X}}\right)\mathrm{d}W, \quad \Delta\bar{\tilde{X}}(0) = 0.$$

By the classical estimates of SDEs (cf. Lemma A.1 in Hu and Wang [9]), we can obtain the estimates of $|\Delta\bar{\tilde{X}}^{\lambda}|$:

$$\begin{aligned} & E\left[\sup_{t\in[0,\tau]}\left|\Delta\bar{\tilde{X}}(t)\right|^4\right] \\ & \leqslant \mathbb{C}E\left[\left(\int_0^{\tau}\left|\Delta\tilde{\mu}\bar{\tilde{X}}^{\lambda}\right|\mathrm{d}t\right)^4 + \left(\int_0^{\tau}\left|\Delta\tilde{\nu}\bar{\tilde{X}}^{\lambda}\right|^2\mathrm{d}t\right)^2\right] \\ & \leqslant \mathbb{C}\left|\lambda - \lambda'\right|^4 E\left[\left(\int_0^{\tau}\left|\bar{\tilde{X}}^{\lambda}\right|\mathrm{d}t\right)^4 + \left(\int_0^{\tau}\left|\bar{\tilde{X}}^{\lambda}\right|^2\mathrm{d}t\right)^2\right] \\ & \leqslant \mathbb{C}\left|\lambda - \lambda'\right|^4 E\left[\sup_{t\in[0,\tau]}\left|\bar{\tilde{X}}^{\lambda}(t)\right|^4\right] \\ & \leqslant \mathbb{C}\left|\lambda - \lambda'\right|^4\left[|x|^4 + \left(\int_0^{\tau}|\tilde{b}|\mathrm{d}t\right)^4 + \left(\int_0^{\tau}|\tilde{\sigma}|^2\mathrm{d}t\right)^2\right] \\ & \leqslant \mathbb{C}\left|\lambda - \lambda'\right|^4. \end{aligned}$$

(ii) Estimate $|\Delta\bar{X}|$ on $[\tau, T]$. Plugging $\bar{v}$ into $\bar{X}$, we can arrange $\Delta\bar{X}$ as

$$\mathrm{d}\Delta\bar{X} = \left(\Delta\mu\bar{X}^{\lambda} + \mu^{\lambda'}\Delta\bar{X}\right)\mathrm{d}s + \left(\Delta\nu\bar{X}^{\lambda} + \nu^{\lambda'}\Delta\bar{X}\right)\mathrm{d}W, \quad \Delta\bar{X}(\tau) = K^{\top}\Delta\bar{\tilde{X}}(\tau).$$

According to the estimates in (i) and by similar methods handling $|\Delta\bar{\tilde{X}}|$, it holds that

$$E\left[\sup_{t\in[\tau,T]}\left|\Delta\bar{X}(t)\right|^4\right]$$

$$
\begin{aligned}
&\leqslant \mathbb{C}E\left[\left|K^\top \Delta \bar{X}(\tau)\right|^4 + \left(\int_\tau^T |\Delta\mu \bar{X}^\lambda| \mathrm{d}t\right)^4 + \left(\int_\tau^T |\Delta\nu \bar{X}^\lambda|^2 \mathrm{d}t\right)^2\right] \\
&\leqslant \mathbb{C}|\lambda-\lambda'|^4 E\left[\mathbb{C} + \sup_{t\in[\tau,T]} \left|\bar{X}^\lambda(t)\right|^4\right] \\
&\leqslant \mathbb{C}|\lambda-\lambda'|^4 E\left[\mathbb{C} + \left|K^\top \bar{\bar{X}}(\tau)\right|^4 + \left(\int_\tau^T |b| \mathrm{d}t\right)^4 + \left(\int_\tau^T |\sigma|^2 \mathrm{d}t\right)^2\right] \\
&\leqslant \mathbb{C}|\lambda-\lambda'|^4 \left\{\mathbb{C}\left[1 + |x|^4 + \left(\int_0^\tau |\tilde{b}| \mathrm{d}t\right)^4 + \left(\int_0^\tau |\tilde{\sigma}|^2 \mathrm{d}t\right)^2\right]\right. \\
&\quad \left. + \left(\int_\tau^T |b| \mathrm{d}t\right)^4 + \left(\int_\tau^T |\sigma|^2 \mathrm{d}t\right)^2\right\} \\
&\leqslant \mathbb{C}|\lambda-\lambda'|^4.
\end{aligned}
$$

