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• **Supplementary File** •

Time-delay effects on the dynamical behavior of switched nonlinear time-delay systems

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Appendix A Proof of Lemma 1

For any $\varepsilon > 0$, define the following function

$$F_\varepsilon(t) = \begin{cases} H_{i_k}(t)\exp[-(\varrho+\varepsilon)(t-t_k)], t \in [t_k, t_{k+1}), \\ H_{i_0}(t), t_0 - \tau \leqslant t \leqslant t_0, \end{cases} \tag{A1}$$

Firstly, for any $q \in [t_0, t_1)$, we have two cases to consider. If $q - \tau \geqslant t_0$, by (A1), we get

$$\begin{aligned} e^{\varepsilon(q-t_0)}D^+F_\varepsilon(q) &= e^{\varepsilon(q-t_0)}[D^+H_{i_k}(q)e^{-(\varrho+\varepsilon)(q-t_0)} - (\varrho+\varepsilon)H_{i_k}(q)e^{-(\varrho+\varepsilon)(q-t_0)}] \\ &\leqslant D^+H_{i_k}(q)e^{-\varrho(q-t_0)} - (\varrho+\varepsilon)H_{i_k}(q)e^{-\varrho(q-t_0)} \\ &\leqslant [\alpha_{i_0}H_{i_0}(q) + \beta_{i_0}H_{i_0}(q-\tau)]e^{-\varrho(q-t_0)} - (\varrho+\varepsilon)H_{i_0}(q)e^{-\varrho(q-t_0)} \\ &\leqslant \alpha_{i_0}H_{i_0}(q)e^{-\varrho(q-t_0)} + \beta_{i_0}H_{i_0}(q-\tau)e^{-\varrho(q-t_0)} - (\varrho+\varepsilon)H_{i_0}(q)e^{-\varrho(q-t_0)} \\ &\leqslant (\alpha_{i_0} - \varrho - \varepsilon)H_{i_0}(q)e^{-\varrho(q-t_0)} + \beta_{i_0}H_{i_0}(q-\tau)e^{-\varrho(q-t_0)} \\ &\leqslant (\alpha_{i_0} - \varrho - \varepsilon)F(q) + \beta_{i_0}F(q-\tau)e^{-\varrho\tau} \\ &\leqslant (\alpha - \varrho - \varepsilon + \beta)F(q) \\ &\leqslant -\varepsilon F(q). \end{aligned} \tag{A2}$$

Moreover, if $q - \tau < t_0$, so $t_0 - \tau \leqslant q - \tau < t_0$, then we obtain that

$$\begin{aligned} e^{\varepsilon(q-t_0)}D^+F_\varepsilon(q) &\leqslant (\alpha_{i_0} - \varrho - \varepsilon)F(q) + F(q-\tau)e^{-\varrho(q-t_0)} \\ &\leqslant (\alpha - \varrho - \varepsilon + \beta)F(q) \\ &\leqslant -\varepsilon F(q). \end{aligned} \tag{A3}$$

Secondly, for any $q \in [t_k, t_{k+1})$, we also have the following two cases. Suppose that $q - \tau \geqslant t_k$, we see that

$$\begin{aligned} e^{\varepsilon(q-t_0)}D^+F_\varepsilon(q) &\leqslant (\alpha_{i_k} - \varrho - \varepsilon)H_{i_k}(q)e^{-\varrho(q-t_k)} + \beta_{i_k}H_{i_k}(q-\tau)e^{-\varrho(q-t_k)} \\ &\leqslant (\alpha_{i_k} - \varrho - \varepsilon)F(q) + \beta_{i_k}F(q-\tau)e^{-\varrho\tau} \\ &\leqslant (\alpha - \varrho - \varepsilon + \beta)F(q) \\ &\leqslant -\varepsilon F(q). \end{aligned} \tag{A4}$$

Furthermore, if $q - \tau < t_k$, due to the fact that the delay is small, thus $t_{k-1} \leqslant q - \tau < t_k$, we get

$$\begin{aligned} e^{\varepsilon(q-t_0)}D^+F_\varepsilon(q) &\leqslant (\alpha_{i_k} - \varrho - \varepsilon)H_{i_k}(q)e^{-\varrho(q-t_k)} + \beta_{i_k}H_{i_{k-1}}(q-\tau)e^{-\varrho(q-t_k)} \\ &\leqslant (\alpha_{i_k} - \varrho - \varepsilon)F(q) + \beta_{i_k}F(q-\tau)e^{\varrho(q-\tau-t_k)}e^{-\varrho\tau} \\ &\leqslant (\alpha_{i_k} - \varrho - \varepsilon)F(q) + \beta_{i_k}F(q-\tau)e^{-\varrho\tau} \end{aligned}$$

