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On potential games over probabilistic Boolean dynamics

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As one of the most fundamental concepts in non-cooperative dynamic games, the dynamic Nash equilibrium has garnered great significance in the past few decades. Unfortunately, the calculation of dynamic Nash equilibria is tricky due to the existence of coupling constraints between agents. Within the realm of static games, Rosenthal [1] proposed the notion of potential games by introducing a fictitious function called the potential function, whose pure Nash equilibrium can be directly derived via maximizing the potential function. Inspired by static potential games, a special class of dynamic games, called dynamic potential games, has been proposed by associating with an optimal control problem, the dynamic Nash equilibrium of which can be easily obtained by solving the corresponding optimal control problem [2].

Different from static games, Nash equilibrium strategies in dynamic games vary in terms of the information structure (e.g., open-loop information structure, feedback information structure). Li et al. [3] proposed a mathematical model of logical dynamic games, which incorporate logical networks to describe the evolution of external states. By associating an optimal control problem with a logical dynamic game, the concept of logical dynamic potential games was proposed, along with its verification criteria and dynamic Nash equilibria analysis under a feedback information structure. However, the verification criterion given in [3] fails to provide a display representation of the corresponding optimal control problem, and the logical dynamic games can only depict finite-valued deterministic systems.

In this paper, we consider open-loop potential games over probabilistic Boolean dynamics. The main contributions are summarized as follows. (i) The verification of an open-loop potential probabilistic dynamic Boolean game (OL-PPDBG) is transformed into verifying whether a series of static subgames are potential games. (ii) A feasible necessary and sufficient condition is proposed for the determination of an OL-PPDBG, along with an algorithm realizing it.

Consider a game played by n agents, where each agent decides in a dynamic environment with available information about the state of the system. The system evolves according to the following probabilistic Boolean networks:

$$x_i(t+1) = f_i(x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)), \qquad (1)$$

where $x_i(t) \in X_i^t = \mathcal{D}$, $i = 1, \dots, m$ and $u_j(t) \in U_j^t = \mathcal{D}$, $j = 1, \dots, n$ are the states of subsystem i and actions of agent j. $f_i : \mathcal{D}^m \times \mathcal{D}^n \to \mathcal{D}$, $i = 1, \dots, m$ are logical functions chosen from a finite candidate set $\mathcal{F}_i = \{f_i^1, \dots, f_i^{l_i}\}$ with a given probability $P_r(f_i = f_i^j) = p_i^j$ satisfying $\sum_{j=1}^{l_i} p_i^j = 1$. Denote the state vector and action vector as $x(t) = [x_1(t), \dots, x_m(t)]$ and $u(t) = [u_1(t), \dots, u_n(t)]$. We focus on a finite-horizon optimization problem over time periods $\{0, 1, \dots, T\}$. The objective function for agent i is

$$J_i(x(0), u) = \mathbb{E}\left[\sum_{t=0}^{T} w_i(x(t), u(t)) + \phi_i(x(T+1))\right], \qquad (2)$$

where $x(0)$ is the initial state, $u = \{u(0), \dots, u(T)\}$ is an admissible action sequence, $\mathbb{E}$ is the expectation, $w_i : \mathcal{D}^m \times \mathcal{D}^n \to \mathbb{R}$ is the stage cost function for agent i, and $\phi_i : \mathcal{D}^m \to \mathbb{R}$ is the terminal cost function. The probabilistic Boolean dynamics (1) with an objective function (2) is called a PDBG.

The action $u_i(t)$ taken by agent i is determined by a mapping $\mu_i(\cdot, \cdot) : \mathcal{D}^m \times \mathbb{N} \to \mathcal{D}$, i.e., $u_i(t) = \mu_i(x(0), t)$, in which case the information structure of agent i is open-loop pattern. All agents take their own strategy based on a given $x(0)$, hence we no longer distinguish $u_i(t)$ and $\mu_i(x(0), t)$. Denote $u^i = [u_i(0), \dots, u_i(T)]$ and $u = (u^i, u^{-i}) \in U$, where U is the admissible joint action sequence set.

Definition 1 ([4])**.** An admissible action sequence $u^* = (u^{1*}, \dots, u^{n*})$ is called a pure strategy open-loop Nash equilibrium (OLNE) of the dynamic game (1) and (2) if

$$J_i(x(0), u^{i*}, u^{-i*}) \leqslant J_i(x(0), u^i, u^{-i*}), \quad i \in N. \qquad (3)$$

By Definition 1, the existence of OLNE is equivalent to solutions of the following optimal control problem:

$$\begin{aligned} &\min_{u^i} \mathbb{E}\left[\sum_{t=0}^{T} w_i(x(t), u_i(t), u_{-i}^*(t)) + \phi_i(x(T+1))\right] \\ &\text{s.t. } x(t+1) = f(x(t), u(t)), \end{aligned} \qquad (4)$$

where $i \in N$, $f = [f_1, \dots, f_m]$.