(iii) Estimate $|\Delta \bar{Y}_1|$ and $|\Delta \bar{Y}_2|$. By (9) and (10), $\Delta \bar{Y}_\theta = \Delta \mathbb{J}_{\theta,1}[\bar{v}] + \Delta \mathbb{J}_{\theta,2}[\bar{v}]$. First, estimate $\Delta \mathbb{J}_{\theta,1}[\bar{v}]$,

$$
\begin{aligned}
&|\Delta \mathbb{J}_{\theta,1}[\bar{v}]| \\
&\quad = \left|\frac{1}{2}E\left\{\int_0^\tau \left[\Delta\left\langle \tilde{a}_\theta \bar{\bar{X}}_\theta, \bar{\bar{X}}_\theta\right\rangle + \Delta\left\langle \tilde{c}_\theta \bar{v}, \bar{v}\right\rangle\right]\mathrm{d}t + \Delta\left\langle \tilde{G}_\theta \bar{\bar{X}}_\theta(\tau), \bar{\bar{X}}_\theta(\tau)\right\rangle\right\}\right| \\
&\quad \leqslant \frac{1}{2}E\left\{\int_0^\tau \left[\left|\Delta\left\langle \tilde{a}_\theta \bar{\bar{X}}_\theta, \bar{\bar{X}}_\theta\right\rangle\right| + \left|\Delta\left\langle \tilde{c}_\theta \bar{v}, \bar{v}\right\rangle\right|\right]\mathrm{d}t + \left|\Delta\left\langle \tilde{G}_\theta \bar{\bar{X}}_\theta(\tau), \bar{\bar{X}}_\theta(\tau)\right\rangle\right|\right\}.
\end{aligned}
$$

We only give the estimates of the first term on the right side of the inequality above. As $\bar{v}$ is a linear feedback of the optimal state, the estimates of the rest of the terms are similar. By Young's inequality and Fubini's theorem, among others, we have

$$
\begin{aligned}
&E\left[\int_0^\tau \left|\Delta\left\langle \tilde{a}_\theta \bar{\bar{X}}_\theta, \bar{\bar{X}}_\theta\right\rangle\right| \mathrm{d}t\right] \\
&\quad = E\left[\int_0^\tau \left|\left\langle \tilde{a}_\theta \bar{\bar{X}}_\theta^\lambda, \bar{\bar{X}}_\theta^\lambda\right\rangle - \left\langle \tilde{a}_\theta \bar{\bar{X}}_\theta^{\lambda'}, \bar{\bar{X}}_\theta^{\lambda'}\right\rangle\right| \mathrm{d}t\right] \\
&\quad = E\left[\int_0^\tau \left|\left\langle \tilde{a}_\theta \left(\bar{\bar{X}}_\theta^\lambda + \bar{\bar{X}}_\theta^{\lambda'}\right), \Delta\bar{\bar{X}}_\theta\right\rangle\right| \mathrm{d}t\right] \\
&\quad \leqslant \mathbb{C}E\left\{\int_0^\tau \left|\bar{\bar{X}}_\theta^\lambda + \bar{\bar{X}}_\theta^{\lambda'}\right| \left|\Delta\bar{\bar{X}}_\theta\right| \mathrm{d}t\right\} \\
&\quad \leqslant \mathbb{C}\int_0^\tau \left\{E\left[\left|\bar{\bar{X}}_\theta^\lambda\right|^2 + \left|\bar{\bar{X}}_\theta^{\lambda'}\right|^2\right]\right\}^{\frac{1}{2}} \left\{E\left[\left|\Delta\bar{\bar{X}}_\theta\right|^2\right]\right\}^{\frac{1}{2}} \mathrm{d}t \\
&\quad \leqslant \mathbb{C}\left|\lambda-\lambda'\right|.
\end{aligned}
$$

Thus, $|\Delta \mathbb{J}_{\theta,1}[\bar{v}]| \leqslant \mathbb{C}\left|\lambda-\lambda'\right|$. Similarly, dealing with $|\Delta \mathbb{J}_{\theta,2}[\bar{v}]|$, we have

$$
|\Delta \bar{Y}_\theta| \leqslant |\Delta \mathbb{J}_{\theta,1}[\bar{v}]| + |\Delta \mathbb{J}_{\theta,2}[\bar{v}]| \leqslant \mathbb{C}\left|\lambda-\lambda'\right|.
$$

Therefore, $\bar{Y}_1^\lambda$ and $\bar{Y}_2^\lambda$ are continuous w.r.t. λ.