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$$\begin{aligned}&\leqslant (\alpha_{i_k}-\varrho-\varepsilon)F(q)+\beta_{i_k}\frac{F(q)}{\gamma}e^{-\varrho(t_k-t_{k-1})}e^{-\varrho\tau}\\&\leqslant (\alpha_{i_k}-\varrho-\varepsilon+\frac{\beta_{i_k}}{\gamma}e^{-\varrho\tau})F(q)\\&\leqslant -\varepsilon F(q).\end{aligned}\tag{A5}$$

Therefore, no matter $t\in[t_0,t_1)$ or $q\in[t_k,t_{k+1})$, we derive that

$$e^{\varepsilon(q-t_0)}D^+F_\varepsilon(q)\leqslant -\varepsilon F(q),\tag{A6}$$

which can imply that, for any $q\in[t_k,t_{k+1})$,

$$\begin{aligned}D^+F(q)&=D^+F_\varepsilon(q)e^{\varepsilon(q-t_k)}+\varepsilon F_\varepsilon(q)e^{\varepsilon(q-t_k)}\\&\leqslant -\varepsilon F(q)+\varepsilon F_\varepsilon(q)e^{\varepsilon(q-t_k)}\\&\leqslant 0.\end{aligned}\tag{A7}$$

This concludes the proof.

Remark 1. To illustrate the existence of the constant ϱ in Lemma 1, we define the function $\Psi(s)=\alpha+\frac{\beta}{\gamma}e^{-\tau s}-s$, then the derivative of $\Psi(s):\Psi'(s)=-\frac{\tau\beta}{\gamma}e^{-\tau s}-1<0$, that is, the function $\Psi(s)$ is monotonically decreasing. Moreover, we have $\Psi(0)=\alpha+\frac{\beta}{\gamma}>\alpha+\beta$. So there exist some ϱ such that $\varrho>\alpha+\beta$ and $\alpha+\frac{\beta}{\gamma}e^{-\varrho\tau}-\varrho<0$.

Appendix B Proof of Lemma 2

Firstly, we will demonstrate that

$$H_{i_k}(t)\leqslant \varpi_k e^{\varrho(t-t_0)}, t\in[t_k,t_{k+1}), k\in\mathbb{N},\tag{B1}$$

where ϱ is defined in Lemma 1. According to the definition of function $F(t)$, it is equivalent to proving that

$$F(t)\leqslant \varpi_k e^{\varrho(t_k-t_0)}, t\in[t_k,t_{k+1}).\tag{B2}$$

We claim that (B1) is true for $k=0$, i.e., $F(t)\leqslant\varpi_0=\tilde{H}_{i_0}(t_0)$, $t\in[t_0,t_1)$. It is obvious that $F(t_0)=H_{i_0}(t_0)\leqslant \tilde{H}_{i_0}(t_0)\leqslant\varpi_0$. If (B2) is not valid for any $t\in[t_0,t_1)$, there must be some $t'\in(t_0,t_1)$ such that $F(t')=\varpi_0$, $F(t)\leqslant\varpi_0$, $t\in[t_0,t')$, and $D^+F(t')\geqslant 0$. So $F(s)\leqslant F(t')$ for $t'\in[t_0,t_1)$, $s\in[t_0,t')$. Thus, by Lemma 1, we obtain $D^+F(t')<0$, resulting in a contradiction. Utilizing the mathematical induction, we suppose that (B2) is true for any $t\in[t_k,t_{k+1})$. Next, we will demonstrate (B2) holds for $t\in[t_{k+1},t_{k+2})$.

Therefore

$$\begin{aligned}F(t_{k+1})&=H_{i_{k+1}}(t_{k+1})\\&\leqslant \gamma_{i_k}H_{i_k}(t_{k+1}^-)\\&=\gamma_{i_k}F(t_{k+1}^-)e^{\varrho(t_{k+1}-t_k)}\\&\leqslant \gamma_{i_k}\varpi_k e^{\varrho(t_k-t_0)}e^{\varrho(t_{k+1}-t_k)}\\&\leqslant \gamma_{i_k}\varpi_k e^{\varrho(t_{k+1}-t_0)}\\&\leqslant \varpi_{k+1}e^{\varrho(t_{k+1}-t_0)}.\end{aligned}\tag{B3}$$

So, it is clear that (B2) is valid for $t=t_{k+1}$.