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Note that it is difficult to find an OLNE since n optimal control problems in (4) are coupled with each other, let alone that the state variables $x(t)$ (constrained by the system dynamics) are also included in (4). In this paper, we focus on a special class of PDBGs that admits a potential game structure, called OL-PPDBGs.

The OL-PPDBG is defined based on the existence of the following optimal control problem that must be solvable and satisfy some conditions. An optimal control problem with probabilistic Boolean dynamics is described as

$$\begin{aligned}&\min_u J(x(0),u)=\mathbb{E}\left[\sum_{t=0}^{T} g_t(x(t),u(t))+h_{T+1}(x(T+1))\right]\\&\text{s.t. } \; x(t+1)=f(x(t),u(t)),\end{aligned}\tag{5}$$

where $g_t:\mathcal{D}^m\times\mathcal{D}^n\to\mathbb{R}$ is the stage cost function at each time step, and $h_{T+1}:\mathcal{D}^m\to\mathbb{R}$ is the terminal cost function.

Definition 2. The PDBG (1) and (2) is called an OL-PPDBG if there exists a solvable optimal control problem (5), such that for all $i\in N$, and for $\hat{u}_i(t),u_i(t)\in U_i^t$, $t\in[0:T]$, $\forall u_{-i}(t)\in U_{-i}^t$, $\forall u(0),\dots,u(t-1)$,

$$J_i(x(0),H_t)-J_i(x(0),\hat{H}_t)=J(x(0),H_t)-J(x(0),\hat{H}_t)\tag{6}$$

holds, where

$$H_t=(u(0),\dots,u(t-1),(u_i(t),u_{-i}(t)),u^*(t+1),\dots,u^*(T)),$$
$$\hat{H}_t=(u(0),\dots,u(t-1),(\hat{u}_i(t),u_{-i}(t)),u^*(t+1),\dots,u^*(T))$$

with $u^*=(u^*(0),\dots,u^*(T))$ being the solution of the optimal control problem (5). In such case, Eq. (5) is called the associated optimal control problem for the OL-PPDBG.

In the following, the conditions, under which a PDBG admits a potential structure, are proposed relying on some constructed static subgames. The value function of agent $i\in N$ at time t is constructed according to (4) as

$$\begin{aligned}\mathbb{E}V_t^i(x(t))=\min_{u_i(t),\dots,u_i(T)}\mathbb{E}\Bigg[&\sum_{p=t}^{T} w_i(x(p),u_i(p),u_{-i}^*(p))\\&+\phi_i(x(T+1))\Bigg],\ \forall i=1,\dots,m,\end{aligned}\tag{7}$$

where u^* is an OLNE. An equivalent transformation of (7) yields that $\mathbb{E}V_t^i(x(t))$ satisfies the following dynamic programming equation:

$$\begin{aligned}\mathbb{E}V_t^i(x(t))=\min_{u_i(t)}\mathbb{E}[&V_{t+1}^i(f(x(t),u_i(t),u_{-i}^*(t)))\\&+w_i(x(t),u_i(t),u_{-i}^*(t))],\ \forall i=1,\dots,m,\end{aligned}\tag{8}$$

and $\mathbb{E}V_{T+1}^i(x(T+1))=\mathbb{E}\phi_i(x(T+1))$, $\forall i=1,\dots,m$. For each time $t=0,1,\dots,T$, a static subgame, denoted by $G_t(\mathbb{E}x(t))$, is defined as $G_t(\mathbb{E}x(t))=(N,\{U_i^t\},\{r_t^i(\mathbb{E}x(t),\cdot)\})$, where $\mathbb{E}x(t)\in X_t$ is the expectation of the state at time t, the cost function of agent i is

$$r_t^i(\mathbb{E}x(t),u(t))=\mathbb{E}[w_i(x(t),u(t))+V_{t+1}^i(f(x(t),u(t)))].\tag{9}$$

Theorem 1 (Subgame condition)**.** The PDBG (1) and (2) is an OL-PPDBG, if and only if for any $t\in[0:T]$, there exists a function $v_t(\mathbb{E}x(t),u(t))$ such that $G_t(\mathbb{E}x(t))$ is a potential game for every $\mathbb{E}x(t)\in X_t$ with $v_t(\mathbb{E}x(t),u(t))$ being its potential function.