The proof is complete.

Now, we discuss the optimal parameter $\bar{\lambda}$ satisfying (5) in two cases.

Case 1. $\bar{Y}_1^0 \leqslant \bar{Y}_2^0$ or $\bar{Y}_1^1 \geqslant \bar{Y}_2^1$. Then, it is easily verified that

$$
(\bar{\lambda}, \bar{v}) = \begin{cases} (0, \bar{v}^0), \text{if } \bar{Y}_1^0 \leqslant \bar{Y}_2^0, \\ (1, \bar{v}^1), \text{if } \bar{Y}_1^1 \geqslant \bar{Y}_2^1 \end{cases}
$$

satisfies (5).

Case 2. $\bar{Y}_1^0 > \bar{Y}_2^0$ and $\bar{Y}_1^1 < \bar{Y}_2^1$. By the intermediate value theorem of continuous functions, there exists a constant $\bar{\lambda} \in (0,1)$ s.t. $\bar{Y}_1^{\bar{\lambda}} = \bar{Y}_2^{\bar{\lambda}}$. In this case, $(\bar{\lambda}, \bar{v}^{\bar{\lambda}})$ still satisfies (5).

Now, we further characterize $\bar{\lambda}$ by explicitly computing $\bar{Y}_1^{\lambda}$ and $\bar{Y}_2^{\lambda}$.

Define

$$\mathbb{K} = -\left(R^{\lambda} + D^\top P D\right)^{-1}\left(B^\top P + D^\top P C\right),$$

$$\mathbb{H} = -\left(R^{\lambda} + D^\top P D\right)^{-1}\left(B^\top \phi + D^\top P \sigma\right).$$

Similarly, define $\tilde{\mathbb{K}}$ and $\tilde{\mathbb{H}}$.

Set $V_1(\bar{X}) = \frac{1}{2}\left[\bar{X}^\top \mathbb{X}\bar{X} + \mathbb{Y}\bar{X}\right]$, where

$$\begin{cases} \dot{\mathbb{X}} + \mathbb{X}\left(\mathcal{A} + B\mathbb{K}\right) + \left(\mathcal{A} + B\mathbb{K}\right)^\top \mathbb{X} \\ \quad + \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} + \mathbb{K}^\top c_1 \mathbb{K} + \left(C + D\mathbb{K}\right)^\top \mathbb{X}\left(C + D\mathbb{K}\right) = 0, \\ \mathbb{X}(T) = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{cases} \tag{36}$$

$$\begin{cases} \dot{\mathbb{Y}} + \mathbb{Y}\left(\mathcal{A} + B\mathbb{K}\right) \\ \quad +2\left(D\mathbb{H} + \sigma\right)^\top \mathbb{X}\left(C + D\mathbb{K}\right) + 2\left(B\mathbb{H} + b\right)^\top \mathbb{X} + 2\mathbb{H}^\top c_1 \mathbb{K} = 0, \\ \mathbb{Y}(T) = 0. \end{cases} \tag{37}$$

Then, let $V_2(\bar{\tilde{X}}) = \frac{1}{2}[\bar{\tilde{X}}^\top \tilde{\mathbb{X}}\bar{\tilde{X}} + \tilde{\mathbb{Y}}\bar{\tilde{X}}]$, where

$$\begin{cases} \dot{\tilde{\mathbb{X}}} + \tilde{\mathbb{X}}\left(\tilde{\mathcal{A}} + \tilde{B}\tilde{\mathbb{K}}\right) + \left(\tilde{\mathcal{A}} + \tilde{B}\tilde{\mathbb{K}}\right)^\top \tilde{\mathbb{X}} \\ \quad + \begin{bmatrix} \tilde{a}_1 & 0 \\ 0 & 0 \end{bmatrix} + \tilde{\mathbb{K}}^\top \tilde{c}_1 \tilde{\mathbb{K}} + \left(\tilde{C} + \tilde{D}\tilde{\mathbb{K}}\right)^\top \tilde{\mathbb{X}}\left(\tilde{C} + \tilde{D}\tilde{\mathbb{K}}\right) = 0, \\ \tilde{\mathbb{X}}(\tau) = \begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & 0 \end{bmatrix} + K\mathbb{X}(\tau)K^\top, \end{cases} \tag{38}$$