Suppose that (B2) is not satisfied for $t\in(t_{k+1},t_{k+2})$, meaning there exists at least one $\bar{t}\in(t_{k+1},t_{k+2})$ such that $F(\bar{t})=\varpi_{k+1}e^{\varrho(t_{k+1}-t_0)}$, $F(t)\leqslant F(\bar{t})$, $t\in(t_{k+1},\bar{t})$, and $D^+F(\bar{t})\geqslant 0$. Under this circumstance, for any $s\in[t_k,t_{k+1})$, we obtain that

$$\begin{aligned}F(s)&\leqslant \varpi_k e^{\varrho(t_k-t_0)}\\&\leqslant \frac{\varpi_{k+1}}{\gamma}e^{\varrho(t_{k+1}-t_0)}e^{-\varrho(t_{k+1}-t_k)}\\&\leqslant \frac{F(\bar{t})}{\gamma}e^{-\varrho(t_{k+1}-t_k)}.\end{aligned}\tag{B4}$$

Similarly, by Lemma 1, we have $D^+F(\bar{t})<0$, this is another contradiction. Therefore, $F(t)\leqslant\varpi_{k+1}e^{\varrho(t_{k+1}-t_0)}$ is satisfied for all $t\in(t_{k+1},t_{k+2})$.

In summary, we have (B2) holds for all $t\in[t_k,t_{k+1})$, $k\in\mathbb{N}$. Let $\varrho=\varrho^*$, and $\varepsilon\to 0$, we obtain that (B1) is true, this concludes the proof.

Remark 2. It follows from Lemma 2 that the sizes of time delays affect the rate functions of Lyapunov functions. Meanwhile, we know that the function $s(\tau)$ is monotonously decreasing with respect to τ, that is, to increase the delay τ, we can decrease the value of s.

Appendix C Proof of Theorem 1

It can be concluded that, for any $t \in [t_k, t_{k+1})$,

$$\begin{aligned} H_{i_k}(t) &\leqslant \gamma^k e^{\varrho^*(t-t_0)} \tilde{H}_{i_0}(t_0) \\ &\leqslant \gamma^{\sum\limits_{i=1}^{m}(N_{0i}+\frac{T_i(t,t_0)}{\tau_{ai}})} e^{\varrho^*(t-t_0)} \tilde{H}_{i_0}(t_0) \\ &\leqslant \gamma^{\sum\limits_{i=1}^{m} N_{0i}} e^{(\frac{\ln\gamma}{\sum\limits_{i=1}^{m}\tau_{ai}}+\varrho^*)(t-t_0)} \tilde{H}_{i_0}(t_0). \end{aligned} \tag{C1}$$

In addition, based on (C1), and $a_i\|x(t)\|^\eta \leqslant H_i(t) \leqslant b_i\|x(t)\|^\eta$, we can imply that

$$\|x(t)\| \leqslant (\frac{b_{i_0}}{a_i}\gamma^{\sum\limits_{i=1}^{m} N_{0i}})^{\frac{1}{\eta}} \|\phi\| e^{-\xi(t-t_0)}, t \geqslant t_0, \tag{C2}$$

where $\xi = -(\frac{\ln\gamma}{\sum\limits_{i=1}^{m}\tau_{ai}} + \varrho^*)/\eta$. Thus, the SNTDS is GUES.

Remark 3. In contrast with the existing results in Ref. [1], the stability analysis is different. Ref. [1] investigated the asymptotic stability property for switched systems, but we study the exponential stability of a SNTDS, which increases the complexity and challenge. Meanwhile, although some findings have been made on the stability analysis of switched systems with time delays (e.g., see [2–5]), the effects of time delays on the dynamical behavior has not appeared in the previous literature. Thus, the methods in these papers are no longer applicable to our results.