Remark 1. The proposed result is still valid in the case that the game considered degenerates into a deterministic logical dynamic game studied in [3]. The results proposed in [3] require that the optimal control problem (5) is known, while the proof of Theorem 1 gives a way to constructing such an optimal control problem.

Proposition 1. An action sequence $u^*=(u^{1*},\dots,u^{n*})$ is an OLNE of the OL-PPDBG, if and only if, $u^*(t)=[u_1^*(t),\dots,u_n^*(t)]$ is a pure Nash equilibrium of the subgame $G_t(\mathbb{E}x(t))$.

In what follows, we propose a feasible approach to the determination of OL-PPDBGs via potential equations. To this end, the algebraic form of (1) obtained by using semi-tensor product (STP) is

$$\mathbb{E}\vec{x}(t+1)=L\mathbb{E}\vec{x}(t)\vec{u}(t),\tag{10}$$

where $\vec{x}(t)=\ltimes_{i=1}^m\vec{x}_i(t)$, $\vec{u}(t)=\ltimes_{i=1}^n\vec{u}_i(t)$, and $L\in\Upsilon_{2^m\times 2^{m+n}}$. For each subgame $G_t(\mathbb{E}x(t))$, $t\in[0:T]$, the cost of agent i can be expressed by

$$r_t^i(\mathbb{E}x(t),u(t))=V_{w_i}\mathbb{E}\vec{x}(t)\vec{u}(t)+V_{t+1}^iL\mathbb{E}\vec{x}(t)\vec{u}(t),$$

where $V_{T+1}^i=V_{\phi_i}$, V_{w_i} and V_{ϕ_i} are the structure vectors of w_i and ϕ_i. Then we can get the payoff vector of $G_t(\mathbb{E}x(t))$ for agent i, i.e., $V^i_{G_t(\mathbb{E}x(t))}=V_{w_i}\mathbb{E}\vec{x}(t)+V_{t+1}^iL\mathbb{E}\vec{x}(t)$.

Theorem 2 (Potential equation condition)**.** The PDBG (1) and (2) is an OL-PPDBG, if and only if for $\forall t\in[0:T]$, the following two conditions hold.

(1) For each $\mathbb{E}x(t)\in X_t$ leading to a subgame $G_t(\mathbb{E}x(t))$, the following matrix equation is solvable:

$$\Xi\xi_{G_t(\mathbb{E}x(t))}=V^{\mathrm{T}}_{G_t(\mathbb{E}x(t))},\tag{11}$$

where $\Xi_i=I_{2^{i-1}}\otimes\mathbf{1}_2\otimes I_{2^{n-i}}, i=1,2,\dots,n$, $V_{G_t(\mathbb{E}x(t))}=[V^2_{G_t(\mathbb{E}x(t))}-V^1_{G_t(\mathbb{E}x(t))},\dots,V^n_{G_t(\mathbb{E}x(t))}-V^1_{G_t(\mathbb{E}x(t))}]$,

$$\Xi=\begin{bmatrix}-\Xi_1&\Xi_2&\mathbf{0}&\cdots&\mathbf{0}\\-\Xi_1&\mathbf{0}&\Xi_3&\cdots&\mathbf{0}\\\vdots&\vdots&\vdots&&\vdots\\-\Xi_1&\mathbf{0}&\mathbf{0}&\cdots&\Xi_n\end{bmatrix},\quad \xi_{G_t(\mathbb{E}x(t))}=\begin{bmatrix}\xi^1_{G_t(\mathbb{E}x(t))}\\\xi^2_{G_t(\mathbb{E}x(t))}\\\vdots\\\xi^n_{G_t(\mathbb{E}x(t))}\end{bmatrix}.$$

(2) The following equation is solvable:

$$V_t\mathbb{E}\vec{x}(t)\vec{u}(t)=V_{\mathbb{E}x(t)}\vec{u}(t),\tag{12}$$

where $\mathbb{E}x(t)\in X_t, u(t)\in\prod_{j=1}^n U_j^t$, $V_{\mathbb{E}x(t)}$ is the structure vector of the potential function of subgame $G_t(\mathbb{E}x(t))$, and obtained by

$$V_{\mathbb{E}x(t)}=V^1_{G_t(\mathbb{E}x(t))}-(\xi^1_{G_t(\mathbb{E}x(t))})^{\mathrm{T}}(\mathbf{1}_2^{\mathrm{T}}\otimes I_{2^{n-1}}).$$

Based on Theorem 2, an algorithm is presented in Appendix C to verify whether a given PDBG (1) and (2) is an OL-PPDBG or not.

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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