$$\begin{cases} \dot{\tilde{\mathbb{Y}}} + \tilde{\mathbb{Y}}\left(\tilde{\mathcal{A}} + \tilde{B}\tilde{\mathbb{K}}\right) \\ \quad +2\left(\tilde{D}\tilde{\mathbb{H}} + \tilde{\sigma}\right)^\top \tilde{\mathbb{X}}\left(\tilde{C} + \tilde{D}\tilde{\mathbb{K}}\right) + 2\left(\tilde{B}\tilde{\mathbb{H}} + \tilde{b}\right)^\top \tilde{\mathbb{X}} + 2\tilde{\mathbb{H}}^\top \tilde{c}_1 \tilde{\mathbb{K}} = 0, \\ \tilde{\mathbb{Y}}(\tau) = \mathbb{Y}(\tau)K^\top. \end{cases} \tag{39}$$

Applying Itô's formula to $V_1(\bar{X})$ and $V_2(\bar{\tilde{X}})$, we obtain

$$\begin{aligned} \bar{Y}_1^{\lambda} = & \frac{1}{2}\left[x^\top \tilde{\mathbb{X}}(0)x + \tilde{\mathbb{Y}}(0)x\right] \\ & + \frac{1}{2}\int_0^{\tau}\left[\left(\tilde{D}\tilde{\mathbb{H}} + \tilde{\sigma}\right)^\top \tilde{\mathbb{X}}\left(\tilde{D}\tilde{\mathbb{H}} + \tilde{\sigma}\right) + \tilde{\mathbb{Y}}\left(\tilde{B}\tilde{\mathbb{H}} + \tilde{b}\right) + \tilde{\mathbb{H}}^\top \tilde{c}_1 \tilde{\mathbb{H}}\right]\mathrm{d}t \\ & + \frac{1}{2}\int_{\tau}^{T}\left[\left(D\mathbb{H} + \sigma\right)^\top \mathbb{X}\left(D\mathbb{H} + \sigma\right) + \mathbb{Y}\left(B\mathbb{H} + b\right) + \mathbb{H}^\top c_1 \mathbb{H}\right]\mathrm{d}t. \end{aligned} \tag{40}$$

Similarly, we can derive that

$$\begin{aligned} \bar{Y}_2^{\lambda} = & \frac{1}{2}\left[x^\top \tilde{\mathcal{X}}(0)x + \tilde{\mathcal{Y}}(0)x\right] \\ & + \frac{1}{2}\int_0^{\tau}\left[\left(\tilde{D}\tilde{\mathbb{H}} + \tilde{\sigma}\right)^\top \tilde{\mathcal{X}}\left(\tilde{D}\tilde{\mathbb{H}} + \tilde{\sigma}\right) + \tilde{\mathcal{Y}}\left(\tilde{B}\tilde{\mathbb{H}} + \tilde{b}\right) + \tilde{\mathbb{H}}^\top \tilde{c}_2 \tilde{\mathbb{H}}\right]\mathrm{d}t \end{aligned}$$

$$+\frac{1}{2}\int_{\tau}^{T}\left[(D\mathbb{H}+\sigma)^{\top}\mathcal{X}(D\mathbb{H}+\sigma)+\mathcal{Y}(B\mathbb{H}+b)+\mathbb{H}^{\top}c_2\mathbb{H}\right]\mathrm{d}t, \tag{41}$$

where

$$\begin{cases}\dot{\mathcal{X}}+\mathcal{X}(\mathcal{A}+B\mathbb{K})+(\mathcal{A}+B\mathbb{K})^{\top}\mathcal{X}\\ \quad+\begin{bmatrix}0&0\\0&a_2\end{bmatrix}+\mathbb{K}^{\top}c_2\mathbb{K}+(C+D\mathbb{K})^{\top}\mathcal{X}(C+D\mathbb{K})=0,\\ \mathcal{X}(T)=\begin{bmatrix}0&0\\0&G_2\end{bmatrix},\end{cases} \tag{42}$$