Appendix D Proof of Theorem 2

Consider the following equation

$$\begin{cases} D^+\varphi(t) = \alpha_i\varphi(t) + \beta_i\varphi(t-\tau), t \neq t_k, \\ \varphi(t) = \gamma_i\varphi(t^-), t = t_k, \\ \varphi(t) = \tilde{H}(t_0), t \in [t_0-\tau, t_0], \end{cases} \tag{D1}$$

where $\varphi(t)$ is the solution and can be solved as

$$\varphi(t) = w(t,t_0)\varphi(t_0) + \int_{t_0}^{t} w(t,s)\beta\varphi(t-s)ds, t \geqslant t_0, \tag{D2}$$

and

$$\begin{aligned} w(t,s) =& e^{\alpha(t-s)} \prod_{i=1}^{m} \gamma_i^{N_{\sigma i}(t,s)} \\ &\leqslant \prod_{i=1}^{m} \gamma_i^{N_{0i}} e^{(\alpha+\sum\limits_{i=1}^{m}\frac{\ln\gamma_i}{\tau_{ai}})(t-s)}. \end{aligned} \tag{D3}$$

Let $c = -(\alpha + \sum\limits_{i=1}^{m}\frac{\ln\gamma_i}{\tau_{ai}})$, we know $c > 0$, so

$$\varphi(t) \leqslant \prod_{i=1}^{m}\gamma_i^{N_{0i}}\varphi(t_0)e^{-c(t-t_0)} + \int_{t_0}^{t}\prod_{i=1}^{m}\gamma_i^{N_{0i}}e^{-c(t-s)}\beta\varphi(s-\tau)ds. \tag{D4}$$

We will proceed to demonstrate the following inequality

$$\varphi(t) < \prod_{i=1}^{m}\gamma_i^{N_{0i}}\varphi(t_0)e^{-\lambda(t-t_0)}, t \geqslant t_0, \tag{D5}$$

where $\lambda = c - \prod\limits_{i=1}^{m}\gamma_i^{N_{0i}}\beta e^{\lambda\tau}$. If it does not hold, there are some $t^* > t_0$ meeting

$$\varphi(t^*) \geqslant \prod_{i=1}^{m}\gamma_i^{N_{0i}}\varphi(t_0)e^{-\lambda(t^*-t_0)}, \tag{D6}$$

and

$$\varphi(t) < \prod_{i=1}^{m}\gamma_i^{N_{0i}}\varphi(t_0)e^{-\lambda(t-t_0)}, t < t^*. \tag{D7}$$

Then, by (D4) and (D7), we obtain

$$\begin{aligned}\varphi(t^*) &\leqslant \prod_{i=1}^{m} \gamma_i^{N_{0i}} e^{-c(t^*-t_0)}\varphi(t_0) + \int_{t_0}^{t^*} \prod_{i=1}^{m} \gamma_i^{N_{0i}} e^{-c(t^*-s)} \beta \prod_{i=1}^{m} \gamma_i^{N_{0i}} \varphi(t_0) e^{-\lambda(s-\tau-t_0)} ds \\ &< \prod_{i=1}^{m} \gamma_i^{N_{0i}} \varphi(t_0)[e^{-c(t^*-t_0)} + \prod_{i=1}^{m} \gamma_i^{N_{0i}} \beta e^{\lambda\tau} e^{-ct^*+\lambda t_0} e^{(c-\lambda)s} \frac{1}{c-\lambda}|_{t_0}^{t^*}]. \end{aligned} \tag{D8}$$

By the fact that $\lambda = c - \prod_{i=1}^{m} \gamma_i^{N_{0i}} \beta e^{\lambda\tau}$, we can imply that

$$\begin{aligned}\varphi(t^*) &< \prod_{i=1}^{m} \gamma_i^{N_{0i}} \varphi(t_0)[e^{-c(t^*-t_0)} + (e^{(c-\lambda)t^*} - e^{(c-\lambda)t_0}) e^{-ct^*+\lambda t_0}] \\ &< \prod_{i=1}^{m} \gamma_i^{N_{0i}} \varphi(t_0)[e^{-c(t^*-t_0)} + e^{-\lambda(t^*-t_0)} - e^{-c(t^*-t_0)}] \\ &< \prod_{i=1}^{m} \gamma_i^{N_{0i}} \varphi(t_0) e^{-\lambda(t^*-t_0)}, \end{aligned} \tag{D9}$$

which contradicts (D6). Hence, we conclude that inequality (D5) is valid. According to the comparison principle, we get

$$H_{i_k}(t) \leqslant \varphi(t) < \prod_{i=1}^{m} \gamma_i^{N_{0i}} \varphi(t_0) e^{-\lambda(t-t_0)}. \tag{D10}$$

Then, we can conclude that the SNTDS is GUES.