$$\begin{cases}\dot{\mathcal{Y}}+\mathcal{Y}(\mathcal{A}+B\mathbb{K})\\ \quad+2(D\mathbb{H}+\sigma)^{\top}\mathcal{X}(C+D\mathbb{K})+2(B\mathbb{H}+b)^{\top}\mathcal{X}+2\mathbb{H}^{\top}c_2\mathbb{K}=0,\\ \mathcal{Y}(T)=0,\end{cases} \tag{43}$$

$$\begin{cases}\dot{\tilde{\mathcal{X}}}+\tilde{\mathcal{X}}\left(\tilde{\mathcal{A}}+\tilde{B}\tilde{\mathbb{K}}\right)+\left(\tilde{\mathcal{A}}+\tilde{B}\tilde{\mathbb{K}}\right)^{\top}\tilde{\mathcal{X}}\\ \quad+\begin{bmatrix}0&0\\0&\tilde{a}_2\end{bmatrix}+\tilde{\mathbb{K}}^{\top}\tilde{c}_2\tilde{\mathbb{K}}+\left(\tilde{C}+\tilde{D}\tilde{\mathbb{K}}\right)^{\top}\tilde{\mathcal{X}}\left(\tilde{C}+\tilde{D}\tilde{\mathbb{K}}\right)=0,\\ \tilde{\mathcal{X}}(\tau)=\begin{bmatrix}0&0\\0&\tilde{G}_2\end{bmatrix}+K\mathcal{X}(\tau)K^{\top},\end{cases} \tag{44}$$

$$\begin{cases}\dot{\tilde{\mathcal{Y}}}+\tilde{\mathcal{Y}}\left(\tilde{\mathcal{A}}+\tilde{B}\tilde{\mathbb{K}}\right)\\ \quad+2\left(\tilde{D}\tilde{\mathbb{H}}+\tilde{\sigma}\right)^{\top}\tilde{\mathcal{X}}\left(\tilde{C}+\tilde{D}\tilde{\mathbb{K}}\right)+2\left(\tilde{B}\tilde{\mathbb{H}}+\tilde{b}\right)^{\top}\tilde{\mathcal{X}}+2\tilde{\mathbb{H}}^{\top}\tilde{c}_2\tilde{\mathbb{K}}=0,\\ \tilde{\mathcal{Y}}(\tau)=\mathcal{Y}(\tau)K^{\top}.\end{cases} \tag{45}$$

Eqs. (36), (38), (42), and (44) are Lyapunov equations; thus, they are uniquely solvable with nonnegative definite solutions (cf. Lemma 7.3 in Yong and Zhou [5]). Therefore, the linear differential equations (37), (43), (39), and (45) are uniquely solvable.

The analysis above is summarized in the following theorem.

Theorem 2. Let Assumption 1 hold. Then, $\bar{\lambda}\in[0,1]$ s.t. $\bar{v}$ defined by (30) is a robust optimal control of the TSLQU problem.

Furthermore, (i) $\bar{\lambda}=0$ if $\bar{Y}_1^0\leqslant\bar{Y}_2^0$, (ii) $\bar{\lambda}=1$ if $\bar{Y}_1^1\geqslant\bar{Y}_2^1$, and (iii) $\bar{\lambda}$ takes values in $(0,1)$ s.t. (40) = (41).

4 Numerical simulation

By the complicated representations of (40) and (41), obtaining analytical solutions of (40) and (41) is scarcely possible. Therefore, we present a numerical simulation to illustrate the effectiveness of our proposed robust optimal control.

Set $x_1=x_2=1$, $A_1=A_2=\tilde{A}_1=\tilde{A}_2=1$, $B_1=B_2=\tilde{B}_1=\tilde{B}_2=-1$, $C_1=C_2=\tilde{C}_1=\tilde{C}_2=10^{-2}$, $D_1=D_2=\tilde{D}_1=\tilde{D}_2=10^{-4}$, $b_1=b_2=\tilde{b}_1=\tilde{b}_2=\sigma_1=\sigma_2=\tilde{\sigma}_1=\tilde{\sigma}_2=10^{-2}$, $K_1=1$, $K_2=1.2$, $G_1=G_2=\tilde{G}_1=\tilde{G}_2=0.5$, $a_1=\tilde{a}_1=0.5$, $a_2=\tilde{a}_2=5$, $c_1=\tilde{c}_1=1$, $c_2=\tilde{c}_2=0.6$.

Using numerical algorithms, we draw curves of $\bar{Y}_1^{\lambda}$ and $\bar{Y}_2^{\lambda}$ in Figure 1. At $\bar{\lambda}=0.49$, we obtain that $\bar{Y}_1^{\bar{\lambda}}=\bar{Y}_2^{\bar{\lambda}}=1.293$. In Figure 2, we present the relation between $\bar{Y}_1^{\lambda}$ and $\bar{Y}_2^{\lambda}$.

We explain the tradeoff between robustness and optimality in Figure 1. Suppose that the true system situation is $\theta=2$, which is not known by the decision maker. On the one hand, the optimal value of the system under $\theta=2$ is $\bar{Y}_2^0=1.248$, which is smaller than $\bar{Y}_2^{\lambda}=1.293$. Then $\bar{v}^{\bar{\lambda}}$ is not optimal for the system under $\theta=2$. On the other hand, if the decision maker misjudges the system state as 1 and therefore chooses $\bar{v}^1$ as an optimal control, then his cost in this circumstance is $\bar{Y}_2^1=Y_2[\bar{v}^1]=1.648$, which is obviously larger than $\bar{Y}_2^0=1.248$. Now, suppose the decision maker chooses the robust control

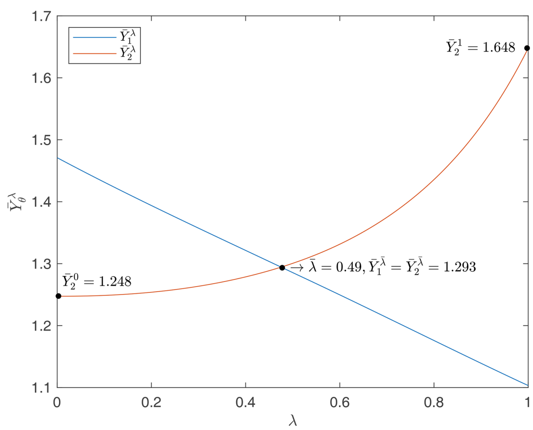

Figure 1 (Color online) Curves of $\bar{Y}_1^{\lambda}$ and $\bar{Y}_2^{\lambda}$.

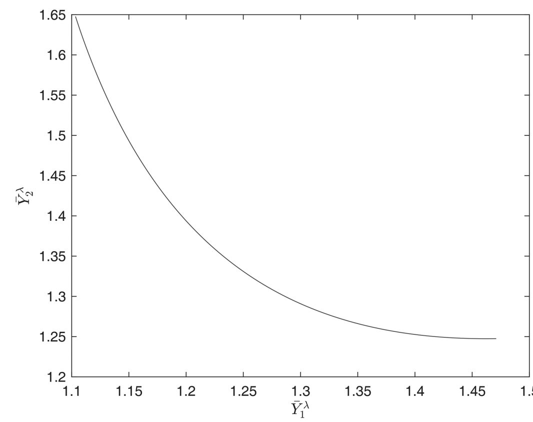

Figure 2 Relation between $\bar{Y}_1^{\lambda}$ and $\bar{Y}_2^{\lambda}$.

$\bar{v}^{\bar{\lambda}}$ instead. Then, we can see that $\bar{Y}_2^{\bar{\lambda}} - \bar{Y}_2^0 = 0.045$, which is the system's loss caused by the robust optimal control $\bar{v}^{\bar{\lambda}}$, is significantly smaller than $\bar{Y}_2^1 - \bar{Y}_2^0 = 0.4$, which is the loss caused by the misjudging of the system state and the choice of $\bar{v}^1$. Similarly, the analysis of $\theta = 1$ can be obtained. Therefore, $\bar{v}^{\bar{\lambda}}$ is effective in dealing with system uncertainty.

5 Conclusion

This paper was devoted to solving the TSLQU problem. Using the convex variational method and convergence technique, a necessary and sufficient optimality condition was obtained. The existence of a robust optimal control was verified, and an optimal parameter was further characterized. Nonetheless, the uniqueness of the optimal parameter was unknown. In the future, we hope to further study the uniqueness of the optimal parameter using methods of functional analysis. Another interesting topic is whether the approaches in our paper still hold for a partially observable state equation with abruptly jumping states. We aim to come back for these topics in subsequent studies.

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