Remark 4. Different from the existing results in [2–5] investigating the stability property of the SNTDS with small delays, Theorem 2 is used to research the SNTDS with large delays, which is an important system and rarely discussed in the existing literatures. Once the small delays and large delays are confirmed, we can utilize Theorem 1 and Theorem 2 to judge whether the SNTDS is GUES. The works in [6–8] just investigated the effect of large delays on linear systems, without studying nonlinear systems, so our results are common. Based on the switching techniques, the results in [9–11] viewed time-delay systems and the delayed neural network as the switched system to investigate their stability, which are the special circumstances of our findings.

Appendix E Simulation Example

In this example, the following SNTDS with two switched subsystems $S = \{1, 2\}$ is investigated:

$$f_1(t, x(t), x(t-1.6)) = \begin{bmatrix} -3.6x_1(t-1.6) - 0.6|\cos(t)|x_2(t) \\ -5x_1(t) - \cos(t)x_2(t-1.6) \end{bmatrix},$$

$$f_2(t, x(t), x(t-1.6)) = \begin{bmatrix} -\frac{2x_1^2(t)}{1+x_1^2(t)} x_1(t) + 0.1\sin(t)x_2(t-1.6) \\ -1.2\sin(t)x_1(t-1.6) + 0.26x_2(t) \end{bmatrix},$$

$$g_1(t, x(t), x(t-1.6)) = \begin{bmatrix} -\sin(t) & 0.8 \\ 1.1 & -\cos(t) \end{bmatrix}, g_2(t, x(t), x(t-1.6)) = \begin{bmatrix} -1.9 & \cos(t) \\ \sin(t) & -0.3 \end{bmatrix},$$

$$u_1(t) = \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix}, u_2(t) = \begin{bmatrix} 0 \\ x_2(t) \end{bmatrix},$$

where $x(t) = [x_1(t)\ x_2(t)]^T$, and $x_0 = [-10\ 10]^T$. Define the multiple Lyapunov functions $H_1(t) = x_1^2(t) + x_2^2(t)$, and $H_2(t) = 0.75x_1^2(t) + 0.75x_2^2(t)$, then $d_1 = 0.6$, $d_2 = 0.32$. So, we derive that

$$\begin{aligned} D^+H_1(t) &= 2x_1(t)[-3.6x_1(t-1.6) - 0.6|\cos(t)|x_2(t) - \sin(t)x_1(t)] + 2x_2(t)[-5x_1(t) \\ &\quad - \cos(t)x_2(t-1.6) + \cos(t)x_2(t)] \\ &\leqslant -3.2x_1^2(t) - 3.2x_2^2(t) + x_1^2(t-1.6) + x_2^2(t-1.6). \end{aligned} \tag{E1}$$

Meanwhile, we conclude that

$$\begin{aligned} D^+H_2(t) &= 1.5x_1(t)[-\frac{2x_1^2(t)}{1+x_1^2(t)} x_1(t) + 0.1\sin(t)x_2(t-1.6) + \cos(t)x_2(t)] \\ &\quad + 1.5x_1(t)[-1.2\sin(t)x_1(t-1.6) + 0.26x_2(t) - 0.3x_2(t)] \\ &\leqslant -1.5x_1^2(t) - 1.5x_2^2(t) + 0.9x_1^2(t-1.6) + 0.9x_2^2(t-1.6). \end{aligned} \tag{E2}$$

Thus, we can get $\bar{d} = 0.6$, $d^* = 3$, $c = 0.5$, $\xi = 2$, and $\eta^* = -0.6$. So we can conclude that conditions iii), iv), and v) of Theorem 1 are satisfied. Meanwhile, state trajectories and the corresponding switching signal are decipted in the Figure E1 from which it can be implied that system is GUES.

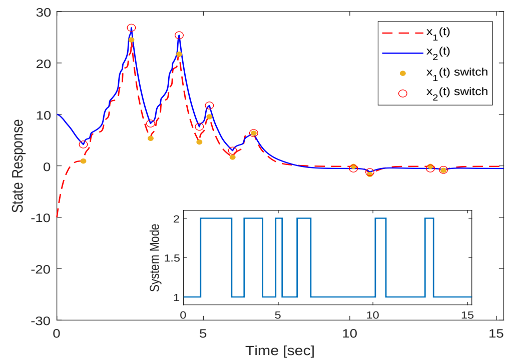

Figure E1 State trajectories and switching signal of the SNTDS.

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